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Logistic and semi-logistic processes *

Barry C. Arnold
Department of Statistics, University of California, Riverside, CA 92521, United States

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Abstract

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Random geometric minima of logistic random variables are again logistic. Certain multivariate logistic distributions share this property. This phenomenon is exploited to develop and study a variety of stationary k-dimensional processes with logistic marginals and semi-logistic marginals.

Keywords: min-stable distribution, min-geometric stable distribution, autoregressive processes, multivariate logistic distribution.

1. Introduction

A random variable X is said to have a *logistic distribution* if its survival function is of the form

$$\bar{F}_X(x) = P(X > x) = \left[1 + \exp\left(\frac{x-\mu}{\sigma}\right)\right]^{-1}, \quad x \in \mathbb{R}, \tag{1.1}$$

in which $\mu \in \mathbb{R}$ is a location parameter and $\sigma > 0$ is a scale parameter. In such a case, we write $X \sim \mathscr{L}(\mu, \sigma)$. The standard logistic distribution corresponds to the choice $\mu = 0$, $\sigma = 1$. The logistic density is quite similar to the normal density and logistic models are viable competitors to normal models in a variety of settings. Variables which are frequently well described by logistic models include a variety of biological measurements and the logarithms of many economic variables (such as exchange rates, income and wealth, etc.). A broad spectrum of multivariate logistic distributions have been studied (see [5]). The focus of this paper is the study of stationary stochastic processes with univariate or multivariate logistic marginal distributions. The goal is to provide logistic alternatives to the perhaps overused normal processes,

Correspondence to: Prof. B.C. Arnold, Department of Statistics, University of California, Riverside, CA 92521, United States.
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described, for example, in [10]. The models introduced are, to some extent, parallel to the exponential processes developed in [11] and elsewhere and the analogous Linnik processes introduced in [1]. The exponential and Linnik models have a structure determined by the geometric compounding closure property of exponential and Linnik variables. The parallel geometric minimization closure property is exploited to develop logistic models. These concepts will be introduced in Section 2. The following sections discuss a variety of related logistic models. The emphasis is on the multivariate case. Appropriate references will be given to earlier papers in which univariate models have been described.

2. Geometric minimization

Suppose that $X_1, X_2, \ldots$ are independent identically distributed (i.i.d.) random variables each having a logistic(μ, σ) distribution. Suppose also that N, independent of the X_i's, has a geometric(p) distribution (i.e., $P(N=n)=pq^{n-1}$, $n=1,2,\ldots$). If one defines

$$Y = \min_{1 \leqslant i \leqslant N} X_i, \tag{2.1}$$

then by a simple conditioning argument it can be verified that

$$Y \sim \mathscr{L}(\mu + \sigma \log p, \sigma). \tag{2.2}$$

Consequently we have

$$Y - \sigma \log p \overset{\mathrm{d}}{=} X_1. \tag{2.3}$$

In fact, if (2.3) holds for every $p \in (0,1)$, then the common distribution of the X_i's must be logistic. Technically it is enough for (2.3) to hold for two distinct values of p, say p_1 and p_2 provided that $\{p_1^j/p_2^k\colon j=0,1,2,\ldots,k=0,1,2,\ldots\}$ is dense in $\mathbb{R}^+$. Alternatively, having (2.3) hold for one value of p and invoking a regularity condition on the rate at which $F_X(x)$ decreases to 0 as x decreases to $-\infty$ is enough to guarantee that the X_i's and consequently Y must be logistic variates (a convenient early reference for these observations is [7]).

There are multivariate analogs to this. Suppose now that $\underline{X}_1, \underline{X}_2, \ldots$ are independent identically distributed k-dimensional random vectors and that N, independent of the $\underline{X}_i$'s, is a geometric(p) random variable. Using a coordinatewise definition of the minimum of random vectors, we may define

$$\underline{Y} = \min_{1 \leqslant i \leqslant N} \underline{X}_i\,. \tag{2.4}$$

Suppose that for any p there exists a vector $\underline{c}(p) > 0$ such that

$$\underline{Y} + \underline{c}(p) \overset{\mathrm{d}}{=} \underline{X}_1. \tag{2.5}$$

In such a case the common distribution of the $\underline{X}_i$'s is said to be *min-geometric stable* (paralleling the definition of max-geometric stability provided in [13]).

