Journal of Computational and Applied Mathematics 367 (2020) 112451

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Super-critical and critical traveling waves in a three-component delayed disease system with mixed diffusion

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ARTICLE INFO

Article history:
Received 26 January 2019
Received in revised form 18 July 2019

MSC:
35C07
92D30
92B05

Keywords:
Three-component disease system
Time delay
Mixed diffusion
Traveling waves

ABSTRACT

This paper deals with traveling waves of a three-component delayed disease system with mixed diffusion. When the basic reproduction number of the corresponding spatial-homogenous delayed differential system $R_0 > 1$ and the wave velocity $c \geq c^*$ (c^* is critical velocity), we obtain that the system admits a positive traveling wave solution. When $R_0 \leq 1$ and $c \in \mathbb{R}$ or $R_0 > 1$ and $c < c^*$, we prove that the system has no positive traveling wave solutions. Our theoretical results may be conducive to disease prevention and control.

1. Introduction

There have been a large amount of work on traveling wave solutions for local and nonlocal diffusion disease systems since they play crucial roles in describing the spatial transmission patterns of infectious diseases [1–27]. In reality, the susceptible and recovered individuals can move randomly over a wide range and have different diffusion distances, while the infected individuals are usually limited in local areas. In addition, many diseases cannot be spread to others instantly after being infected and have an incubation period (or time delay). Based on these facts, we suggest a three-component delayed disease system with mixed diffusion

$$\begin{cases} \partial_t S(x,t) = d_1\big[K_{\lambda_1} * S(x,t) - S(x,t)\big] - \dfrac{\beta S(x,t)I(x,t-\tau)}{S(x,t)+I(x,t-\tau)+R(x,t)}, \\ \partial_t I(x,t) = d_2\partial_{xx}I(x,t) + \dfrac{\beta S(x,t)I(x,t-\tau)}{S(x,t)+I(x,t-\tau)+R(x,t)} - (\gamma+\delta)I(x,t), \\ \partial_t R(x,t) = d_3\big[K_{\lambda_2} * R(x,t) - R(x,t)\big] + \gamma I(x,t), \end{cases} \tag{1.1}$$

where $K_{\lambda_1} * S(x,t) = \int_{\mathbb{R}} K_{\lambda_1}(y)S(x-y,t)dy$, $K_{\lambda_2} * R(x,t) = \int_{\mathbb{R}} K_{\lambda_2}(y)R(x-y,t)dy$ and $K_{\lambda_i}(y) = \frac{1}{\lambda_i}K\left(\frac{y}{\lambda_i}\right)$ for $i = 1, 2$. In (1.1), $S(x,t)$, $I(x,t)$ and $R(x,t)$ denote the densities of susceptible, infected and recovered individuals at location x and time t, respectively. The coefficients $d_i > 0$ ($i = 1, 2, 3$) represent the diffusion rates of each class, the parameters $\lambda_i > 0$ ($i = 1, 2$) refer to the nonlocal diffusion distances of susceptible and recovered individuals, $\beta > 0$ is the transmission rate,

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https://doi.org/10.1016/j.cam.2019.112451

$\gamma > 0$ denotes the recovery rate, $\delta \geq 0$ is the disease-induced death rate and $\tau > 0$ is the time delay. The convolution operators $K_{\lambda_1} * S(x,t) - S(x,t)$ and $K_{\lambda_2} * R(x,t) - R(x,t)$ depict that the rate of susceptible and recovered individuals in position x and at time t depend on the influence of neighboring S and R in all other positions y [21,28–32]. Moreover, model (1.1) with standard incidence function $\beta SI/(S+I+R)$ describes that some of infected individuals will be removed from the population due to disease-induced death, but other recovered individuals will return into the community, which reflects the essential propagation dynamics of infectious diseases [2,8,18,19]. Throughout this paper, the kernel function $K(x)$ satisfies

$$K(x) \in C(\mathbb{R}),\ K(x) = K(-x) \geq 0,\ \int_{-\infty}^{\infty} K(x)dx = 1,\ K(x) \text{ is compactly supported.} \tag{K}$$

As stated by Li [33], a standard kernel function

$$K(x) = \begin{cases} C\exp\left(\dfrac{1}{x^2-1}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where $C > 0$ is some appropriate constant, fulfills the assumption (K).

A traveling wave solution to (1.1) is a special solution in the form of $(S(x,t), I(x,t), R(x,t)) = (S(z), I(z), R(z))$, $z = x+ct$, which satisfies

$$d_1\left[\int_{-\infty}^{\infty} K_{\lambda_1}(y)S(z-y)dy - S(z)\right] - cS'(z) - \frac{\beta S(z)I(z-c\tau)}{S(z)+I(z-c\tau)+R(z)} = 0, \tag{1.2}$$

$$d_2I''(z) - cI'(z) + \frac{\beta S(z)I(z-c\tau)}{S(z)+I(z-c\tau)+R(z)} - (\gamma+\delta)I(z) = 0, \tag{1.3}$$

$$d_3\left[\int_{-\infty}^{\infty} K_{\lambda_2}(y)R(z-y)dy - R(z)\right] - cR'(z) + \gamma I(z) = 0. \tag{1.4}$$

Obviously, system (1.2)–(1.4) has infinitely many equilibrium $(S, 0, R)$ with arbitrary constants $S \geq 0$ and $R \geq 0$ $(S+R \neq 0)$. The purpose of the present paper is to look for positive solutions $(S(z), I(z), R(z))$ of (1.2)–(1.4) satisfying the following asymptotic boundary

$$\begin{aligned} &S(-\infty) := S_{-\infty} = S_1, \quad I(-\infty) := I_{-\infty} = 0, \quad R(-\infty) := R_{-\infty} = 0, \\ &S(\infty) := S_\infty < S_1, \qquad I(\infty) := I_\infty = 0, \qquad R(\infty) := R_\infty = \frac{\gamma(S_1 - S_\infty)}{\gamma+\delta}, \end{aligned}$$

where $S_1 > 0$ is a given constant.

As far as we know, the research on mixed diffusion equations is quite few [34–36]. Recently, Wu et al. [36] investigated a non-delayed disease system with mixed diffusion

$$\begin{cases} \partial_t S(x,t) = d_1\big[K * S(x,t) - S(x,t)\big] + \omega - \mu S(x,t) - h(S(x,t))g(I(x,t)), \\ \partial_t I(x,t) = d_2\partial_{xx}I(x,t) + h(S(x,t))g(I(x,t)) - \mu I(x,t) - d(I(x,t)), \\ \partial_t R(x,t) = d_3\big[K * R(x,t) - R(x,t)\big] + d(I(x,t)) - \mu R(x,t), \end{cases} \tag{1.5}$$

where $K * S(x,t) = \int_{\mathbb{R}} K(y)S(x-y,t)dy$, $K * R(x,t) = \int_{\mathbb{R}} K(y)R(x-y,t)dy$, $\omega > 0$ is the constant external supplies, $\mu > 0$ denotes the natural death rate, $h(S)$, $g(I)$ and $d(I)$ are abstract nonlinear functions. The corresponding ordinary differential system admits a disease-free equilibrium $(\frac{\omega}{\mu}, 0, 0)$. If $R_0 = h(\frac{\omega}{\mu})g'(0)/[\mu + d'(0)] > 1$, then (1.5) has a unique positive endemic equilibrium (S^*, I^*, R^*). Since that the first two equations in (1.5) form a closed system, Wu et al. only consider the subsystem for S-component and I-component. Under certain assumptions on functions $K(\cdot)$, $h(\cdot)$, $g(\cdot)$ and $d(\cdot)$, they applied Schauder's fixed point theorem together with the upper–lower solutions method to obtain that the subsystem has a positive traveling wave solution if $R_0 > 1$ and $c \geq c^* = 2\sqrt{d_2[h(\frac{\omega}{\mu})g'(0) - \mu - d'(0)]}$. Meanwhile, they used the squeeze theorem and the Lyapunov functional technique to deduce the asymptotic boundary for S and I. They also utilized the polar coordinates transform to establish that the subsystem admits no nonnegative traveling wave solutions if $R_0 > 1$ and $0 < c < c^*$.

Compared with system (1.5), we would like to point out that system (1.1) has four differences. Firstly, the first two equations in (1.5) are independent of the third one, while the three equations in (1.1) are not independent. Secondly, (1.5) introduces the same nonlocal diffusion distances of susceptible and recovered individuals, whereas (1.1) concerns the different ones. Thirdly, (1.5) does not consider the fact that the biological development of many diseases have an incubation period/time delay, while (1.1) incorporates this important biological factor. Finally, (1.5) has two stationary solutions, whereas (1.1) admits infinitely many stationary solutions. Therefore, the methods in [36] cannot be directly utilized to investigate our model. Enlighted by [3,8,14,36], we will employ Schauder's fixed point theorem together with the upper–lower solutions method to obtain that (1.1) has super-critical and critical traveling wave solutions on the real line. By contradictory arguments coupled with the theory of classical analysis, we will derive the positiveness of traveling wave solutions. Applying the upper–lower solutions and squeeze theorem, we will get the asymptotic boundary of traveling wave solutions at minus infinity. With the aid of Fluctuation Lemma and subtle analysis method, we will deduce the asymptotic boundary of traveling wave solutions at plus infinity. By virtue of bilateral Laplace transform, we will establish the nonexistence of positive traveling waves.

The paper is organized as follows. Section 2 shows the preliminaries and main results. Sections 3 and 4 establish the existence of super-critical and critical positive traveling wave solutions, respectively. Section 5 is devoted to the nonexistence of nontrivial and positive traveling wave solutions. Section 6 implements some numerical simulations. Short conclusions and discussions are given in Section 7.

2. Preliminaries and main results

To begin with, we present the following two lemmas.

Lemma 2.1. *Let*

$$\Psi(\rho, c) := d_2\rho^2 - c\rho + \beta e^{-\rho c\tau} - \gamma - \delta.$$

Then the following assertions are valid for $R_0 := \beta/(\gamma+\delta) > 1$.

(i) There exist two positive constants ρ^* and c^* such that $\Psi(\rho^*, c^*) = \Psi_\rho(\rho^*, c^*) = 0$.
(ii) If $0 < c < c^*$, then $\Psi(\rho, c) > 0$ for $\rho \in [0, \infty)$.
(iii) If $c > c^*$, then equation $\Psi(\rho, c) = 0$ admits two positive roots $\rho_1(c) := \rho_1$ and $\rho_2(c) := \rho_2$ with $\rho_1 < \rho^* < \rho_2$ such that $\Psi(\rho, c) < 0$ for $\rho \in (\rho_1, \rho_2)$ and $\Psi(\rho, c) > 0$ for $\rho \in [0, \rho_1) \cup (\rho_2, \infty)$.

Proof. Since $R_0 > 1$, we have $\Psi(\rho, 0) = d_2\rho^2 + \beta - \gamma - \delta > 0$ and $\Psi(0, c) = \beta - \gamma - \delta > 0$. For each fixed $\rho > 0$, we deduce $\lim_{c\to\infty} \Psi(\rho, c) = -\infty$ and $\Psi_c(\rho, c) = -\rho - \beta\rho\tau e^{-\rho c\tau} < 0$. For any fixed $c > 0$, it follows that $\lim_{\rho\to\infty} \Psi(\rho, c) = \infty$ and $\Psi_\rho(0, c) = -c - \beta c\tau < 0$. Note that $\Psi_{\rho\rho}(\rho, c) = 2d_2 + \beta c^2\tau^2 e^{-\rho c\tau} > 0$ for any $(\rho, c) \in \mathbb{R} \times \mathbb{R}$. Then the assertions hold. □

Lemma 2.2. *Let*

$$\Phi(\rho, c) := c\rho - d_3 \int_{-\infty}^{\infty} K(y) e^{-\rho\lambda_2 y} dy + d_3.$$

Then for each fixed $c > 0$, there exists some $\tilde{\rho} > 0$ such that $\Phi(\rho, c) > 0$ for any $\rho \in (0, \tilde{\rho})$.

Proof. By basic calculations, we obtain $\Phi(0, c) = 0$, $\Phi_\rho(0, c) = c > 0$, $\Phi_{\rho\rho}(\rho, c) = -d_3\lambda_2^2 \int_{\mathbb{R}} y^2 K(y) e^{-\rho\lambda_2 y} dy < 0$ and $\Phi(\infty, c) = -\infty$ for each fixed $c > 0$. The claim is shown. □

Now we state our results.

Theorem 2.1. *Assume that $R_0 > 1$ and $c \geq c^*$. Then system (1.2)–(1.4) has a positive solution $(S(z), I(z), R(z))$ satisfying $S_{-\infty} = S_1$, $S_\infty < S_1$, $I_{\pm\infty} = 0$ and $R_{-\infty} = 0$. Moreover, if $R(z)$ is bounded on $\mathbb{R}$, then $R_\infty = \frac{\gamma(S_1 - S_\infty)}{\gamma+\delta}$. Additionally, as $z \to -\infty$, $I(z) = O(e^{\rho_1 z})$ for $c > c^*$ and $I(z) = O(-ze^{\rho^* z})$ for $c = c^*$.*

Theorem 2.2. *Assume that $R_0 \leq 1$ and $c \in \mathbb{R}$ or $R_0 > 1$ and $c < c^*$. Then system (1.2)–(1.4) admits no positive solutions $(S(z), I(z), R(z))$ satisfying $S_{-\infty} = S_1 > S_\infty$, $I_{\pm\infty} = 0$, $R_{-\infty} = 0$ and $\sup_{z\in\mathbb{R}} R(z) < \infty$.*

Remark 2.1. Theorem 2.2 indicates that (1.1) has no positive traveling wave solutions for $R_0 > 1$ and $c \leq 0$. While Wu et al. [36] just established the nonexistence results of traveling wave solutions with $0 < c < c^*$. We think that our method adopted in Section 5 can be used to obtain the nonexistence results in (1.5) for nonpositive wave velocity.

3. Super-critical traveling waves

In this section, we restrict our attention to the existence result for $R_0 > 1$ and $c > c^*$ in Theorem 2.1. For $z \in \mathbb{R}$, define the following nonnegative continuous functions by

$$S_+(z) := S_1, \qquad S_-(z) := \begin{cases} S_1(1 - \epsilon_1^{-1} e^{\epsilon_1 z}), & z < z_2, \\ 0, & z \geq z_2, \end{cases}$$
$$I_+(z) := \begin{cases} e^{\rho_1 z}, & z < z_1, \\ I_1, & z \geq z_1, \end{cases} \qquad I_-(z) := \begin{cases} e^{\rho_1 z} - L_1 e^{(\rho_1+\epsilon_2)z}, & z < z_3, \\ 0, & z \geq z_3, \end{cases}$$
$$R_+(z) := L_2 e^{\epsilon_2 z}, \qquad R_-(z) := 0,$$

where $I_1 = \frac{(\beta-\gamma-\delta)S_1}{\gamma+\delta}$, $z_1 = \rho_1^{-1} \ln I_1$, $z_2 = \epsilon_1^{-1} \ln \epsilon_1$, $z_3 = -\epsilon_2^{-1} \ln L_1$, ρ_1 is defined in Lemma 2.1, L_i and ϵ_i $(i = 1, 2)$ are positive constants which will be specified in the following lemma.

Lemma 3.1. *For given sufficiently small $\epsilon_1 > 0$, $\epsilon_2 > 0$ and sufficiently large $L_1 > 0$, $L_2 > 0$, the functions $S_\pm(z)$, $I_\pm(z)$ and $R_\pm(z)$ satisfy*

$$d_1\left[\int_{-\infty}^{\infty} K_{\lambda_1}(y)S_+(z-y)dy - S_+(z)\right] - cS'_+(z) - \frac{\beta S_+(z)I_-(z-c\tau)}{S_+(z)+I_-(z-c\tau)+R_+(z)} \le 0, \quad z\in\mathbb{R}, \tag{3.1}$$

$$d_2 I''_+(z) - cI'_+(z) + \frac{\beta S_+(z)I_+(z-c\tau)}{S_+(z)+I_+(z-c\tau)+R_-(z)} - (\gamma+\delta)I_+(z) \le 0, \quad z\neq z_1, \tag{3.2}$$

$$d_3\left[\int_{-\infty}^{\infty} K_{\lambda_2}(y)R_+(z-y)dy - R_+(z)\right] - cR'_+(z) + \gamma I_+(z) \le 0, \quad z\in\mathbb{R}, \tag{3.3}$$

$$d_1\left[\int_{-\infty}^{\infty} K_{\lambda_1}(y)S_-(z-y)dy - S_-(z)\right] - cS'_-(z) - \frac{\beta S_-(z)I_+(z-c\tau)}{S_-(z)+I_+(z-c\tau)+R_-(z)} \ge 0, \quad z\neq z_2, \tag{3.4}$$

$$d_2 I''_-(z) - cI'_-(z) + \frac{\beta S_-(z)I_-(z-c\tau)}{S_-(z)+I_-(z-c\tau)+R_+(z)} - (\gamma+\delta)I_-(z) \ge 0, \quad z\neq z_3, \tag{3.5}$$

$$d_3\left[\int_{-\infty}^{\infty} K_{\lambda_2}(y)R_-(z-y)dy - R_-(z)\right] - cR'_-(z) + \gamma I_-(z) \ge 0, \quad z\in\mathbb{R}. \tag{3.6}$$

Proof. Proof of (3.1). If $z\in\mathbb{R}$, then $\int_{\mathbb{R}} K_{\lambda_1}(y)S_+(z-y)dy = S_1$, $I_-(z)\ge 0$ and $R_+(z) > 0$. Obviously, we have

$$\begin{aligned}
&d_1\left[\int_{-\infty}^{\infty} K_{\lambda_1}(y)S_+(z-y)dy - S_+(z)\right] - cS'_+(z) - \frac{\beta S_+(z)I_-(z-c\tau)}{S_+(z)+I_-(z-c\tau)+R_+(z)}\\
&= -\frac{\beta S_1 I_-(z-c\tau)}{S_1+I_-(z-c\tau)+R_+(z)}\\
&\le 0 \quad \text{for } z\in\mathbb{R}.
\end{aligned}$$

Proof of (3.2). If $z < z_1$, then $I_+(z) = e^{\rho_1 z}$, $I_+(z-c\tau) = e^{\rho_1(z-c\tau)}$ and $R_-(z) = 0$. By Lemma 2.1, we get

$$\begin{aligned}
&d_2 I''_+(z) - cI'_+(z) + \frac{\beta S_+(z)I_+(z-c\tau)}{S_+(z)+I_+(z-c\tau)+R_-(z)} - (\gamma+\delta)I_+(z)\\
&\le d_2 I''_+(z) - cI'_+(z) + \beta I_+(z-c\tau) - (\gamma+\delta)I_+(z)\\
&= e^{\rho_1 z}\left(d_2\rho_1^2 - c\rho_1 + \beta e^{-\rho_1 c\tau} - \gamma - \delta\right)\\
&= 0 \quad \text{for } z < z_1.
\end{aligned}$$

If $z > z_1$, then $I_+(z-c\tau) \le I_+(z) = I_1 = \frac{(\beta-\gamma-\delta)S_1}{\gamma+\delta}$ and $R_-(z) = 0$. It follows that

$$\begin{aligned}
&d_2 I''_+(z) - cI'_+(z) + \frac{\beta S_+(z)I_+(z-c\tau)}{S_+(z)+I_+(z-c\tau)+R_-(z)} - (\gamma+\delta)I_+(z)\\
&\le \frac{\beta S_1 I_1}{S_1+I_1} - (\gamma+\delta)I_1\\
&= 0 \quad \text{for } z > z_1.
\end{aligned}$$

Proof of (3.3). If $z < z_1$, then $I_+(z) = e^{\rho_1 z}$ and $R_+(z) = L_2 e^{\epsilon_2 z}$. We derive from Lemma 2.2 that

$$\begin{aligned}
&d_3\left[\int_{-\infty}^{\infty} K_{\lambda_2}(y)R_+(z-y)dy - R_+(z)\right] - cR'_+(z) + \gamma I_+(z)\\
&= d_3\int_{-\infty}^{\infty}\frac{1}{\lambda_2}K\left(\frac{y}{\lambda_2}\right)L_2 e^{\epsilon_2(z-y)}dy - d_3 L_2 e^{\epsilon_2 z} - cL_2\epsilon_2 e^{\epsilon_2 z} + \gamma e^{\rho_1 z}\\
&= L_2 e^{\epsilon_2 z}\left[d_3\int_{-\infty}^{\infty} K(y)e^{-\epsilon_2\lambda_2 y}dy - d_3 - c\epsilon_2 + \frac{\gamma}{L_2}e^{(\rho_1-\epsilon_2)z}\right]\\
&\le L_2 e^{\epsilon_2 z}\left[\frac{\gamma}{L_2}e^{(\rho_1-\epsilon_2)z_1} - \Phi(\epsilon_2, c)\right]\\
&\le 0 \quad \text{for } z < z_1,
\end{aligned}$$

which is valid for $\epsilon_2 \in (0, \min\{\tilde{\rho}, \rho_1\})$ and sufficiently large $L_2 > 0$. If $z \geq z_1$, then $R_+(z) = L_2 e^{\epsilon_2 z}$ and $I_+(z) = I_1$. Thus we obtain

$$\begin{aligned} & d_3\left[\int_{-\infty}^{\infty} K_{\lambda_2}(y)R_+(z-y)dy - R_+(z)\right] - cR'_+(z) + \gamma I_+(z) \\ & = L_2 e^{\epsilon_2 z}\left[d_3 \int_{-\infty}^{\infty} K(y)e^{-\epsilon_2\lambda_2 y}dy - d_3 - c\epsilon_2 + \frac{\gamma}{L_2} I_1 e^{-\epsilon_2 z}\right] \\ & \leq L_2 e^{\epsilon_2 z}\left[\frac{\gamma}{L_2} I_1 e^{-\epsilon_2 z_1} - \Phi(\epsilon_2, c)\right] \\ & \leq 0 \quad \text{for } z \geq z_1, \end{aligned}$$

which is true for $\epsilon_2 \in (0, \tilde{\rho})$ and sufficiently large $L_2 > 0$.

Proof of (3.4). By the definition of $S_-(z)$ and (K), we deduce for $\mathbb{R}$ that

$$\int_{-\infty}^{\infty} K_{\lambda_1}(y)S_-(z-y)dy \geq \max\left\{S_1 - S_1\epsilon_1^{-1}e^{\epsilon_1 z}\int_{-\infty}^{\infty} K(y)e^{-\epsilon_1\lambda_1 y}dy,\ 0\right\}. \tag{3.7}$$

Let $\epsilon_1 \in (0, \rho_1)$ be sufficiently small such that $z_2 < z_1$ and

$$d_1 S_1 \epsilon_1^{-1}\int_{-\infty}^{\infty} K(y)(1 - e^{-\epsilon_1\lambda_1 y})dy + cS_1 - \beta e^{(\rho_1-\epsilon_1)z} \geq 0 \quad \text{for } z < z_2. \tag{3.8}$$

If $z < z_2$, then $S_-(z) = S_1(1 - \epsilon_1^{-1}e^{\epsilon_1 z})$ and $I_+(z - c\tau) \leq e^{\rho_1 z}$. We infer from (3.7) and (3.8) that

$$\begin{aligned} & d_1\left[\int_{-\infty}^{\infty} K_{\lambda_1}(y)S_-(z-y)dy - S_-(z)\right] - cS'_-(z) - \frac{\beta S_-(z)I_+(z-c\tau)}{S_-(z) + I_+(z-c\tau) + R_-(z)} \\ & \geq d_1 S_1\left[\epsilon_1^{-1}e^{\epsilon_1 z} - \epsilon_1^{-1}e^{\epsilon_1 z}\int_{-\infty}^{\infty} K(y)e^{-\epsilon_1\lambda_1 y}dy\right] + cS_1 e^{\epsilon_1 z} - \beta e^{\rho_1 z} \\ & \geq e^{\epsilon_1 z}\left[d_1 S_1\epsilon_1^{-1}\int_{-\infty}^{\infty} K(y)(1 - e^{-\epsilon_1\lambda_1 y})dy + cS_1 - \beta e^{(\rho_1-\epsilon_1)z}\right] \\ & \geq 0 \quad \text{for } z < z_2. \end{aligned}$$

If $z > z_2$, then $S_-(z) = 0$. From (3.7), we have

$$\begin{aligned} & d_1\left[\int_{-\infty}^{\infty} K_{\lambda_1}(y)S_-(z-y)dy - S_-(z)\right] - cS'_-(z) - \frac{\beta S_-(z)I_+(z-c\tau)}{S_-(z) + I_+(z-c\tau) + R_-(z)} \\ & = d_1\int_{-\infty}^{\infty} K_{\lambda_1}(y)S_-(z-y)dy \\ & \geq 0 \quad \text{for } z > z_2. \end{aligned}$$

Proof of (3.5). If $z > z_3$, then (3.5) holds trivially. Choose sufficiently small $\epsilon_2 \in (0, \min\{\epsilon_1, \rho_2 - \rho_1\})$ and sufficiently large $L_1 > 1$ such that $z_3 < z_2$ and $1 - \epsilon_1^{-1}e^{\epsilon_1 z_3} \geq 1/2$. Then we get for $z < z_3$ that

$$I_-(z) = e^{\rho_1 z} - L_1 e^{(\rho_1+\epsilon_2)z}, \quad S_-(z) = S_1(1 - \epsilon_1^{-1}e^{\epsilon_1 z}) \geq \frac{1}{2}S_1 \quad \text{and} \quad R_+(z) = L_2 e^{\epsilon_2 z}. \tag{3.9}$$

Due to $\epsilon_2 < \epsilon_1 < \rho_1$, we get

$$e^{(\rho_1-\epsilon_2)z} < 1 \quad \text{for } z < z_3 < 0. \tag{3.10}$$

Note that

$$\Psi(\rho_1, c) = 0 \quad \text{and} \quad \Psi(\rho_1 + \epsilon_2, c) < 0 \tag{3.11}$$

since $\rho_1 < \rho_1 + \epsilon_2 < \rho_2$. Using (3.9)–(3.11), we derive for $z < z_3$ that

$$\begin{aligned}
& d_2 I_-''(z) - cI_-'(z) + \frac{\beta S_-(z) I_-(z-c\tau)}{S_-(z) + I_-(z-c\tau) + R_+(z)} - (\gamma+\delta) I_-(z) \\
&= d_2 I_-''(z) - cI_-'(z) + \beta I_-(z-c\tau) - (\gamma+\delta) I_-(z) - \frac{\beta[I_-^2(z-c\tau) + I_-(z-c\tau)R_+(z)]}{S_-(z) + I_-(z-c\tau) + R_+(z)} \\
&\geq e^{\rho_1 z}\left(d_2\rho_1^2 - c\rho_1 + \beta e^{-\rho_1 c\tau} - \gamma - \delta\right) - \frac{\beta[I_-^2(z-c\tau) + I_-(z-c\tau)R_+(z)]}{S_-(z)} \\
&\quad - L_1 e^{(\rho_1+\epsilon_2)z}\left[d_2(\rho_1+\epsilon_2)^2 - c(\rho_1+\epsilon_2) + \beta e^{-(\rho_1+\epsilon_2)c\tau} - \gamma - \delta\right] \\
&= e^{\rho_1 z}\Psi(\rho_1, c) - L_1 e^{(\rho_1+\epsilon_2)z}\Psi(\rho_1+\epsilon_2, c) - \frac{\beta\left[e^{\rho_1(z-c\tau)} - L_1 e^{(\rho_1+\epsilon_2)(z-c\tau)}\right]^2 + \beta\left[e^{\rho_1(z-c\tau)} - L_1 e^{(\rho_1+\epsilon_2)(z-c\tau)}\right]L_2 e^{\epsilon_2 z}}{S_1(1-\epsilon_1^{-1}e^{\epsilon_1 z})} \\
&\geq -L_1 e^{(\rho_1+\epsilon_2)z}\Psi(\rho_1+\epsilon_2, c) - \frac{2\beta\left[e^{2\rho_1 z} + L_2 e^{(\rho_1+\epsilon_2)z}\right]}{S_1} \\
&= -e^{(\rho_1+\epsilon_2)z}\Psi(\rho_1+\epsilon_2, c)\left\{L_1 - \frac{2\beta[e^{(\rho_1-\epsilon_2)z} + L_2]}{-\Psi(\rho_1+\epsilon_2, c)S_1}\right\} \\
&\geq -e^{(\rho_1+\epsilon_2)z}\Psi(\rho_1+\epsilon_2, c)\left[L_1 - \frac{2\beta(1+L_2)}{-\Psi(\rho_1+\epsilon_2, c)S_1}\right] \\
&\geq 0
\end{aligned}$$

for sufficiently large $L_1 > 1$.