In one dimension we have seen that only logistic distributions will satisfy (2.5). In fact, using [13], we can also characterize the class of k-dimensional distributions for $\underline{X}$ which satisfy (2.5). They turn out to have logistic marginals and provide a potentially rich collection of k-variate logistic distributions, many of which are as yet unexplored. Before identifying the class of solutions to (2.5) we will make two comments. First, a remark on the interchangeability of maximum and minimum in many logistic models, and second, a remark on semi-logistic distributions, the distributions encountered when (2.3) (or (2.5)) holds for only one value of p.

The logistic density is symmetric (about μ). Consequently a geometric minimum of logistic(μ, σ) random variables has the same distribution as a geometric maximum of logistic$(-\mu, \sigma)$ random variables. Consequently, if the X_i's are independent identically distributed logistic random variables and N is geometric(p) (independent of the X_i's), then defining

$$Y = \min_{1 \leq i \leq N} X_i$$

and

$$Z = \max_{1 \leq i \leq N} X_i,$$

there exists $c(p) > 0$ such that

$$Y + c(p) \stackrel{d}{=} X_1, \tag{2.6}$$

$$Z - c(p) \stackrel{d}{=} X_1 \tag{2.7}$$

and

$$Y \stackrel{d}{=} Z - 2c(p). \tag{2.8}$$

Any one of three properties can be used to characterize the logistic distribution (see [9]). Naturally, k-dimensional analogs could be readily stated. Important for our purposes is the ability to easily translate max-geometric stable and max-stable results to corresponding min-geometric stable and min-stable statements.

Now we turn to semi-logistic distributions. Suppose that $X_1, X_2, \ldots$ are independent identically distributed random variables such that (2.3) holds for one particular value of p. It is not difficult to verify that $\bar{F}$, the common survival function of the X_i's, must satisfy

$$\bar{F}(x) = [1 + \psi(x)]^{-1}, \tag{2.9}$$

where the function ψ is nondecreasing, right-continuous and satisfies

$$\psi(x) = \frac{1}{p}\psi(x + \sigma \log p), \tag{2.10}$$

for some $p \in (0, 1)$ and some $\sigma > 0$. Distributions of this type are called *semi-logistic distributions* (Pillai [12] introduced closely related semi-Pareto distributions). The function ψ which appears in (2.9) can be quite arbitrary. To construct such a function, first define ψ^* to be a completely arbitrary nondecreasing right-continuous function on the interval $[0, -\sigma \log p)$ subject only to $\psi^*(-\sigma \log p) \leq \psi^*(0)/p$. Then use (2.10) to extend the definition of ψ^* over the entire real line. The simplest and best behaved solution is

$$\psi(x) = ab^x. \tag{2.11}$$

This of course brings us back to the logistic distribution.

Now we turn to identifying the class of distributions satisfying (2.5). Let $\bar{F}$ denote the common survival function of the X_i's (i.e., $\bar{F}(x) = P(X_i > x)$). From [13, Proposition (3.3)] we conclude that $\bar{F}$ is a min-geometric stable survival function if and only if $\exp(1 - 1/\bar{F})$ is the survival function of a min-stable distribution. Then we refer to [14, Chapter 5] for the observation that such min-stable distributions must be multivariate extreme value distributions. Next, our earlier one-dimensional results assure us that the marginal distributions of $\bar{F}$ must be logistic. The accompanying min-stable survival function $\exp(1 - 1/F)$ must have double exponential marginal survival functions (i.e., its marginal survival function must be of the form $\exp(-e^x)$). But a convenient characterization of such survival functions is available [14, Proposition (5.11)]. Eventually we obtain the following characterization of multivariate-min-geometric stable survival functions, i.e., survival functions $\bar{F}_{\underline{X}}$ such that (2.5) holds for every $p \in (0, 1)$. There must exist nonnegative integrable functions $f_i(s)$, $i = 1, 2, \ldots, k$, on [0, 1] satisfying