Proof of (3.6). By the definitions of $R_-(z)$ and $I_-(z)$, it is easy to see that (3.6) holds. □

Now we introduce a functional space

$$B_\mu(\mathbb{R}, \mathbb{R}^3) := \left\{\varphi(z) = (\varphi_1(z), \varphi_2(z), \varphi_3(z)) \in C(\mathbb{R}, \mathbb{R}^3) : \sup_{z\in\mathbb{R}} |\varphi_i(z)| e^{-\mu|z|} < \infty,\ i = 1, 2, 3\right\}$$

equipped with the norm $|\varphi|_\mu := \max\left\{\sup_{z\in\mathbb{R}} |\varphi_i(z)| e^{-\mu|z|},\ i = 1, 2, 3\right\}$, where μ is a constant satisfying

$$\epsilon_2 < \mu < \min\left\{\frac{d_1+\alpha}{c}, \frac{d_3}{c}, \frac{\sqrt{c^2 + 4d_2(\gamma+\delta)} - c}{2d_2}\right\}$$

and α will be determined later. Define a cone by

$$\mathcal{S} := \left\{ (S(z), I(z), R(z)) \in B_\mu(\mathbb{R}, \mathbb{R}^3) \;\middle|\; \begin{array}{l} S_-(z) \leq S(z) \leq S_+(z), \\ I_-(z) \leq I(z) \leq I_+(z), \\ R_-(z) \leq R(z) \leq R_+(z) \end{array} \right\}.$$

Clearly, the cone $\mathcal{S}$ is nonempty, bounded, closed and convex in $B_\mu(\mathbb{R}, \mathbb{R}^3)$. Define a function $A[S, I, R](z) : \mathcal{S} \mapsto C(\mathbb{R})$ by

$$A[S, I, R](z) := \begin{cases} \dfrac{\beta S(z) I(z-c\tau)}{S(z) + I(z-c\tau) + R(z)}, & S(z)I(z-c\tau) \neq 0, \\ 0, & S(z)I(z-c\tau) = 0. \end{cases}$$

Given a constant α satisfying $\alpha > \beta$, we define three functions by

$$\begin{aligned}
H_1[S, I, R](z) &:= d_1 \int_{-\infty}^{\infty} K_{\lambda_1}(y) S(z-y) dy + \alpha S(z) - A[S, I, R](z), \\
H_2[S, I, R](z) &:= A[S, I, R](z), \\
H_3[S, I, R](z) &:= d_3 \int_{-\infty}^{\infty} K_{\lambda_2}(y) R(z-y) dy + \gamma I(z).
\end{aligned}$$

For $z \in \mathbb{R}$, one can see that $H_1[S, I, R](z)$ is monotonically decreasing with respect to $I(z)$ and monotonically increasing in both $S(z)$ and $R(z)$; $H_2[S, I, R](z)$ is monotonically decreasing with respect to $R(z)$ and monotonically increasing in both $S(z)$ and $I(z)$; $H_3[S, I, R](z)$ is monotonically increasing in both $I(z)$ and $R(z)$. For any $(S(z), I(z), R(z)) \in \mathcal{S}$, define a nonlinear map $\mathcal{M} := (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ on the space $B_\mu(\mathbb{R}, \mathbb{R}^3)$ by

$$\mathcal{M}_1[S, I, R](z) := \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{d_1+\alpha}{c}(z-\eta)} H_1[S, I, R](\eta)d\eta,$$
$$\mathcal{M}_2[S, I, R](z) := \frac{1}{\Lambda}\left\{\int_{-\infty}^{z} e^{\sigma_1(z-\eta)} H_2[S, I, R](\eta)d\eta + \int_{z}^{\infty} e^{\sigma_2(z-\eta)} H_2[S, I, R](\eta)d\eta\right\},$$
$$\mathcal{M}_3[S, I, R](z) := \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{d_3}{c}(z-\eta)} H_3[S, I, R](\eta)d\eta,$$

where

$$\sigma_1 = \frac{c-\sqrt{c^2+4d_2(\gamma+\delta)}}{2d_2}, \quad \sigma_2 = \frac{c+\sqrt{c^2+4d_2(\gamma+\delta)}}{2d_2} \quad \text{and} \quad \Lambda = d_2(\sigma_2-\sigma_1).$$

Lemma 3.2. $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$: $\mathcal{S} \mapsto \mathcal{S}$.

Proof. By the monotonicity of H_i $(i = 1, 2, 3)$ and definition of $B_\mu(\mathbb{R}, \mathbb{R}^3)$, we need to show for any $(S, I, R) \in \mathcal{S}$ that

$$S_-(z) \le \mathcal{M}_1[S_-, I_+, R_-](z) \le \mathcal{M}_1[S, I, R](z) \le \mathcal{M}_1[S_+, I_-, R_+](z) \le S_+(z), \tag{3.12}$$
$$I_-(z) \le \mathcal{M}_2[S_-, I_-, R_+](z) \le \mathcal{M}_2[S, I, R](z) \le \mathcal{M}_2[S_+, I_+, R_-](z) \le I_+(z) \tag{3.13}$$

and

$$R_-(z) \le \mathcal{M}_3[S, I_-, R_-](z) \le \mathcal{M}_3[S, I, R](z) \le \mathcal{M}_3[S, I_+, R_+](z) \le R_+(z). \tag{3.14}$$

Proof of (3.12). It follows from (3.1) that

$$\begin{aligned}\mathcal{M}_1[S_+, I_-, R_+](z) &= \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{d_1+\alpha}{c}(z-\eta)} H_1[S_+, I_-, R_+](\eta)d\eta \\ &\le \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{d_1+\alpha}{c}(z-\eta)} \big[cS_+'(\eta) + (d_1+\alpha)S_+(\eta)\big]d\eta \\ &= S_+(z) \quad \text{for } z \in \mathbb{R}.\end{aligned}$$

On the other hand, we infer from (3.4) that

$$\begin{aligned}\mathcal{M}_1[S_-, I_+, R_-](z) &= \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{d_1+\alpha}{c}(z-\eta)} H_1[S_-, I_+, R_-](\eta)d\eta \\ &\ge \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{d_1+\alpha}{c}(z-\eta)} \big[cS_-'(\eta) + (d_1+\alpha)S_-(\eta)\big]d\eta \\ &= S_-(z) \quad \text{for } z \ne z_2.\end{aligned}$$

Then $\mathcal{M}_1[S_-, I_+, R_-](z) \ge S_-(z)$ for $z \in \mathbb{R}$ since the continuity of both $\mathcal{M}_1[S_-, I_+, R_-](z)$ and $S_-(z)$ at the point z_2.

Proof of (3.13). Note that

$$-d_2\sigma_1\sigma_2 = \gamma+\delta, \quad e^{\rho_1 z_1} = I_1 \quad \text{and} \quad c\rho_1 - d_2\rho_1^2 + \gamma + \delta = -d_2(\rho_1-\sigma_1)(\rho_1-\sigma_2). \tag{3.15}$$

Then by (3.2) and (3.15), we derive that

$$\begin{aligned}\mathcal{M}_2[S_+, I_+, R_-](z) &= \frac{1}{\Lambda}\left\{\int_{-\infty}^{z} e^{\sigma_1(z-\eta)} H_2[S_+, I_+, R_-](\eta)d\eta + \int_{z}^{\infty} e^{\sigma_2(z-\eta)} H_2[S_+, I_+, R_-](\eta)d\eta\right\} \\ &\le \frac{1}{\Lambda}\int_{-\infty}^{z} e^{\sigma_1(z-\eta)}\big[cI_+'(\eta) - d_2 I_+''(\eta) + (\gamma+\delta)I_+(\eta)\big]d\eta \\ &\quad + \frac{1}{\Lambda}\int_{z}^{\infty} e^{\sigma_2(z-\eta)}\big[cI_+'(\eta) - d_2 I_+''(\eta) + (\gamma+\delta)I_+(\eta)\big]d\eta \\ &= \frac{1}{\Lambda}\int_{-\infty}^{z_1} e^{\sigma_1(z-\eta)}\big(c\rho_1 - d_2\rho_1^2 + \gamma + \delta\big)e^{\rho_1\eta}d\eta \\ &\quad + \frac{(\beta-\gamma-\delta)S_1}{\Lambda}\int_{z_1}^{z} e^{\sigma_1(z-\eta)}d\eta + \frac{(\beta-\gamma-\delta)S_1}{\Lambda}\int_{z}^{\infty} e^{\sigma_2(z-\eta)}d\eta \\ &= \frac{\sigma_2-\rho_1}{\sigma_2-\sigma_1}I_1 e^{\sigma_1(z-z_1)} + \frac{\sigma_2}{\sigma_2-\sigma_1}I_1 - \frac{\sigma_2}{\sigma_2-\sigma_1}I_1 e^{\sigma_1(z-z_1)} - \frac{\sigma_1}{\sigma_2-\sigma_1}I_1 \\ &= I_1 - \frac{\rho_1}{\sigma_2-\sigma_1}I_1 e^{\sigma_1(z-z_1)} \\ &\le I_1 \quad \text{for } z > z_1\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}_2[S_+,I_+,R_-](z) &\le \frac{1}{\Lambda}\int_{-\infty}^{z} e^{\sigma_1(z-\eta)}\big[cI_+'(\eta)-d_2I_+''(\eta)+(\gamma+\delta)I_+(\eta)\big]d\eta\\
&\quad+\frac{1}{\Lambda}\int_{z}^{\infty} e^{\sigma_2(z-\eta)}\big[cI_+'(\eta)-d_2I_+''(\eta)+(\gamma+\delta)I_+(\eta)\big]d\eta\\
&=\frac{1}{\Lambda}\int_{-\infty}^{z} e^{\sigma_1(z-\eta)}\big(c\rho_1-d_2\rho_1^2+\gamma+\delta\big)e^{\rho_1\eta}d\eta\\
&\quad+\frac{1}{\Lambda}\int_{z}^{z_1} e^{\sigma_2(z-\eta)}\big(c\rho_1-d_2\rho_1^2+\gamma+\delta\big)e^{\rho_1\eta}d\eta+\frac{(\beta-\gamma-\delta)S_1}{\sigma_2\Lambda}e^{\sigma_2(z-z_1)}\\
&=e^{\rho_1 z}-\frac{\rho_1}{\sigma_2-\sigma_1}I_1e^{\sigma_2(z-z_1)}\\
&\le e^{\rho_1 z}\quad\text{for } z<z_1.
\end{aligned}$$

Using (3.5) and the fact that $e^{\rho_1 z_3}=L_1e^{(\rho_1+\epsilon_2)z_3}$, we obtain that

$$\begin{aligned}
\mathcal{M}_2[S_-,I_-,R_+](z) &= \frac{1}{\Lambda}\left\{\int_{-\infty}^{z} e^{\sigma_1(z-\eta)}H_2[S_-,I_-,R_+](\eta)d\eta+\int_{z}^{\infty} e^{\sigma_2(z-\eta)}H_2[S_-,I_-,R_+](\eta)d\eta\right\}\\
&\ge \frac{1}{\Lambda}\int_{-\infty}^{z_3} e^{\sigma_1(z-\eta)}[cI_-'(\eta)-d_2I_-''(\eta)+(\gamma+\delta)I_-(\eta)]d\eta\\
&=\frac{1}{\Lambda}\int_{-\infty}^{z_3} e^{\sigma_1(z-\eta)}(c\rho_1-d_2\rho_1^2+\gamma+\delta)e^{\rho_1\eta}d\eta\\
&\quad+\frac{1}{\Lambda}\int_{-\infty}^{z_3} e^{\sigma_1(z-\eta)}[-c(\rho_1+\epsilon_2)+d_2(\rho_1+\epsilon_2)^2-\gamma-\delta]L_1e^{(\rho_1+\epsilon_2)\eta}d\eta\\
&\ge \frac{\epsilon_2}{\sigma_2-\sigma_1}e^{\rho_1z_3+\sigma_1(z-z_3)}\\
&\ge 0\quad\text{for } z>z_3
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}_2[S_-,I_-,R_+](z) &\ge \frac{1}{\Lambda}\int_{-\infty}^{z} e^{\sigma_1(z-\eta)}\big[cI_-'(\eta)-d_2I_-''(\eta)+(\gamma+\delta)I_-(\eta)\big]d\eta\\
&\quad+\frac{1}{\Lambda}\int_{z}^{z_3} e^{\sigma_2(z-\eta)}\big[cI_-'(\eta)-d_2I_-''(\eta)+(\gamma+\delta)I_-(\eta)\big]d\eta\\
&=\frac{1}{\Lambda}\int_{-\infty}^{z} e^{\sigma_1(z-\eta)}(c\rho_1-d_2\rho_1^2+\gamma+\delta)e^{\rho_1\eta}d\eta\\
&\quad+\frac{1}{\Lambda}\int_{-\infty}^{z} e^{\sigma_1(z-\eta)}[-c(\rho_1+\epsilon_2)+d_2(\rho_1+\epsilon_2)^2-\gamma-\delta]L_1e^{(\rho_1+\epsilon_2)\eta}d\eta\\
&\quad+\frac{1}{\Lambda}\int_{z}^{z_3} e^{\sigma_2(z-\eta)}(c\rho_1-d_2\rho_1^2+\gamma+\delta)e^{\rho_1\eta}d\eta\\
&\quad+\frac{1}{\Lambda}\int_{z}^{z_3} e^{\sigma_2(z-\eta)}[-c(\rho_1+\epsilon_2)+d_2(\rho_1+\epsilon_2)^2-\gamma-\delta]L_1e^{(\rho_1+\epsilon_2)\eta}d\eta\\
&=e^{\rho_1 z}-L_1e^{(\rho_1+\epsilon_2)z}+\frac{\epsilon_2}{\sigma_2-\sigma_1}e^{\rho_1z_3+\sigma_2(z-z_3)}\\
&\ge e^{\rho_1 z}-L_1e^{(\rho_1+\epsilon_2)z}\quad\text{for } z<z_3.
\end{aligned}$$

Applying the continuity of $\mathcal{M}_2[S_\pm,I_\pm,R_\mp](z)$ and $I_\pm(z)$, one can have (3.13) holds.

Proof of (3.14). Using (3.3) and (3.6), we deduce that

$$\begin{aligned}
\mathcal{M}_3[S,I_+,R_+](z) &= \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{d_3}{c}(z-\eta)}H_3[S,I_+,R_+](\eta)d\eta\\
&\le \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{d_3}{c}(z-\eta)}\big[cR_+'(\eta)+d_3R_+(\eta)\big]d\eta\\
&=R_+(z)\quad\text{for } z\in\mathbb{R}
\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_3[S,I_-,R_-](z) &= \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{d_3}{c}(z-\eta)}H_3[S,I_-,R_-](\eta)d\eta \\ &\geq \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{d_3}{c}(z-\eta)}\big[cR_-'(\eta)+d_3R_-(\eta)\big]d\eta \\ &= R_-(z) \quad \text{for} \quad z\in\mathbb{R}. \quad \Box\end{aligned}$$

Lemma 3.3. *The map $\mathcal{M}=(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3)$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^3)$.*

Proof. For any $(S_1,I_1,R_1)\in\mathcal{S}$ and $(S_2,I_2,R_2)\in\mathcal{S}$, we derive that

$$\begin{aligned}&\big|H_1(S_1,I_1,R_1)(z)-H_1(S_2,I_2,R_2)(z)\big|e^{-\mu|z|} \\ &\leq d_1e^{-\mu|z|}\int_{-\infty}^{\infty}K_{\lambda_1}(y)\big|S_1(z-y)-S_2(z-y)\big|dy+\alpha\big|S_1(z)-S_2(z)\big|e^{-\mu|z|} \\ &\quad+2\beta\big|S_1(z)-S_2(z)\big|e^{-\mu|z|}+2\beta\big|I_1(z)-I_2(z)\big|e^{-\mu|z|}+\beta\big|R_1(z)-R_2(z)\big|e^{-\mu|z|} \\ &=\frac{d_1}{\lambda_1}\int_{-\infty}^{\infty}K\left(\frac{z-y}{\lambda_1}\right)\big|S_1(y)-S_2(y)\big|e^{-\mu|y|}e^{-\mu|z|+\mu|y|}dy+(\alpha+2\beta)\big|S_1(z)-S_2(z)\big|e^{-\mu|z|} \\ &\quad+2\beta\big|I_1(z-c\tau)-I_2(z-c\tau)\big|e^{-\mu|z-c\tau|}e^{\mu c\tau}+\beta\big|R_1(z)-R_2(z)\big|e^{-\mu|z|} \\ &\leq\left[\frac{d_1}{\lambda_1}\int_{-\infty}^{\infty}K\left(\frac{z-y}{\lambda_1}\right)e^{\mu|z-y|}dy\right]\big|S_1-S_2\big|_\mu+(\alpha+2\beta)\big|S_1-S_2\big|_\mu \\ &\quad+2\beta e^{\mu c\tau}\big|I_1-I_2\big|_\mu+\beta\big|R_1-R_2\big|_\mu \\ &\leq\left[d_1\int_{-\infty}^{\infty}K(y)e^{\mu\lambda_1|y|}dy+\alpha+2\beta\right]\big|S_1-S_2\big|_\mu+2\beta e^{\mu c\tau}\big|I_1-I_2\big|_\mu+\beta\big|R_1-R_2\big|_\mu,\end{aligned}$$

$$\big|H_2(S_1,I_1,R_1)(z)-H_2(S_2,I_2,R_2)(z)\big|e^{-\mu|z|}\leq 2\beta\big|S_1-S_2\big|_\mu+2\beta e^{\mu c\tau}\big|I_1-I_2\big|_\mu+\beta\big|R_1-R_2\big|_\mu$$

and

$$\big|H_3(S_1,I_1,R_1)(z)-H_3(S_2,I_2,R_2)(z)\big|e^{-\mu|z|}\leq\left[d_3\int_{-\infty}^{\infty}K(y)e^{\mu\lambda_2|y|}dy\right]\big|R_1-R_2\big|_\mu+\gamma\big|I_1-I_2\big|_\mu,$$

which coupled with (K) indicates that there exist constants $l_i>0$ $(i=1,2,3)$ such that

$$\big|H_i(S_1,I_1,R_1)-H_i(S_2,I_2,R_2)\big|_\mu\leq l_i\big[|S_1-S_2|_\mu+|I_1-I_2|_\mu+|R_1-R_2|_\mu\big].$$

Then we obtain that

$$\begin{aligned}\big|\mathcal{M}_1(S_1,I_1,R_1)(z)-\mathcal{M}_1(S_2,I_2,R_2)(z)\big|e^{-\mu|z|} &\leq\frac{1}{c}\big|H_1(S_1,I_1,R_1)-H_1(S_2,I_2,R_2)\big|_\mu\int_{-\infty}^{z}e^{-\frac{d_1+\alpha}{c}(z-\eta)}e^{\mu|\eta|-\mu|z|}d\eta \\ &\leq\frac{1}{c}\big|H_1(S_1,I_1,R_1)-H_1(S_2,I_2,R_2)\big|_\mu\int_{-\infty}^{z}e^{-\frac{d_1+\alpha}{c}(z-\eta)}e^{\mu|\eta-z|}d\eta \\ &=\frac{l_1}{d_1+\alpha-c\mu}\left[\big|S_1-S_2\big|_\mu+\big|I_1-I_2\big|_\mu+\big|R_1-R_2\big|_\mu\right],\end{aligned}$$

$$\begin{aligned}&\big|\mathcal{M}_2(S_1,I_1,R_1)(z)-\mathcal{M}_2(S_2,I_2,R_2)(z)\big|e^{-\mu|z|} \\ &\leq\frac{1}{\Lambda}\big|H_2(S_1,I_1,R_1)-H_2(S_2,I_2,R_2)\big|_\mu\left[\int_{-\infty}^{z}e^{\sigma_1(z-\eta)}e^{\mu|\eta|-\mu|z|}d\eta+\int_{z}^{\infty}e^{\sigma_2(z-\eta)}e^{\mu|\eta|-\mu|z|}d\eta\right] \\ &\leq\frac{1}{\Lambda}\big|H_2(S_1,I_1,R_1)-H_2(S_2,I_2,R_2)\big|_\mu\left[\int_{-\infty}^{z}e^{\sigma_1(z-\eta)}e^{\mu|\eta-z|}d\eta+\int_{z}^{\infty}e^{\sigma_2(z-\eta)}e^{\mu|\eta-z|}d\eta\right] \\ &\leq\frac{l_2(2\mu+\sigma_1-\sigma_2)}{d_2(\sigma_2-\sigma_1)(\sigma_1+\mu)(\sigma_2-\mu)}\left[\big|S_1-S_2\big|_\mu+\big|I_1-I_2\big|_\mu+\big|R_1-R_2\big|_\mu\right]\end{aligned}$$

and

$$\begin{aligned}\left|\mathcal{M}_3(S_1,I_1,R_1)(z)-\mathcal{M}_3(S_2,I_2,R_2)(z)\right|e^{-\mu|z|} &\le \frac{1}{c}\left|H_3(S_1,I_1,R_1)-H_3(S_2,I_2,R_2)\right|_\mu \int_{-\infty}^{z} e^{-\frac{d_3}{c}(z-\eta)}e^{\mu|\eta|-\mu|z|}d\eta \\ &\le \frac{1}{c}\left|H_3(S_1,I_1,R_1)-H_3(S_2,I_2,R_2)\right|_\mu \int_{-\infty}^{z} e^{-\frac{d_3}{c}(z-\eta)}e^{\mu|\eta-z|}d\eta \\ &= \frac{l_3}{d_3-c\mu}\left[\left|I_1-I_2\right|_\mu+\left|R_1-R_2\right|_\mu\right]\end{aligned}$$

for $0<\mu<\min\{(d_1+\alpha)/c, d_3/c, -\sigma_1\}$. □

Lemma 3.4. *The map $\mathcal{M}=(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3)$ is compact with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^3)$.*

Proof. This proof can be carried out in a similar manner as that in [4] and we give the details for completeness. For any $(S,I,R)\in\mathcal{S}$, we derive from (K) that $H_1(S,I,R)(z)\le(d_1+\alpha)S_1$, $H_2(S,I,R)(z)\le\beta S_1$ and

$$\begin{aligned}H_3(S,I,R)(z) &\le d_3L_2e^{\epsilon_2 z}\int_{-\infty}^{z}K_{\lambda_2}(y)e^{-\epsilon_2 y}dy+\gamma I_1 \\ &= d_3L_2e^{\epsilon_2 z}\int_{-\infty}^{\infty}K(y)e^{-\epsilon_2\lambda_2 y}dy+\gamma I_1 \\ &= \kappa_0e^{\epsilon_2 z}+\gamma I_1 \quad \text{for } z\in\mathbb{R},\end{aligned}$$

where $\kappa_0=d_3L_2\int_{\mathbb{R}}K(y)e^{-\epsilon_2\lambda_2 y}dy>0$ is a constant. Then a direct computation yields

$$\begin{aligned}\left|\frac{d\mathcal{M}_1[S,I,R](z)}{dz}\right| &= \left|-\frac{d_1+\alpha}{c^2}e^{-\frac{d_1+\alpha}{c}z}\int_{-\infty}^{z}e^{\frac{d_1+\alpha}{c}\eta}H_1(S,I,R)(\eta)d\eta+\frac{1}{c}H_1(S,I,R)(z)\right| \\ &\le \frac{2(d_1+\alpha)S_1}{c},\end{aligned} \tag{3.16}$$

$$\begin{aligned}\left|\frac{d\mathcal{M}_2[S,I,R](z)}{dz}\right| &= \frac{1}{\Lambda}\left|\int_{-\infty}^{z}\sigma_1e^{\sigma_1(z-\eta)}H_2(S,I,R)(\eta)d\eta+\int_{z}^{\infty}\sigma_2e^{\sigma_2(z-\eta)}H_2(S,I,R)(\eta)d\eta\right| \\ &\le \frac{\beta S_1}{\Lambda}\left[\int_{-\infty}^{z}|\sigma_1|e^{\sigma_1(z-\eta)}d\eta+\int_{z}^{\infty}\sigma_2e^{\sigma_2(z-\eta)}d\eta\right] \\ &\le \frac{2\beta S_1}{d_2(\sigma_2-\sigma_1)}\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\left|\frac{d\mathcal{M}_3[S,I,R](z)}{dz}\right| &= \left|-\frac{d_3}{c^2}e^{-\frac{d_3}{c}z}\int_{-\infty}^{z}e^{\frac{d_3}{c}\eta}H_3(S,I,R)(\eta)d\eta+\frac{1}{c}H_3(S,I,R)(z)\right| \\ &\le \left|\frac{d_3}{c^2}e^{-\frac{d_3}{c}z}\int_{-\infty}^{z}e^{\frac{d_3}{c}\eta}(\kappa_1e^{\epsilon_2\eta}+\gamma I_1)d\eta+\frac{1}{c}(\kappa_1e^{\epsilon_2 z}+\gamma I_1)\right| \\ &= \frac{2d_3\kappa_0+c\epsilon_2\kappa_0}{cd_3+c^2\epsilon_2}e^{\epsilon_2 z}+\frac{2\gamma I_1}{c} \quad \text{for } z\in\mathbb{R}.\end{aligned} \tag{3.18}$$

Meanwhile, Lemma 3.2 shows that $\left|\mathcal{M}_1[S,I,R](z)\right|+\left|\mathcal{M}_2[S,I,R](z)\right|+\left|\mathcal{M}_3[S,I,R](z)\right|\le S_1+I_1+L_2e^{\epsilon_2 z}$ for $z\in\mathbb{R}$. Recall that $\mu>\epsilon_2$. Then for any $\varepsilon>0$, there exists a large enough number $N>0$ such that

$$\begin{aligned}\left\{\left|\mathcal{M}_1[S,I,R](z)\right|+\left|\mathcal{M}_2[S,I,R](z)\right|+\left|\mathcal{M}_3[S,I,R](z)\right|\right\}e^{-\mu|z|} &\le (S_1+I_1+L_2e^{\epsilon_2 z})e^{-\mu|z|} \\ &< (S_1+I_1)e^{-\mu N}+L_2e^{(\epsilon_2-\mu)N} \\ &< \varepsilon, \quad \text{for } |z|>N.\end{aligned} \tag{3.19}$$

Utilizing (3.16)–(3.18) and Arzerà–Ascoli theorem, we can choose finite elements in $\mathcal{M}(\mathcal{S})$ such that they are a finite ε-net of $\mathcal{M}(\mathcal{S})(z)$ for $z\in[-N,N]$ with the supremum norm, which is also a finite ε-net of $\mathcal{M}(\mathcal{S})(z)$ for $z\in\mathbb{R}$ with the decay norm $|\cdot|_\mu$ (see (3.19)). Therefore, the map $\mathcal{M}$ is compact with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^3)$. □

According to Lemmas 3.2–3.4 and Schauder's fixed point theorem, we obtain that the map $\mathcal{M}$ has a fixed point $(S(z),I(z),R(z))\in\mathcal{S}$, which satisfies

$$S_-(z)\le S(z)\le S_+(z),\quad I_-(z)\le I(z)\le I_+(z)\quad\text{and}\quad R_-(z)\le R(z)\le R_+(z),\quad z\in\mathbb{R}. \tag{3.20}$$

In the following, we investigate some properties of the fixed point $(S(z), I(z), R(z))$ of $\mathcal{M}$.

Lemma 3.5. *The fixed point $(S(z), I(z), R(z))$ of $\mathcal{M}$ is the solution of (1.2)–(1.4) and satisfies the following assertions.*

(i) $0 < S(z) < S_1$, $0 < I(z) < I_1$ *and* $R(z) > 0$;
(ii) $(S_{-\infty}, I_{-\infty}, R_{-\infty}) = (S_1, 0, 0)$, $\lim_{z\to-\infty} I(z)e^{-\rho_1 z} = 1$ *and* $I_\infty = 0$;
(iii) The limit S_∞ *exists and* $S_\infty < S_1$;
(iv) $(\gamma+\delta)\int_{\mathbb{R}} I(z)dz = \int_{\mathbb{R}} \frac{\beta S(z)I(z-c\tau)}{S(z)+I(z-c\tau)+R(z)}dz = c(S_1 - S_\infty)$;
(v) If $\limsup_{z\to\infty} R(z) < \infty$, *then* $R_\infty = \frac{\gamma(S_1-S_\infty)}{\gamma+\delta}$.

Proof. (i) Recalling that $\alpha > \beta$ and $S_-(z) \le S(z) \le S_+(z)$ in $\mathbb{R}$, we have

$$\begin{aligned}
S(z) &= \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{d_1+\alpha}{c}(z-\eta)}\left[d_1\int_{-\infty}^{\infty} K_{\lambda_1}(y)S(\eta-y)dy + \alpha S(\eta) - A[S,I,R](\eta)\right]d\eta \\
&\ge \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{d_1+\alpha}{c}(z-\eta)}\left[d_1\int_{-\infty}^{\infty} K_{\lambda_1}(y)S(\eta-y)dy + (\alpha-\beta)S(\eta)\right]d\eta \\
&\ge \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{d_1+\alpha}{c}(z-\eta)}\left[d_1\int_{-\infty}^{\infty} K_{\lambda_1}(y)S_-(\eta-y)dy + (\alpha-\beta)S_-(\eta)\right]d\eta \\
&> 0 \quad \text{for } z \in \mathbb{R}.
\end{aligned}$$

Analogously, one can obtain that $I(z) > 0$ and $R(z) > 0$ in $\mathbb{R}$. This together with the definition of $\mathcal{M}$ implies that the fixed point $(S(z), I(z), R(z))$ of $\mathcal{M}$ is the solution of (1.2)–(1.4). This rest of the proof is based on the way of contradiction. Suppose that there exists some $\acute{z} \in \mathbb{R}$ such that $S(\acute{z}) = S_1$, then $S'(\acute{z}) = 0$. By (1.2), (K), $I(z) > 0$, $R(z) > 0$ and $S(z) \le S_1$ for $z \in \mathbb{R}$, we deduce

$$\begin{aligned}
0 &= d_1\left[\int_{-\infty}^{\infty} K_{\lambda_1}(y)S(\acute{z}-y)dy - S(\acute{z})\right] - cS'(\acute{z}) - \frac{\beta S(\acute{z})I(\acute{z}-c\tau)}{S(\acute{z})+I(\acute{z}-c\tau)+R(\acute{z})} \\
&= d_1\left[\int_{-\infty}^{\infty} K_{\lambda_1}(y)S(\acute{z}-y)dy - S_1\right] - \frac{\beta S_1 I(\acute{z}-c\tau)}{S_1+I(\acute{z}-c\tau)+R(\acute{z})} \\
&< 0.
\end{aligned}$$

A contradiction occurs, so $S(z) < S_1$ for $z \in \mathbb{R}$. Assume that there is some $\grave{z} \in \mathbb{R}$ such that $I(\grave{z}) = I_1$, then $I'(\grave{z}) = 0$ and $I''(\grave{z}) \le 0$. Using (1.3), $S(z) < S_1$, $R(z) > 0$ and $I(z) \le I_1$ for $z \in \mathbb{R}$, we obtain

$$\begin{aligned}
0 &= d_2 I''(\grave{z}) - cI'(\grave{z}) + \frac{\beta S(\grave{z})I(\grave{z}-c\tau)}{S(\grave{z})+I(\grave{z}-c\tau)+R(\grave{z})} - (\gamma+\delta)I(\grave{z}) \\
&< \frac{\beta S(\grave{z})I(\grave{z}-c\tau)}{S(\grave{z})+I(\grave{z}-c\tau)} - (\gamma+\delta)I(\grave{z}) \\
&< \frac{\beta S_1 I_1}{S_1+I_1} - (\gamma+\delta)I_1 \\
&= 0,
\end{aligned}$$

which leads to a contradiction. Hence $I(z) < I_1$ for $z \in \mathbb{R}$.