$$\int_0^1 f_i(s)\,\mathrm{d}s = 1, \quad i = 1, 2, \ldots, k,$$

and

$$\bar{F}_{\underline{X}}(\underline{x}) = \left[1 + \int_0^1 \max_{1 \leqslant i \leqslant k} \left[f_i(s)\,\mathrm{e}^{(x_i - \mu_i)/\sigma_i}\right] \mathrm{d}s\right]^{-1}. \tag{2.12}$$

The logistic character of the marginals is readily apparent in (2.12). Setting $\underline{\mu} = \underline{0}$ and $\underline{\sigma} = \underline{1}$ in (2.12) yields a standardized version of the distribution. The corresponding value of $\underline{c}(p)$ in (2.5) is then $(-\log p)\underline{1}$. Not every k-dimensional logistic distribution can be written in the form (2.12) (for example, many of those described in [5] do not have this character). Two examples which do satisfy (2.12) are

$$\bar{F}_{\underline{X}}(\underline{x}) = \left[1 + \left(\sum_{i=1}^k \mathrm{e}^{\alpha x_i}\right)^{1/\alpha}\right]^{-1} \tag{2.13}$$

and

$$\bar{F}_{\underline{X}}(\underline{x}) = \left[1 + \sum_{i=1}^k \mathrm{e}^{x_i} - \left[\sum_{i=1}^k \mathrm{e}^{-x_i}\right]^{-1}\right]^{-1}. \tag{2.14}$$

It is easy to directly verify that in both these cases we have $\underline{Y} + (-\log p)\underline{1} \stackrel{\mathrm{d}}{=} \underline{X}$.

Semi-logistic versions of (2.12) are readily described. They are of the form

$$\bar{F}_{\underline{X}}(\underline{x}) = \left[1 + \int_0^1 \max_{0 \leqslant i \leqslant k} [f_i(s) g_i(x_i)]\,\mathrm{d}s\right]^{-1}, \tag{2.15}$$

where the f_i's integrate to 1 as before and the g_i's are nondecreasing, right-continuous functions which satisfy $g_i(x_i) = g_i(x_i + \log p)/p$.

3. Autoregressive logistic processes

We concentrate on standard logistic processes; location and scale parameters $\underline{\mu}$ and $\underline{\sigma}$ can be introduced later in an obvious way. Thus we are interested in stationary stochastic processes with k-dimensional state space whose stationary distribution has logistic marginals. Since our constructions will involve geometric minimization, we additionally will require that the stationary distribution be min-geometric stable, that is to say, its survival function is of the form (2.12) with $\underline{\mu} = \underline{0}$ and $\underline{\sigma} = \underline{1}$. Semi-logistic versions involving (2.15) could of course also be described.

Let $\underline{\epsilon}_0, \underline{\epsilon}_1, \underline{\epsilon}_2, \ldots$ be a sequence of independent identically distributed "innovation" random vectors with common survival function

$$\bar{F}_{\underline{\epsilon}}(\underline{x}) = \left[1 + \int_0^1 \left[\max_{1 \leqslant i \leqslant k} f_i(s)\, \mathrm{e}^{x_i}\right] \mathrm{d}s\right]^{-1}. \tag{3.1}$$

Now define a Markov process $\{\underline{X}_n\}_{n=1}^{\infty}$ as follows:

$$\underline{X}_0 = \underline{\epsilon}_0,$$

and for $n \geqslant 1$,

$$\begin{aligned} \underline{X}_n &= \underline{X}_{n-1} + (-\log p)\underline{1}, && \text{with probability } p, \\ &= \min\{\underline{X}_{n-1} + (-\log p)\underline{1}, \underline{\epsilon}_n\}, && \text{with probability } 1-p, \end{aligned} \tag{3.2}$$

where $p \in [0, 1)$. p is a dependence parameter. A one-dimensional version of this process was introduced in [15]. Yeh-Shu [16] discussed a multivariate Pareto process closely related to a special case of the present process corresponding to the simple survival function

$$\bar{F}_{\underline{\epsilon}}(\underline{x}) = \left(1 + \sum_{i=1}^{k} \mathrm{e}^{x_i}\right)^{-1}. \tag{3.3}$$

It is not difficult to verify that the process (3.2) has (3.1) as its long-run distribution regardless of the initial distribution of $\underline{X}_0$. The choice of $\underline{X}_0 = \underline{\epsilon}_0$ yields the desired stationary process called a *k-dimensional autoregressive logistic process*. The marginal processes $X_n(i)$, $i = 1, 2, \ldots, k$, are of course autoregressive logistic processes of the kind studied in [8]. It is not difficult to verify that for each n and i,

$$P(X_n(i) > X_{n-1}(i)) = \tfrac{1}{2}(1 + p). \tag{3.4}$$

This observation may be used to derive a simple consistent estimate of p based on a realization from the process $\underline{X}_n$, namely

$$\hat{p} = \frac{2}{nk}\left[\sum_{j=1}^{n} \sum_{i=1}^{k} I(X_j(i) > X_{j-1}(i))\right] - 1. \tag{3.5}$$

Location and scale parameters may be consistently estimated using marginal sample moments. The general problem of estimating the structural functions $f_1(s), \ldots, f_k(s)$ appearing in (3.1) would appear to be quite challenging. If these f_i's are assumed to be known up to a few unknown parameters, then it will generally be possible to construct consistent estimates based on mixed sample moments from the observed series. In one dimension, Arnold and Robertson [8] observed that the autocorrelation, provided p is not too small, was approximately given by

$$\rho(X_n, X_{n+k}) \doteq p^{k/2}. \tag{3.6}$$

Simulations lend support to this assertion. If p is close to zero, the process (3.2) behaves like a sequence of i.i.d. k-dimensional logistic variables. Here again, one-dimensional simulations support this claim, using both estimated spectra and estimated bi-spectra. In one dimension, sample paths exhibit stretches of regular increase interspersed by sporadic drops. The character of the sample paths of the k-dimensional processes is a little harder to describe, even though, marginally, the pattern of regular increases with sporadic drops is necessarily present.

A possibly distressing feature associated with the autoregressive logistic process (3.2) is that the joint distribution of $(\underline{X}_n, \underline{X}_{n+1})$ is singular. In practice this means that, given a realization from the process, one could actually determine the value of p exactly, by looking for tied marginal increments. A simple modification which alleviates this problem is to make the value p in (3.2) a random variable. Specifically we postulate that, in addition to the i.i.d. sequence $\{\underline{\epsilon}_n\}_{n=0}^{\infty}$, we have a further i.i.d. sequence $\{B_n\}_{n=1}^{\infty}$ independent of the $\underline{\epsilon}$'s whose common distribution function has support [0, 1). We then define our new k-dimensional logistic process by

$$\underline{X}_0 = \underline{\epsilon}_0,$$

and given $B_n = p$ and $\underline{X}_n = \underline{x}_n$ we define

$$\begin{aligned} \underline{X}_{n+1} &= \underline{x}_n + (-\log p)\underline{1}, && \text{with probability } p, \\ &= \min\{\underline{x}_n + (-\log p)\underline{1}, \underline{\epsilon}_{n+1}\}, && \text{with probability } 1-p. \end{aligned} \tag{3.7}$$

It is not difficult to verify that this yields a stationary k-dimensional logistic process with absolutely continuous joint distributions for $(\underline{X}_n, \underline{X}_{n+1})$. See [8] for more detailed discussion of univariate processes of this type, especially the so-called power-logistic processes which correspond to the choice $F_B(p) = p^{\delta}$ as a common distribution for the $\{B_n\}$ sequence. Included in that paper is a discussion of the potential of such processes for modelling currency exchange rate series.