(ii) Applying squeeze theorem in (3.20) leads to $(S_{-\infty}, I_{-\infty}, R_{-\infty}) = (S_1, 0, 0)$ and $\lim_{z\to-\infty} I(z)e^{-\rho_1 z} = 1$. Integrating (1.2) from η to ξ ($\eta < \xi$) and utilizing (K) and $0 < S(z) < S_1$ for $z \in \mathbb{R}$, we get

$$\begin{aligned}
\int_{\eta}^{\xi} \frac{\beta S(z)I(z-c\tau)}{S(z)+I(z-c\tau)+R(z)}dz &= d_1\int_{\eta}^{\xi}\int_{-\infty}^{\infty} K_{\lambda_1}(y)[S(z-y)-S(z)]dydz - \int_{\eta}^{\xi} cS'(z)dz \\
&= -d_1\int_{\eta}^{\xi}\int_{-\infty}^{\infty} K_{\lambda_1}(y)y\int_0^1 S'(z-\theta y)d\theta dydz - cS(\xi) + cS(\eta) \\
&= d_1\int_{-\infty}^{\infty} K_{\lambda_1}(y)y\int_0^1 [S(\eta-\theta y)-S(\xi-\theta y)]d\theta dy - cS(\xi) + cS(\eta) \\
&= d_1\int_{0}^{\infty} K_{\lambda_1}(y)y\int_0^1 [S(\eta-\theta y)-S(\xi-\theta y)]d\theta dy \\
&\quad + d_1\int_{0}^{\infty} K_{\lambda_1}(y)y\int_0^1 [S(\xi+\theta y)-S(\eta+\theta y)]d\theta dy - cS(\xi) + cS(\eta) \\
&< 2d_1 S_1\int_0^{\infty} K_{\lambda_1}(y)ydy + cS_1,
\end{aligned}$$

for any $\eta, \xi \in \mathbb{R}$, which implies that

$$\int_{-\infty}^{\infty} \frac{\beta S(z)I(z-c\tau)}{S(z)+I(z-c\tau)+R(z)}dz < \infty. \tag{3.21}$$

Using (1.3) and the boundedness of $I(z)$ on $\mathbb{R}$, we have

$$I(z) = \frac{1}{\Lambda}\left[\int_{-\infty}^{z} e^{\sigma_1(z-\eta)} \frac{\beta S(\eta)I(\eta-c\tau)}{S(\eta)+I(\eta-c\tau)+R(\eta)}d\eta + \int_{z}^{\infty} e^{\sigma_2(z-\eta)} \frac{\beta S(\eta)I(\eta-c\tau)}{S(\eta)+I(\eta-c\tau)+R(\eta)}d\eta\right]. \tag{3.22}$$

Then by (3.17) and (3.22), we deduce

$$|I'(z)| \le \frac{2\beta S_1}{d_2(\sigma_2-\sigma_1)} \quad \text{for } z \in \mathbb{R}. \tag{3.23}$$

Integrating (1.3) from η to ξ $(\eta < \xi)$ and using (K) and $0 < I(z) < I_1$ in $\mathbb{R}$, we get

$$\begin{aligned}(\gamma+\delta)\int_{\eta}^{\xi} I(z)dz &= d_2[I'(\xi)-I'(\eta)] - c[I(\xi)-I(\eta)] + \int_{\eta}^{\xi} \frac{\beta S(z)I(z-c\tau)}{S(z)+I(z-c\tau)+R(z)}dz \\ &< \frac{4\beta S_1}{\sigma_2-\sigma_1} + cI_1 + 2d_1S_1\int_0^{\infty} K_{\lambda_1}(y)ydy + cS_1\end{aligned}$$

for any $\eta, \xi \in \mathbb{R}$, which ensures that

$$\int_{-\infty}^{\infty} I(z)dz < \infty. \tag{3.24}$$

Therefore, we obtain from (3.23) and (3.24) that $I_\infty = 0$.

(iii) To obtain the existence of S_∞, we suppose for the contrary that $\limsup_{z\to\infty} S(z) > \liminf_{z\to\infty} S(z)$. Then Fluctuation Lemma [37] claims that there exists a sequence $\{z_n\}$ satisfying $z_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} S(z_n) = \limsup_{z\to\infty} S(z) := \varrho_1 \quad \text{and} \quad S'(z_n) = 0. \tag{3.25}$$

Meanwhile, there is a sequence $\{\eta_n\}$ satisfying $\eta_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} S(\eta_n) = \liminf_{z\to\infty} S(z) := \varrho_2 < \varrho_1 \quad \text{and} \quad S'(\eta_n) = 0. \tag{3.26}$$

By the similar arguments in [18–20], we can obtain that $S(z_n+y) \to \varrho_1$ and $S(\eta_n+y) \to \varrho_2$ as $n \to \infty$ for any $y \in [-r, r]$, where r is the radius of supp$K$. Note that

$$\lim_{n\to\infty} \int_{\eta_n}^{z_n} \frac{\beta S(z)I(z-c\tau)}{S(z)+I(z-c\tau)+R(z)}dz = 0. \tag{3.27}$$

Integrating (1.2) from η_n to z_n and using (3.25)–(3.27) and (K), we derive

$$\begin{aligned}0 &< c(\varrho_1-\varrho_2) \\ &= c\lim_{n\to\infty}[S(z_n)-S(\eta_n)] \\ &= d_1\lim_{n\to\infty}\int_{\eta_n}^{z_n}\int_{-\infty}^{\infty} K_{\lambda_1}(y)[S(z-y)-S(z)]dydz - \lim_{n\to\infty}\int_{\eta_n}^{z_n}\frac{\beta S(z)I(z-c\tau)}{S(z)+I(z-c\tau)+R(z)}dz \\ &= d_1\lim_{n\to\infty}\int_{\eta_n}^{z_n}\int_{-\infty}^{\infty} K_{\lambda_1}(y)[S(z-y)-S(z)]dydz \\ &= -d_1\lim_{n\to\infty}\int_{\eta_n}^{z_n}\int_{-\infty}^{\infty} yK_{\lambda_1}(y)\int_0^1 S'(z-\theta y)d\theta dydz \\ &= d_1\lim_{n\to\infty}\int_{-\infty}^{\infty} yK_{\lambda_1}(y)\int_0^1 [S(\eta_n-\theta y)-S(z_n-\theta y)]d\theta dy \\ &= 0,\end{aligned}$$

which yields a contradiction. Thus we obtain $\limsup_{z\to\infty} S(z) = \liminf_{z\to\infty} S(z)$, which implies the limit S_∞ exists. We next show that $S_\infty < S_1$. Since $S(z) < S_1$, we have $S_\infty \le S_1$. Suppose that $S_\infty = S_1$, then we get $S_{-\infty} = S_\infty$. Integrating

(1.2) from over $\mathbb{R}$ and utilizing Fubini's theorem give

$$\begin{aligned} 0 &= c(S_\infty - S_{-\infty}) \\ &= d_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\lambda_1}(y)[S(z-y)-S(z)]dydz - \int_{-\infty}^{\infty} \frac{\beta S(z)I(z-c\tau)}{S(z)+I(z-c\tau)+R(z)}dz \\ &= -\int_{-\infty}^{\infty} \frac{\beta S(z)I(z-c\tau)}{S(z)+I(z-c\tau)+R(z)}dz \\ &< 0, \end{aligned} \tag{3.28}$$

a contradiction occurs. Thus we obtain $S_\infty < S_1$.

(iv) One can see from (3.28) that

$$\int_{-\infty}^{\infty} \frac{\beta S(z)I(z-c\tau)}{S(z)+I(z-c\tau)+R(z)}dz = c(S_1 - S_\infty). \tag{3.29}$$

Moreover, it follows from (3.22), (3.24) and Fubini's theorem that

$$(\gamma+\delta)\int_{-\infty}^{\infty} I(z)dz = \int_{-\infty}^{\infty} \frac{\beta S(z)I(z-c\tau)}{S(z)+I(z-c\tau)+R(z)}dz. \tag{3.30}$$

(v) Suppose that $\limsup_{z\to\infty} R(z) > \liminf_{z\to\infty} R(z)$, then we can show the limit R_∞ exists in a similar way as (iii). An integration of (1.4) over $\mathbb{R}$, we have $cR_\infty = \gamma\int_{\mathbb{R}} I(z)dz$, which combined with (3.29) and (3.30) gives that

$$R_\infty = \frac{\gamma(S_1 - S_\infty)}{\gamma+\delta}.$$

The proof of this lemma is completed. □

4. Critical traveling waves

This section is devoted to establishing the existence result for $R_0 > 1$ and $c = c^*$ in Theorem 2.1. For $z \in \mathbb{R}$, we define the following nonnegative continuous functions by

$$S_+^*(z) := S_1, \qquad S_-^*(z) := \begin{cases} S_1(1-\epsilon_4^{-1}e^{\epsilon_4 z}), & z < z_5, \\ 0, & z \ge z_5, \end{cases}$$

$$I_+^*(z) := \begin{cases} -L_3 z e^{\rho^* z}, & z < z_4, \\ I_1, & z \ge z_4, \end{cases} \qquad I_-^*(z) := \begin{cases} [-L_3 z - L_5(-z)^{\frac{1}{2}}]e^{\rho^* z}, & z < z_6, \\ 0, & z \ge z_6, \end{cases}$$

$$R_+^*(z) := L_4 e^{\epsilon_3 z}, \qquad R_-^*(z) := 0,$$

where ρ^* is defined in Lemma 2.1, $I_1 = \frac{(\beta-\gamma-\delta)S_1}{\gamma+\delta}$, $L_3 = e\rho^* I_1$, $z_4 = -\frac{1}{\rho^*}$, $z_5 = \epsilon_4^{-1}\ln\epsilon_4$, $z_6 = -\frac{L_5^2}{L_3^2}$, ϵ_3, ϵ_4, L_4 and L_5 are four positive constants to be determined later.

Lemma 4.1. *For given sufficiently small $\epsilon_3 > 0$, $\epsilon_4 > 0$ and sufficiently large $L_4 > 1$, $L_5 > 1$, the functions $S_\pm^*(z)$, $I_\pm^*(z)$ and $R_\pm^*(z)$ satisfy*

$$d_1\left[\int_{-\infty}^{\infty} K_{\lambda_1}(y)S_+^*(z-y)dy - S_+^*(z)\right] - c^*(S_+^*)'(z) - \frac{\beta S_+^*(z)I_-^*(z-c^*\tau)}{S_+^*(z)+I_-^*(z-c^*\tau)+R_+^*(z)} \le 0, \quad z\in\mathbb{R}, \tag{4.1}$$

$$d_2(I_+^*)''(z) - c^*(I_+^*)'(z) + \frac{\beta S_+^*(z)I_+^*(z-c^*\tau)}{S_+^*(z)+I_+^*(z-c^*\tau)+R_-^*(z)} - (\gamma+\delta)I_+^*(z) \le 0, \quad z \ne z_4, \tag{4.2}$$

$$d_3\left[\int_{-\infty}^{\infty} K_{\lambda_2}(y)R_+^*(z-y)dy - R_+^*(z)\right] - c^*(R_+^*)'(z) + \gamma I_+^*(z) \le 0, \quad z\in\mathbb{R}, \tag{4.3}$$

$$d_1\left[\int_{-\infty}^{\infty} K_{\lambda_1}(y)S_-^*(z-y)dy - S_-^*(z)\right] - c^*(S_-^*)'(z) - \frac{\beta S_-^*(z)I_+^*(z-c^*\tau)}{S_-^*(z)+I_+^*(z-c^*\tau)+R_-^*(z)} \ge 0, \quad z \ne z_5, \tag{4.4}$$

$$d_2(I_-^*)''(z) - c^*(I_-^*)'(z) + \frac{\beta S_-^*(z)I_-^*(z-c^*\tau)}{S_-^*(z)+I_-^*(z-c^*\tau)+R_+^*(z)} - (\gamma+\delta)I_-^*(z) \ge 0, \quad z \ne z_6, \tag{4.5}$$

$$d_3\left[\int_{-\infty}^{\infty} K_{\lambda_2}(y)R_-^*(z-y)dy - R_-^*(z)\right] - c^*(R_-^*)'(z) + \gamma I_-^*(z) \ge 0, \quad z\in\mathbb{R}. \tag{4.6}$$

Proof. Proof of (4.1). Inequality (4.1) holds by the definitions of $S_+^*(z)$, $I_-^*(z)$ and $R_+^*(z)$.

Proof of (4.2). When $z < z_4$, $I_+^*(z) = -L_3 z e^{\rho^* z}$ and $I_+^*(z - c^*\tau) = -L_3(z - c^*\tau)e^{\rho^*(z-c^*\tau)}$. It follows from Lemma 2.1 that

$$\begin{aligned}
&d_2(I_+^*)''(z) - c^*(I_+^*)'(z) + \frac{\beta S_+^*(z) I_+^*(z - c^*\tau)}{S_+^*(z) + I_+^*(z - c^*\tau) + R_-^*(z)} - (\gamma + \delta) I_+^*(z) \\
&\le d_2(I_+^*)''(z) - c^*(I_+^*)'(z) + \beta I_+^*(z - c^*\tau) - (\gamma + \delta) I_+^*(z) \\
&= -L_3 e^{\rho^* z}[\Psi(\rho^*, c^*) z + \Psi_\rho(\rho^*, c^*)] \\
&= 0.
\end{aligned}$$

When $z > z_4$, $I_+^*(z - c^*\tau) \le I_+^*(z) = I_1 = \frac{(\beta-\gamma-\delta)S_1}{\gamma+\delta}$, $S_+^*(z) = S_1$ and $R_-^*(z) = 0$. Then we have

$$d_2(I_+^*)''(z) - c^*(I_+^*)'(z) + \frac{\beta S_+^*(z) I_+^*(z - c^*\tau)}{S_+^*(z) + I_+^*(z - c^*\tau) + R_-^*(z)} - (\gamma + \delta) I_+^*(z) \le \frac{\beta S_1 I_1}{S_1 + I_1} - (\gamma + \delta) I_1 = 0.$$

Proof of (4.3). When $z < z_4$, $R_+^*(z) = L_4 e^{\epsilon_3 z}$ and $I_+^*(z) = -L_3 z e^{\rho^* z}$. Then we deduce from Lemma 2.2 that

$$\begin{aligned}
&d_3\left[\int_{-\infty}^{\infty} K_{\lambda_2}(y) R_+^*(z-y)dy - R_+^*(z)\right] - c^*(R_+^*)'(z) + \gamma I_+^*(z) \\
&= d_3 \int_{-\infty}^{\infty} \frac{1}{\lambda_2} K\left(\frac{y}{\lambda_2}\right) L_4 e^{\epsilon_3(z-y)} dy - d_3 L_4 e^{\epsilon_3 z} - c^* L_4 \epsilon_3 e^{\epsilon_3 z} - \gamma L_3 z e^{\rho^* z} \\
&= L_4 e^{\epsilon_3 z}\left[d_3 \int_{-\infty}^{\infty} K(y) e^{-\epsilon_3 \lambda_2 y} dy - d_3 - c^*\epsilon_3 - \frac{\gamma L_3}{L_4} z e^{(\rho^* - \epsilon_3)z}\right] \\
&= L_4 e^{\epsilon_3 z}\left[-\Phi(\epsilon_3, c^*) - \frac{\gamma L_3}{L_4} z e^{(\rho^* - \epsilon_3)z}\right] \\
&\le 0 \quad \text{for } z < z_4,
\end{aligned}$$

which is true for $\epsilon_3 \in (0, \min\{\tilde{\rho}, \rho^*\})$ and large enough $L_4 > 1$. When $z \ge z_4$, $R_+^* = L_4 e^{\epsilon_3 z}$ and $I_+^*(z) = I_1$. Then we derive that

$$\begin{aligned}
&d_3\left[\int_{-\infty}^{\infty} K_{\lambda_2}(y) R_+^*(z-y)dy - R_+^*(z)\right] - c^*(R_+^*)'(z) + \gamma I_+^*(z) \\
&= L_4 e^{\epsilon_3 z}\left[d_3 \int_{-\infty}^{\infty} K(y) e^{-\epsilon_3 \lambda_2 y} dy - d_3 - c^*\epsilon_3 + \frac{\gamma}{L_4} I_1 e^{-\epsilon_3 z}\right] \\
&\le L_4 e^{\epsilon_3 z}\left[-\Phi(\epsilon_3, c^*) + \frac{\gamma}{L_4} I_1 e^{-\epsilon_3 z_4}\right] \\
&\le 0 \quad \text{for } z \ge z_4,
\end{aligned}$$

which is valid for $\epsilon_3 \in (0, \tilde{\rho})$ and large enough $L_4 > 1$.

Proof of (4.4). When $z > z_5$, $S_-^*(z) = 0$. Then inequality (4.4) holds trivially. Choose $\epsilon_4 \in (0, \rho^*)$ to be small enough such that $z_5 < z_4$ and

$$d_1 S_1 \epsilon_4^{-1} \int_{-\infty}^{\infty} K(y)(1 - e^{-\epsilon_4 \lambda_1 y}) dy + c^* S_1 + \beta L_3(z - c^*\tau) e^{(\rho^* - \epsilon_4)z} \ge 0 \quad \text{for } z < z_5. \tag{4.7}$$

When $z < z_5$, $S_-^*(z) = S_1(1 - \epsilon_4^{-1} e^{\epsilon_4 z})$ and $I_+^*(z - c^*\tau) = -L_3(z - c^*\tau) e^{\rho^*(z-c^*\tau)}$. It follows from (4.7) that

$$\begin{aligned}
&d_1\left[\int_{-\infty}^{\infty} K_{\lambda_1}(y) S_-^*(z-y)dy - S_-^*(z)\right] - c^*(S_-^*)'(z) - \frac{\beta S_-^*(z) I_+^*(z - c\tau)}{S_-^*(z) + I_+^*(z - c\tau) + R_-^*(z)} \\
&\ge d_1\left[\int_{-\infty}^{\infty} K_{\lambda_1}(y) S_-^*(z-y)dy - S_-^*(z)\right] - c^*(S_-^*)'(z) - \beta I_+^*(z - c\tau) \\
&= d_1 S_1\left[\epsilon_4^{-1} e^{\epsilon_4 z} - \epsilon_4^{-1} e^{\epsilon_4 z} \int_{-\infty}^{\infty} K(y) e^{-\epsilon_4 \lambda_1 y} dy\right] + c^* S_1 e^{\epsilon_4 z} + \beta L_3 (z - c^*\tau) e^{\rho^*(z-c^*\tau)} \\
&\ge e^{\epsilon_4 z}\left[d_1 S_1 \epsilon_4^{-1} \int_{-\infty}^{\infty} K(y)(1 - e^{-\epsilon_4 \lambda_1 y}) dy + c^* S_1 + \beta L_3(z - c^*\tau) e^{(\rho^* - \epsilon_4)z}\right] \\
&\ge 0 \quad \text{for } z < z_5.
\end{aligned}$$

Proof of (4.5). When $z > z_6$, $I_-^*(z) = 0$. Obviously, inequality (4.5) holds. Let $\epsilon_4 \in (0, \epsilon_3)$ be small enough and $L_5 > 1$ be large enough such that $z_6 < z_5$, $1 - \epsilon_4^{-1} e^{\epsilon_4 z_6} \ge \frac{1}{2}$ and

$$2S_1^{-1} L_3^2 (-z)^{\frac{3}{2}} (-z + c^*\tau)^2 e^{\rho^*(z-c^*\tau)} + L_3 L_4 (-z)^{\frac{3}{2}} (-z + c^*\tau) e^{\epsilon_3 z} < \frac{1}{16} L_5 (c^*)^2 \tau^2 \quad \text{for } z < z_6. \tag{4.8}$$

Recall that $z_5 < z_4$ and we have for $z < z_6$ that

$$I_-^*(z) = I_+^*(z) - L_5(-z)^{\frac{1}{2}} e^{\rho^* z}, \quad S_-^*(z) = S_1(1 - \epsilon_4^{-1} e^{\epsilon_4 z}) \geq \frac{1}{2} S_1 \quad \text{and} \quad R_+^*(z) = L_4 e^{\epsilon_3 z}. \tag{4.9}$$

A direct computation yields

$$c^*(I_-^*)'(z) = c^*(I_+^*)'(z) + c^* L_5 e^{\rho^* z} \left[\frac{1}{2}(-z)^{-\frac{1}{2}} - \rho^*(-z)^{\frac{1}{2}} \right] \tag{4.10}$$

and

$$\begin{aligned} d_2(I_-^*)''(z) &= d_2(I_+^*)''(z) + d_2 L_5 e^{\rho^* z} \left[\frac{1}{4}(-z)^{-\frac{3}{2}} + \rho^*(-z)^{-\frac{1}{2}} - (\rho^*)^2(-z)^{\frac{1}{2}} \right] \\ &\geq d_2(I_+^*)''(z) + d_2 L_5 e^{\rho^* z} \left[\rho^*(-z)^{-\frac{1}{2}} - (\rho^*)^2(-z)^{\frac{1}{2}} \right]. \end{aligned} \tag{4.11}$$

Utilizing Taylor's theorem, we get for $z < z_6$ that

$$(-z + c^*\tau)^{\frac{1}{2}} \leq (-z)^{\frac{1}{2}} + \frac{1}{2} c^*\tau(-z)^{-\frac{1}{2}} - \frac{1}{8}(c^*)^2\tau^2(-z)^{-\frac{3}{2}} + \frac{1}{16}(c^*)^3\tau^3(-z)^{-\frac{5}{2}}. \tag{4.12}$$

We have from (4.9) that

$$\begin{aligned} &- \beta I_-^*(z - c^*\tau) + \frac{\beta S_-^*(z) I_-^*(z - c^*\tau)}{S_-^*(z) + I_-^*(z - c^*\tau) + R_+^*(z)} \\ &= - \frac{\beta[(I_-^*)^2(z - c^*\tau) + I_-^*(z - c^*\tau) R_+^*(z)]}{S_-^*(z) + I_-^*(z - c^*\tau) + R_+^*(z)} \\ &\geq - \frac{\beta[(I_+^*)^2(z - c^*\tau) + I_+^*(z - c^*\tau) R_+^*(z)]}{S_-^*(z)} \\ &= - \frac{\beta[L_3^2(-z + c^*\tau)^2 e^{2\rho^*(z - c^*\tau)} + L_3 L_4(-z + c^*\tau) e^{\rho^*(z - c^*\tau) + \epsilon_3 z}]}{S_1(1 - \epsilon_4^{-1} e^{\epsilon_4 z})} \\ &\geq -2S_1^{-1}\beta[L_3^2(-z + c^*\tau)^2 e^{2\rho^*(z - c^*\tau)} + L_3 L_4(-z + c^*\tau) e^{\rho^*(z - c^*\tau) + \epsilon_3 z}] \quad \text{for } z < z_6. \end{aligned} \tag{4.13}$$

By $\Psi(\rho^*, c^*) = \Psi_\rho(\rho^*, c^*) = 0$ and (4.8)–(4.13), we derive for $z < z_6$ that

$$\begin{aligned} &d_2(I_-^*)''(z) - c^*(I_-^*)'(z) + \frac{\beta S_-^*(z) I_-^*(z - c^*\tau)}{S_-^*(z) + I_-^*(z - c^*\tau) + R_+^*(z)} - (\gamma + \delta) I_-^*(z) \\ &= d_2(I_-^*)''(z) - c^*(I_-^*)'(z) + \beta I_-^*(z - c^*\tau) - (\gamma + \delta) I_-^*(z) - \beta I_-^*(z - c^*\tau) + \frac{\beta S_-^*(z) I_-^*(z - c^*\tau)}{S_-^*(z) + I_-^*(z - c^*\tau) + R_+^*(z)} \\ &\geq d_2(I_+^*)''(z) - c^*(I_+^*)'(z) + \beta I_+^*(z - c^*\tau) - (\gamma + \delta) I_+^*(z) \\ &\quad + d_2 L_5 e^{\rho^* z} \left[\rho^*(-z)^{-\frac{1}{2}} - (\rho^*)^2(-z)^{\frac{1}{2}} \right] - c^* L_5 e^{\rho^* z} \left[\frac{1}{2}(-z)^{-\frac{1}{2}} - \rho^*(-z)^{\frac{1}{2}} \right] - \beta L_5(-z + c^*\tau)^{\frac{1}{2}} e^{\rho^*(z - c^*\tau)} \\ &\quad + (\gamma + \delta) L_5(-z)^{\frac{1}{2}} e^{\rho^* z} - 2S_1^{-1}\beta[L_3^2(-z + c^*\tau)^2 e^{2\rho^*(z - c^*\tau)} + L_3 L_4(-z + c^*\tau) e^{\rho^*(z - c^*\tau) + \epsilon_3 z}] \\ &\geq -L_3 e^{\rho^* z}[\Psi(\rho^*, c^*) z + \Psi_\rho(\rho^*, c^*)] + d_2 L_5 e^{\rho^* z} \left[\rho^*(-z)^{-\frac{1}{2}} - (\rho^*)^2(-z)^{\frac{1}{2}} \right] - c^* L_5 e^{\rho^* z} \left[\frac{1}{2}(-z)^{-\frac{1}{2}} - \rho^*(-z)^{\frac{1}{2}} \right] \\ &\quad - \beta L_5 \left[(-z)^{\frac{1}{2}} + \frac{1}{2} c^*\tau(-z)^{-\frac{1}{2}} - \frac{1}{8}(c^*)^2\tau^2(-z)^{-\frac{3}{2}} + \frac{1}{16}(c^*)^3\tau^3(-z)^{-\frac{5}{2}} \right] e^{\rho^*(z - c^*\tau)} \\ &\quad + (\gamma + \delta) L_5(-z)^{\frac{1}{2}} e^{\rho^* z} - 2S_1^{-1}\beta[L_3^2(-z + c^*\tau)^2 e^{2\rho^*(z - c^*\tau)} + L_3 L_4(-z + c^*\tau) e^{\rho^*(z - c^*\tau) + \epsilon_3 z}] \\ &= L_5 e^{\rho^* z}(-z)^{-\frac{1}{2}} \left[\frac{1}{2}\Psi_\rho(\rho^*, c^*) + \Psi(\rho^*, c^*) z \right] + \beta L_5 e^{\rho^*(z - c^*\tau)} \left[\frac{1}{8}(c^*)^2\tau^2(-z)^{-\frac{3}{2}} - \frac{1}{16}(c^*)^3\tau^3(-z)^{-\frac{5}{2}} \right] \\ &\quad - 2S_1^{-1}\beta[L_3^2(-z + c^*\tau)^2 e^{2\rho^*(z - c^*\tau)} + L_3 L_4(-z + c^*\tau) e^{\rho^*(z - c^*\tau) + \epsilon_3 z}] \\ &= \beta(-z)^{-\frac{3}{2}} e^{\rho^*(z - c^*\tau)} \left[\frac{1}{16} L_5 (c^*)^2\tau^2 - 2S_1^{-1} L_3^2(-z)^{\frac{3}{2}}(-z + c^*\tau)^2 e^{\rho^*(z - c^*\tau)} - L_3 L_4(-z)^{\frac{3}{2}}(-z + c^*\tau) e^{\epsilon_3 z} \right] \\ &\quad + \frac{1}{16}\beta L_5 (c^*)^2\tau^2(-z)^{-\frac{3}{2}} e^{\rho^*(z - c^*\tau)} \left(1 + \frac{c^*\tau}{z} \right) \\ &\geq 0. \end{aligned}$$

Proof of (4.6). Since $R_-^*(z) = 0$ and $I_-^*(z) \geq 0$ for $z \in \mathbb{R}$, inequality (4.6) holds trivially. □

Using the functions $S^*_\pm(z), I^*_\pm(z), R^*_\pm(z)$ constructed in this section and the similar arguments in Section 3, one can obtain the existence of a nontrivial and positive traveling wave solution of (1.1) with critical velocity. In particular, if $z \to -\infty$, then $I(z) = O(-ze^{\rho^* z})$ for $c = c^*$.