4. Extremes from autoregressive logistic processes

Let $\underline{X}_1, \underline{X}_2, \ldots$ be a realization from an autoregressive standard logistic process of the form (3.2). Define

$$\underline{T}_n = \min_{1 \leq i \leq n} \underline{X}_i \tag{4.1}$$

and

$$\underline{M}_n = \max_{1 \leq i \leq n} \underline{X}_i \,. \tag{4.2}$$

What can be said about the distribution of these extreme vectors?

First consider the minima vector $\underline{T}_n$. Sample trajectories $\underline{X}_n$ increase except when an innovation (an $\underline{\epsilon}_n$) is observed (with probability $1-p$) and it has at least one small coordinate. The number of innovations observed at times $2, 3, \ldots, n$ has a binomial$(n-1, 1-p)$ distribution. $\underline{T}_n$ will exceed $\underline{t}$ if $\underline{X}_1$ and every one of the random number of observed innovations exceeds $\underline{t}$. Consequently

$$\underline{T}_n \stackrel{\mathrm{d}}{=} \min_{i \leq N} \underline{\epsilon}_i, \tag{4.3}$$

where the $\underline{\epsilon}_i$'s are i.i.d. with common distribution (3.1) and $N-1 \sim$ binomial$(n-1, 1-p)$. If we rewrite (3.1) in the abbreviated form

$$\bar{F}_{\underline{\epsilon}}(\underline{x}) = [1 + g(\underline{x})]^{-1}, \tag{4.4}$$

we may verify that

$$\bar{F}_{\underline{T}_n}(\underline{t}) = [1+g(\underline{t})]^{-1}\left[\frac{1+pg(\underline{t})}{1+g(\underline{t})}\right]^{n-1}. \tag{4.5}$$

From this or from well-known results about minima of random numbers of random vectors, the asymptotic distribution of $\underline{T}_n$ is readily obtained.

Next consider the vector of maxima $\underline{M}_n$. In the one-dimensional case Yeh et al. [15] presented a technique which permits a simple evaluation of the distribution of M_n. They dealt with Pareto processes, but a simple logarithmic transformation changes them to logistic processes.

In one dimension, one defines a class of level-crossing processes as follows. For any $t \in \mathbb{R}$, define a (0, 1) process by

$$W_n(t) = \begin{cases} 0, & \text{if } X_n > t, \\ 1, & \text{if } X_n \leqslant t. \end{cases} \tag{4.6}$$

These level-crossing processes turn out to be Markov chains with transition matrices given by

$$\begin{array}{cc} & \begin{array}{cc} \quad 0 \quad & \quad 1 \quad \end{array} \\ \begin{array}{c} 0 \\ \\ 1 \end{array} & \begin{pmatrix} \dfrac{1+p\mathrm{e}^t}{1+\mathrm{e}^t} & \dfrac{(1-p)\mathrm{e}^t}{1+\mathrm{e}^t} \\ \dfrac{1-p}{1+\mathrm{e}^t} & \dfrac{p+\mathrm{e}^t}{1+\mathrm{e}^t} \end{pmatrix} \end{array}. \tag{4.7}$$

We then argue easily that, for any t,

$$\begin{aligned} P(M_n \leqslant t) &= P(W_1(t)=1,\, W_2(t)=1,\dots,W_n(t)=1) \\ &= \frac{\mathrm{e}^t}{1+\mathrm{e}^t}\left(\frac{p+\mathrm{e}^t}{1+\mathrm{e}^t}\right)^{n-1}. \end{aligned} \tag{4.8}$$

From this the asymptotic distribution of M_n is readily obtained.