5. Nonexistence of traveling waves

This section is devoted to proving the nonexistence results in Theorem 2.2. We divide the proofs into three cases: (i) $R_0 \le 1$ and $c \in \mathbb{R}$; (ii) $R_0 > 1$ and $c \le 0$; (iii) $R_0 > 1$ and $0 < c < c^*$. By contradiction, we assume that a continuous functional pair $(S(z), I(z), R(z))$ is a positive solution of (1.2)–(1.4) satisfying

$$S_\infty < S_{-\infty} = S_1, \quad I_{\pm\infty} = 0, \quad R_{-\infty} = 0, \quad \sup_{z\in\mathbb{R}} R(z) < \infty.$$

Since $I(z)$ is continuous on $\mathbb{R}$ and $I_{\pm\infty} = 0$, we have from (1.3) that

$$I(z) = \frac{1}{\Lambda}\left[\int_{-\infty}^{z} e^{\sigma_1(z-\eta)} \frac{\beta S(\eta) I(\eta - c\tau)}{S(\eta) + I(\eta - c\tau) + R(\eta)} d\eta + \int_{z}^{\infty} e^{\sigma_2(z-\eta)} \frac{\beta S(\eta) I(\eta - c\tau)}{S(\eta) + I(\eta - c\tau) + R(\eta)} d\eta\right], \tag{5.1}$$

where

$$\sigma_1 = \frac{c - \sqrt{c^2 + 4d_2(\gamma+\delta)}}{2d_2}, \quad \sigma_2 = \frac{c + \sqrt{c^2 + 4d_2(\gamma+\delta)}}{2d_2} \quad \text{and} \quad \Lambda = d_2(\sigma_2 - \sigma_1).$$

By (5.1), the derivative of $I(z)$ is

$$I'(z) = \frac{\sigma_1}{\Lambda}\int_{-\infty}^{z} e^{\sigma_1(z-\eta)} \frac{\beta S(\eta) I(\eta - c\tau)}{S(\eta) + I(\eta - c\tau) + R(\eta)} d\eta + \frac{\sigma_2}{\Lambda}\int_{z}^{\infty} e^{\sigma_2(z-\eta)} \frac{\beta S(\eta) I(\eta - c\tau)}{S(\eta) + I(\eta - c\tau) + R(\eta)} d\eta. \tag{5.2}$$

Applying L'Hôpital's principle in (5.2) gives $\lim_{z\to\pm\infty} I'(z) := I'_{\pm\infty} = 0$, which together with (1.3) leads to that $\lim_{z\to\pm\infty} I''(z) := I''_{\pm\infty} = 0$. By the analogous arguments as the proof of (3.24), we have that $I(z)$ is integrable on $\mathbb{R}$.

Case 1. $R_0 \le 1$ and $c \in \mathbb{R}$. Integrating (1.3) over $\mathbb{R}$ and using $I_{\pm\infty} = I'_{\pm\infty} = 0$, we obtain

$$\begin{aligned}(\gamma+\delta)\int_{-\infty}^{\infty} I(z)dz &= \beta\int_{-\infty}^{\infty} \frac{S(z)I(z-c\tau)}{S(z)+I(z-c\tau)+R(z)}dz \\ &< \beta\int_{-\infty}^{\infty} I(z-c\tau)dz \\ &\le (\gamma+\delta)\int_{-\infty}^{\infty} I(z)dz,\end{aligned}$$

which yields a contradiction.

Case 2. $R_0 > 1$ and $c \le 0$. It follows from the asymptotic boundary of $(S(z), I(z), R(z))$ at minus infinity that

$$\lim_{z\to-\infty} \frac{\beta S(z)}{S(z)+I(z-c\tau)+R(z)} = \beta,$$

which indicates that there exists a number $z^* \ll 0$ such that

$$\frac{\beta S(z)}{S(z)+I(z-c\tau)+R(z)} > \frac{\beta+\gamma+\delta}{2} \quad \text{for } z < z^*. \tag{5.3}$$

For $z \in \mathbb{R}$, we define a function $F(z) := \int_{-\infty}^{z} I(\eta)d\eta$. Clearly, $F(z) > 0$ is strictly increasing with respect to z. By (1.3) and (5.3), we have

$$cI'(z) \ge d_2 I''(z) + \frac{\beta+\gamma+\delta}{2}[I(z-c\tau) - I(z)] + \frac{\beta-\gamma-\delta}{2} I(z) \quad \text{for } z < z^*. \tag{5.4}$$

Integrating (5.4) from $-\infty$ to z ($z < z^*$) twice and using $I_{-\infty} = I'_{-\infty} = 0$, we derive

$$\begin{aligned}0 \ge cF(z) &\ge d_2 I(z) + \frac{\beta+\gamma+\delta}{2}\int_{-\infty}^{z}[F(\eta-c\tau) - F(\eta)]d\eta + \frac{\beta-\gamma-\delta}{2}\int_{-\infty}^{z} F(\eta)d\eta \\ &= d_2 I(z) - c\tau\frac{\beta+\gamma+\delta}{2}\int_{-\infty}^{z}\int_0^1 F'(\eta - c\tau\theta)d\theta d\eta + \frac{\beta-\gamma-\delta}{2}\int_{-\infty}^{z} F(\eta)d\eta \\ &= d_2 I(z) - c\tau\frac{\beta+\gamma+\delta}{2}\int_0^1 F(z-c\tau\theta)d\theta + \frac{\beta-\gamma-\delta}{2}\int_{-\infty}^{z} F(\eta)d\eta \\ &> 0.\end{aligned}$$

A contradiction occurs.

Case 3. $R_0 > 1$ and $0 < c < c^*$. Integrating (5.4) from $-\infty$ to z ($z < z^*$) and utilizing $I_{-\infty} = I'_{-\infty} = 0$, we get

$$\frac{\beta-\gamma-\delta}{2}F(z) \le cI(z) - d_2 I'(z) - \frac{\beta+\gamma+\delta}{2}[F(z-c\tau)-F(z)] \quad \text{for } z < z^*. \tag{5.5}$$

We obtain from (5.5) and the monotonicity of $F(z)$ on $\mathbb{R}$ that

$$\begin{aligned}\frac{\beta-\gamma-\delta}{2}\int_{-\infty}^{z}F(\eta)d\eta + d_2 I(z) &\le cF(z) - \frac{\beta+\gamma+\delta}{2}\int_{-\infty}^{z}[F(\eta-c\tau)-F(\eta)]d\eta \\ &\le cF(z) + \frac{c\tau(\beta+\gamma+\delta)}{2}\int_{-\infty}^{z}\int_0^1 F'(\eta-c\tau\theta)d\theta d\eta \\ &= cF(z) + \frac{c\tau(\beta+\gamma+\delta)}{2}\int_0^1 F(z-c\tau\theta)d\theta \\ &\le \left[c + \frac{c\tau(\beta+\gamma+\delta)}{2}\right]F(z) \quad \text{for } z < z^*.\end{aligned} \tag{5.6}$$

By (5.6) and the monotonicity of $F(z)$ on $\mathbb{R}$, we have that there is a large enough $\eta_0 > 0$ such that

$$\frac{\eta_0(\beta-\gamma-\delta)}{2}F(z-\eta_0) \le \left[c + \frac{c\tau(\beta+\gamma+\delta)}{2}\right]F(z) \quad \text{for } z < z^*.$$

Then there is a constant $\nu_0 \in (0, 1)$ such that

$$F(z-\eta_0) \le \nu_0 F(z) \quad \text{for } z < z^*. \tag{5.7}$$

Denote $\mu_0 := \frac{1}{\eta_0}\ln\frac{1}{\nu_0} > 0$ and set

$$G(z) := F(z)e^{-\mu_0 z}. \tag{5.8}$$

From (5.7) and (5.8), we deduce $G(z-\eta_0) \le G(z)$ for $z < z^*$, which together with $G(z) \ge 0$ gives that $\lim_{z\to-\infty} G(z)$ exists. Hence there is a positive constant G_0 such that

$$F(z) \le G_0 e^{\mu_0 z} \quad \text{for } z < z^*. \tag{5.9}$$

It follows from (5.6) that

$$d_2 I(z) \le \left[c + \frac{c\tau(\beta+\gamma+\delta)}{2}\right]F(z) \quad \text{for } z < z^*, \tag{5.10}$$

which together with (5.9) implies that $I(z)e^{-\mu_0 z}$ is bounded for $z < z^*$. From (5.5) and (5.9), we have

$$d_2 I'(z) \le cI(z) - \frac{\beta+\gamma+\delta}{2}[F(z-c\tau)-F(z)] \le cI(z) + (\beta+\gamma+\delta)G_0 e^{\mu_0 z} \quad \text{for } z < z^*. \tag{5.11}$$

So $|I'(z)|e^{-\mu_0 z}$ is bounded for $z < z^*$. Moreover, by (5.4) we get $|I''(z)|e^{-\mu_0 z}$ is bounded for $z < z^*$. From the boundedness of $I(z)$, $I'(z)$ and $I''(z)$ on the real line, we obtain

$$\sup_{z\in\mathbb{R}}\left\{I(z)e^{-\mu_0 z}\right\} < \infty, \quad \sup_{z\in\mathbb{R}}\left\{|I'(z)|e^{-\mu_0 z}\right\} < \infty, \quad \sup_{z\in\mathbb{R}}\left\{|I''(z)|e^{-\mu_0 z}\right\} < \infty. \tag{5.12}$$

Let $\upsilon(\cdot) \in C^\infty(\mathbb{R}, [0, 1])$, $\upsilon(z)$ be a nondecreasing function satisfying $\upsilon(z) = 0$ for $z \in (-\infty, -2]$ and $\upsilon(z) = 1$ for $z \in [-1, \infty)$. For $N \in \mathbb{N}$, set $\upsilon_N(z) = \upsilon(\frac{z}{N})$. Multiplying (1.4) by $e^{-\nu z}\upsilon_N(z)$ and integrating the resultant equation over $\mathbb{R}$, we have

$$c\int_{-\infty}^{\infty}R'(z)e^{-\nu z}\upsilon_N(z)dz = d_3\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}K_{\lambda_2}(y)R(z-y)dy - R(z)\right]e^{-\nu z}\upsilon_N(z)dz + \gamma\int_{-\infty}^{\infty}I(z)e^{-\nu z}\upsilon_N(z)dz. \tag{5.13}$$

An elementary computation yields

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K_{\lambda_2}(y)R(z-y)dye^{-\nu z}\upsilon_N(z)dz &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{1}{\lambda_2}K\left(\frac{z-y}{\lambda_2}\right)R(y)dy\right]e^{-\nu z}\upsilon_N(z)dz \\ &= \int_{-\infty}^{\infty}R(y)e^{-\nu y}\left[\int_{-\infty}^{\infty}K(z)e^{-\nu\lambda_2 z}\upsilon_N(\lambda_2 z + y)dz\right]dy.\end{aligned} \tag{5.14}$$

Moreover, we get

$$\int_{-\infty}^{\infty}R'(z)e^{-\nu z}\upsilon_N(z)dz = \nu\int_{-\infty}^{\infty}R(z)e^{-\nu z}\upsilon_N(z)dz - \int_{-\infty}^{\infty}R(z)e^{-\nu z}\upsilon'_N(z)dz. \tag{5.15}$$

It follows from (5.13)–(5.15) that

$$\begin{aligned}&(c\nu+d_3)\int_{-\infty}^{\infty}R(z)e^{-\nu z}\upsilon_N(z)dz-d_3\int_{-\infty}^{\infty}K(y)e^{-\nu\lambda_2 y}dy\int_{-\infty}^{\infty}R(z)e^{-\nu z}dz-c\int_{-\infty}^{\infty}R(z)e^{-\nu z}\upsilon_N'(z)dz\\&\quad\le\gamma\int_{-\infty}^{\infty}I(z)e^{-\nu z}\upsilon_N(z)dz.\end{aligned}\tag{5.16}$$

Recalling that $\Phi(\nu,c)=c\nu+d_3-d_3\int_{\mathbb{R}}K(y)e^{-\nu\lambda_2 y}dy>0$ for $\nu\in(0,\tilde{\rho})$ and passing to the limits in (5.16) as $N\to\infty$, we deduce

$$\int_{-\infty}^{\infty}R(z)e^{-\nu z}dz\le\frac{\gamma}{\Phi(\nu,c)}\int_{-\infty}^{\infty}I(z)e^{-\nu z}dz.$$

Thus we derive

$$\int_{-\infty}^{\infty}R(z)e^{-\nu z}dz<\infty\tag{5.17}$$

for any $\nu\in(0,\hat{\mu})$ with $\hat{\mu}:=\min\{\mu_0,\tilde{\rho}\}$. Then we obtain from (5.12) and (5.17) that

$$\int_{-\infty}^{\infty}e^{-\rho z}\frac{\beta I(z-c\tau)[I(z-c\tau)+R(z)]}{S(z)+I(z-c\tau)+R(z)}dz<\infty$$

for any $\rho\in(0,\mu_0+\hat{\mu})$.

For $\rho\in\mathbb{C}$ with $0<\mathrm{Re}\rho<\mu_0$, define the bilateral Laplace transform of $I(z)$ by $\mathcal{L}(\rho):=\int_{\mathbb{R}}I(z)e^{-\rho z}dz$. Eq. (1.3) is equivalent to

$$d_2I''(z)-cI'(z)+\beta I(z-c\tau)-(\gamma+\delta)I(z)=\frac{\beta I(z-c\tau)[I(z-c\tau)+R(z)]}{S(z)+I(z-c\tau)+R(z)}.\tag{5.18}$$

Taking bilateral Laplace transform on (5.18) yields

$$\Psi(\rho,c)\mathcal{L}(\rho)=\int_{-\infty}^{\infty}e^{-\rho z}\frac{\beta I(z-c\tau)[I(z-c\tau)+R(z)]}{S(z)+I(z-c\tau)+R(z)}dz.\tag{5.19}$$

Note that $\mathcal{L}(\rho)$ on the left-hand side of (5.19) is well-defined for any $\rho\in(0,\mu_0)$, while the bilateral Laplace transform on the right-hand side is well-defined for any $\rho\in(0,\mu_0+\hat{\mu})$. Then the property of Laplace transform [38] enables to conclude that the two integrals in (5.19) can be analytical on the whole right half plane; see similar arguments in [2,8,12,20]. Since $\Psi(\rho,c)\to\infty$ as $\rho\to\infty$, a contradiction appears in (5.19).