In the case of a k-dimensional process we can define level-crossing processes for each $\underline{t} \in \mathbb{R}^k$ by

$$W_n(\underline{t}) = I(\underline{X}_n \leqslant \underline{t}), \tag{4.9}$$

and we can again observe that

$$P(\underline{M}_n \leqslant \underline{t}) = P(W_1(\underline{t})=1,\, W_2(\underline{t})=1,\dots,W_n(\underline{t})=1). \tag{4.10}$$

If the $\{W_n(\underline{t})\}$ processes can be shown to be Markov processes with simple transition matrices analogous to (4.7), then the distribution of $\underline{M}_n$ will be easy to derive. One is tempted to *conjecture* that

$$P(\underline{M}_n \leqslant \underline{t}) = \frac{g(\underline{t})}{1+g(\underline{t})}\left(\frac{p+g(\underline{t})}{1+g(\underline{t})}\right)^{n-1}, \tag{4.11}$$

where $g(\underline{t})$ is as defined in (4.4). At present, the question of the exact distribution of $\underline{M}_n$ remains open.

The Markovian character of the level-crossing processes (in one dimension) was shown in [6] to essentially characterize the autoregressive logistic process among stationary processes of the form $X_n = \min(X_{n-1}, Y_n) + c$. The possibility of a k-dimensional analog of this result is intriguing.

5. Logistic processes involving geometric minimization

In this section we describe a k-dimensional version of the logistic process introduced in [4]. Let $\{U_n\}_{n=0}^{\infty}$ be a sequence of i.i.d. Bernoulli(p) random variables ($P(U_n = 1) = p$), where $p \in (0, 1)$. Also let $\underline{V}_n$ be a sequence of i.i.d. k-dimensional logistic random vectors with common survival function (3.1), i.e.,

$$P(\underline{V}_n \geqslant \underline{v}) = \left[1 + \int_0^1 \left[\max_{1 \leqslant i \leqslant k} f_i(s)\, \mathrm{e}^{v_i}\right] \mathrm{d}s\right]^{-1}. \tag{5.1}$$

Recall that geometric minima of random variables of the form (5.1) have survival functions which differ from (5.1) only by a translation. This is exploited to define our process. A natural sequence of geometric random variables can be associated with the Bernoulli sequence $\{U_n\}$. Define $\{N_n\}_{n=0}^{\infty}$ as follows:

$$\{N_n = 1\} = \{U_n = 1\},$$

and for $i = 2, 3, \ldots,$

$$\{N_n = i\} = \{U_n = 0,\ U_{n+1} = 0, \ldots, U_{n+i-2} = 0,\ U_{n+i-1} = 1\}. \tag{5.2}$$

In words, N_n is the waiting time until "success" among trials $n, n+1, \ldots$. By construction, the $\{N_n\}$'s are dependent geometric random variables. All of them are independent of the $\{\underline{V}_n\}$ sequence (since the U_n's were independent of the $\underline{V}_n$'s). Our stationary standard k-dimensional logistic process may then be defined as follows, for $n = 0, 1, 2, \ldots$:

$$\underline{X}_n = \left(\min_{i \leqslant N_n} \underline{V}_{n+i-1}\right) + (-\log p)\underline{1}. \tag{5.3}$$

Each $\underline{X}_n$ has survival function (5.1). The sample paths of such processes have "flat spots", instances in which $\underline{X}_n = \underline{X}_{n+1} = \cdots = \underline{X}_{n+k}$ (this occurs when $U_n = 0, U_{n+1} = 0, \ldots, U_{n+k} = 0$; so the length of a "flat spot" is geometric). Such phenomena might occur in economic series perhaps, for example, due to market closures. The frequency of ties ($\underline{X}_n = \underline{X}_{n+1}$) in an observed series can be used to generate a simple consistent estimate of p.

Several variations of this scheme show promise of developing flexible families of k-dimensional logistic processes and are currently under investigation.

One possibility involves replacing the i.i.d. Bernoulli sequence $\{U_n\}$ by an i.i.d. multinomial sequence $\{\underline{U}_n\}$ where each $\underline{U}_n$ is a k-dimensional random vector of 0's and 1's such that

$$P(\underline{U}_n = \underline{s}) = p_{\underline{s}}, \quad \underline{s} \in \{0, 1\}^k, \tag{5.4}$$

(where $\sum_s p_s = 1$).