6. Numerical simulations

In this section, we implement some numerical simulations to show the existence and nonexistence of traveling wave solutions in (1.1). Here we apply the finite difference method developed in [39]. Now set

$$d_1=10,\ d_2=5,\ d_3=20,\ \lambda_1=\lambda_2=1,\ \beta=2,\ \gamma=\delta=1/2,\ \tau=1,\ K(y)=e^{-\pi y^2},\ S(-\infty)=1,\ S(\infty)=1/5.$$

In this case, $R_0=2>1$. System (1.2)–(1.4) is reduced to

$$10\left[\int_{-\infty}^{\infty}e^{-\pi y^2}S(z-y)dy-S(z)\right]-cS'(z)-\frac{2S(z)I(z-c)}{S(z)+I(z-c)+R(z)}=0,\tag{6.1}$$

$$5I''(z)-cI'(z)+\frac{2S(z)I(z-c)}{S(z)+I(z-c)+R(z)}-I(z)=0,\tag{6.2}$$

$$20\left[\int_{-\infty}^{\infty}e^{-\pi y^2}R(z-y)dy-R(z)\right]-cR'(z)+\frac{1}{2}I(z)=0.\tag{6.3}$$

Then the solution $(S(z),I(z),R(z))$ of (6.1)–(6.3) satisfies

$$\lim_{z\to-\infty}(S(z),I(z),R(z))=(1,0,0)\quad\text{and}\quad\lim_{z\to\infty}(S(z),I(z),R(z))=(1/5,0,2/5).\tag{6.4}$$

Truncate $\mathbb{R}=(-\infty,\infty)$ by $[-B,B]$ for some large B and adopt the uniform partition $[-B,B]$ as

$$-B=z_1<z_2<\cdots<z_{2n-1}<z_{2n}<z_{2n+1}=B,$$

where $z_i=z_1+(i-1)h$, $h=B/n$, $i=1,2,\ldots,2n+1$. Corresponding the truncation, we have that the asymptotic boundary condition in (6.4) becomes

$$(S(z_1),I(z_1),R(z_1))=(1,0,0)\quad\text{and}\quad(S(z_{2n+1}),I(z_{2n+1}),R(z_{2n+1}))=(1/5,0,2/5).$$

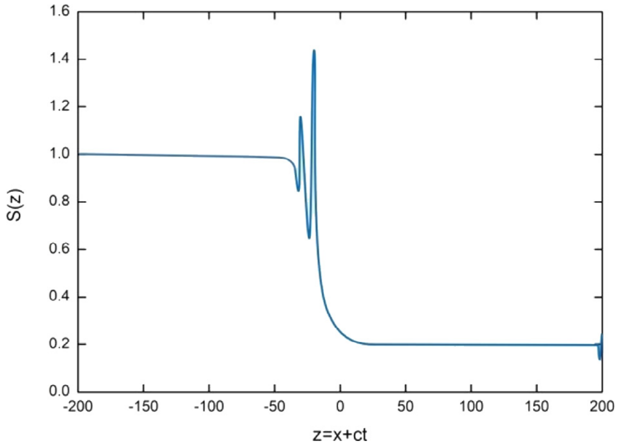

Fig. 1. S oscillates frequently for $c = 1$ in system (6.1)–(6.3).

Utilizing the conventional numeric differentiation and analogous techniques [39] to deal with the nonlocal dispersal terms $\int_{\mathbb{R}} e^{-\pi y^2} S(z-y)dy$ and $\int_{\mathbb{R}} e^{-\pi y^2} R(z-y)dy$, we obtain the following discrete system

$$\begin{aligned}
\frac{c}{2h}S(z_{i+1}) =& \frac{c}{2h}S(z_{i-1}) + 6 - 5\int_{-z_i-B}^{z_i+B} e^{-\pi y^2}dy - \int_{z_i-B}^{B-z_i} e^{-\pi y^2}dy \\
&+ \frac{40h}{3}\sum_{j=1}^{n} S(z_{2j})e^{-\pi(z_i-z_{2j})^2} + \frac{20h}{3}\sum_{j=1}^{n} S(z_{2j+1})e^{-\pi(z_i-z_{2j+1})^2} \\
&+ \frac{10h}{3}e^{-\pi(z_i-z_1)^2} + \frac{2h}{3}e^{-\pi(z_i-z_{2n+1})^2} - 10S(z_i) - \frac{2S(z_i)I(z_i-c)}{S(z_i)+I(z_i-c)+R(z_i)}, \\
\frac{5}{h^2}I(z_{i+1}) &+ \left(\frac{c}{h} + \frac{5}{h^2}\right)I(z_{i-1}) - \left(\frac{c}{h} + \frac{10}{h^2} + 1\right)I(z_i) + \frac{2S(z_i)I(z_i-c)}{S(z_i)+I(z_i-c)+R(z_i)} = 0, \\
\frac{c}{2h}R(z_{i+1}) =& \frac{c}{2h}R(z_{i-1}) + 4 - 4\int_{z_i-B}^{B-z_i} e^{-\pi y^2}dy + \frac{80h}{3}\sum_{j=1}^{n} R(z_{2j})e^{-\pi(z_i-z_{2j})^2} \\
&+ \frac{40h}{3}\sum_{j=1}^{n} R(z_{2j+1})e^{-\pi(z_i-z_{2j+1})^2} + \frac{20h}{3}e^{-\pi(z_i-z_1)^2} + \frac{8h}{3}e^{-\pi(z_i-z_{2n+1})^2} - 20R(z_i) - \frac{1}{2}I(z_i).
\end{aligned}$$

Using Matlab, we get from Lemma 2.1 that $c^* = \sqrt{5\ln 2} \approx 1.8616 < 2$. Figs. 1–3 are simulation results for $c = 1 < c^*$, which show that S, I, R are not monotone and I, R may take negative values. Figs. 4–6 are simulation results for $c = 3 > c^*$, which show that S, I, R are not monotone and S, R oscillate very frequently. See Figs. 7 and 8 for the truncation of S and R in Figs. 4 and 6 by restricting $z \in [-100, 0]$, respectively.

7. Conclusions and discussions

In the current paper, we introduce a three-component delayed disease model with mixed diffusion based on several practical situations. This model can help people better understand how the infectious disease spread in space remotely as well as nearby. Combining with the methods for the nonlocal diffusion and reaction–diffusion systems, we obtained the overall information on threshold-type propagation dynamics. Our main results show that the basic reproduction number R_0 and critical velocity c^* completely determine the transmission dynamics of infectious disease. Specifically, when $R_0 > 1$ and $c \geq c^*$, this model (1.1) has a nontrivial and positive traveling wave solution; when $R_0 > 1$ and $c < c^*$ or $R_0 \leq 1$ and $c \in \mathbb{R}$, this model has no nontrivial and positive traveling wave solutions in system (1.1). Our theoretical analysis may assist people to make strategies on disease prevention and control.

In Theorem 2.1, we need an additional condition (i.e., $R(z)$ is bounded on $\mathbb{R}$) to obtain the existence of R_∞. In the view of mathematical biology, this condition fits reality. However, we do not derive it by rigorous analysis. In fact, when the wave speed is large enough, one can prove the boundedness of $R(z)$ in $\mathbb{R}$, see the similar arguments in [25, Corollary 2.10]. New methods have to be developed in the future to solve this problem.

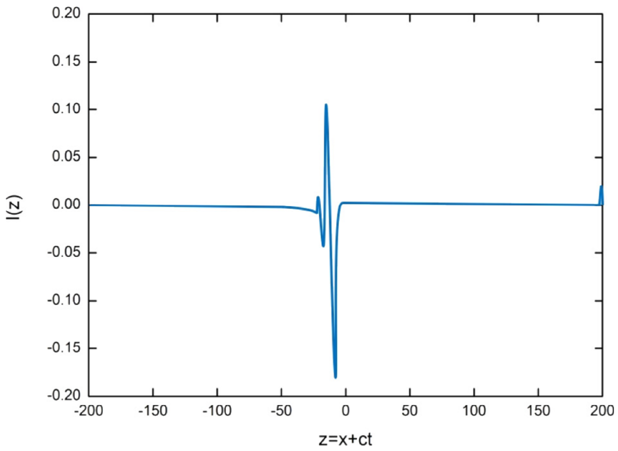

Fig. 2. I oscillates frequently and reaches some negative values for $c = 1$ in system (6.1)–(6.3).

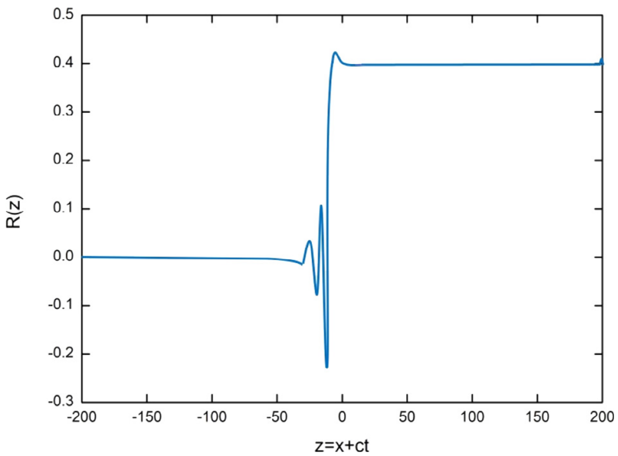

Fig. 3. R oscillates frequently and reaches some negative values for $c = 1$ in system (6.1)–(6.3).

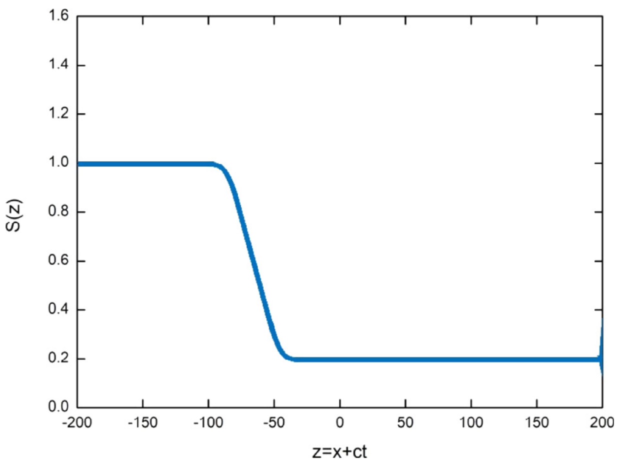

Fig. 4. S oscillates frequently and has a dip for $c = 3$ in system (6.1)–(6.3).

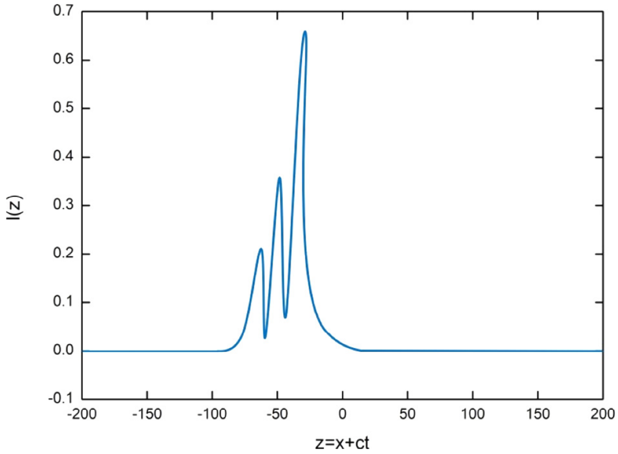

Fig. 5. I is not monotone for $c = 3$ in system (6.1)–(6.3).

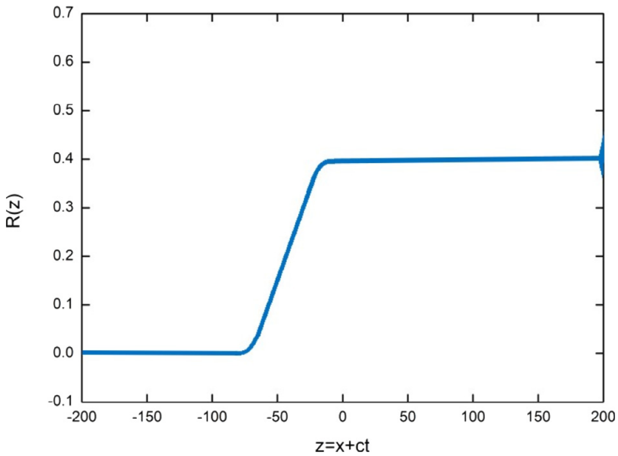

Fig. 6. R oscillates frequently for $c = 3$ in system (6.1)–(6.3).

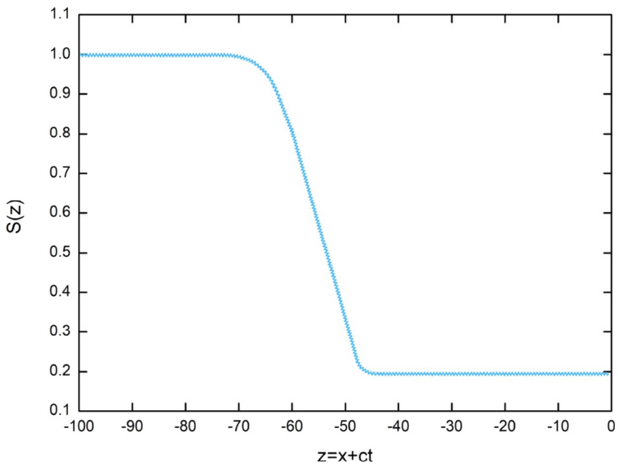

Fig. 7. The truncation of Fig. 4 by restricting $z \in [-100, 0]$.

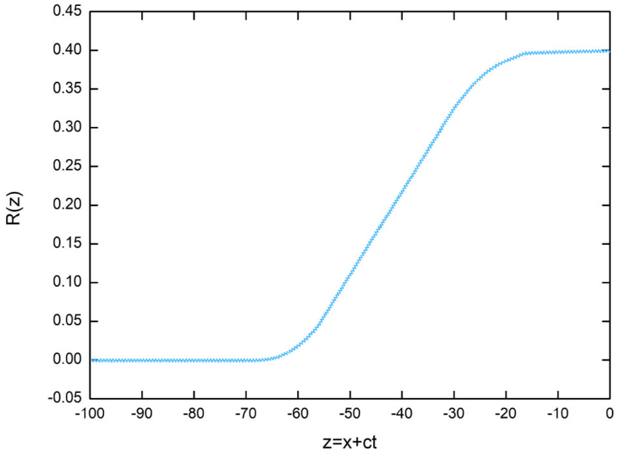

Fig. 8. The truncation of Fig. 6 by restricting $z \in [-100, 0]$.

Acknowledgments

The authors are deeply grateful to the anonymous referees for the valuable suggestions and comments which have greatly improved the presentation of our paper. This work was supported by the China Postdoctoral Science Foundation (grant number 2018M642174) and the National Natural Science Foundation of China (grant number 11731014).

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