Again take the $\underline{V}_n$'s to be i.i.d. with common survival function (5.1). Now define our stationary logistic process in a manner analogous to that used in (5.3) but this time looking ahead coordinatewise for a U_n with a 1 in the given coordinate and then minimizing the appropriate corresponding coordinates of the $\underline{V}_n$'s. Thus for the lth coordinate $X_n(l)$, we define a one-dimensional geometric random variable

$N_n(l)$ = waiting time in trials $n, n+1, \dots$ until a U_n is observed with a 1 in the lth coordinate.

Then defi… (where $V_n(l)$ is the lth coordinate of $\underline{V}_n$)

$$X_n(l) = \min_{i \leqslant N_n(l)} V_{n+i-1}(l) - \log p_l^*, \tag{5.5}$$

where $p_l^* = \Sigma\{p_{\underline{s}}: s_l = 1, \underline{s} \in \{0, 1\}^k\}$. Given a realization of such a process it is possible to estimate all the $p_{\underline{s}}$'s by observing the frequency of various configurations of coordinatewise ties.

A second possibility involves a relaxation of the assumption that the U_n's are i.i.d. Bernoulli random variables. Instead we might postulate that they represent a realization of a stationary Markov chain with state space $\{0, 1\}$. We can still define a sequence of random variables $\{N_n\}$ as in (5.2). However, they will no longer have a common geometric distribution. If the transition matrix for the Markov chain $\{U_n\}$ is of the form

$$\begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1-p_0 & p_0 \\ p_1 & 1-p_1 \end{pmatrix} \end{array}, \tag{5.6}$$

then we may verify that

$$P(N_n = 1) = \frac{p_0}{p_0 + p_1},$$

and for $k \geqslant 2$,

$$P(N_n = k) = \left(\frac{p_0 p_1}{p_0 + p_1}\right)(1 - p_0)^{k-2}. \tag{5.7}$$

Geometric minima of random vectors with survival function (5.1) are again of the same type. However, in the current setting if we use an analog of (5.3) to define our process

$$\underline{X}_n = \min_{i \leqslant N_n} \underline{V}_{n+i-1}, \tag{5.8}$$

then, since our N_n's are not geometric (having distribution (5.7) instead), we have to be devious in selecting a common (nonlogistic) distribution for the $\underline{V}_n$'s to ensure that the resulting process is indeed a stationary k-dimensional logistic process with common survival function (5.1) for the $\underline{X}_n$'s. In one dimension it is shown in [3] that the appropriate common distribution for the V's may be described as follows. Let $\tilde{U}$ be a uniform(0, 1) random variable; then define

$$\tilde{W} = 1 - \frac{p_0}{p_0 + p_1}\tilde{U} - \frac{p_0 p_1}{p_0 + p_1} \frac{\tilde{U}^2}{[1 - (1 - p_0)\tilde{U}]}, \tag{5.9}$$

and finally

$$V = \log\left(\frac{\tilde{W}}{1 - \tilde{W}}\right). \tag{5.10}$$

In that report an analogous development is provided under the assumption that the U_n sequence is a second-order Markov chain. It is possible to estimate the dependence parameters of the model (the elements of the transition matrix of U_n), by observing the frequencies of ties and increases in the observed series X_n.

In principle we can easily extend these results to higher dimensions. If we use $\{N_n\}$ with distribution (5.7) in our definition of $\underline{X}_n$ using (5.8), then, to obtain a logistic process, the common survival function of the $\underline{V}_n$'s must satisfy

$$\left[1+\int_0^1\left[\max_{1\leqslant i\leqslant k} f_i(s)\,\mathrm{e}^{v_i}\right]\mathrm{d}s\right]^{-1} = \frac{p_0}{p_0+p_1}\bar{F}_{\underline{V}}(\underline{v}) + \frac{p_0 p_1}{p_0+p_1}\frac{\left[\bar{F}_{\underline{V}}(\underline{v})\right]^2}{\left[1-(1-p_0)\bar{F}_{\underline{V}}(\underline{v})\right]}. \tag{5.11}$$

Consequently, $\bar{F}_{\underline{V}}(\underline{v})$ is the solution of a quadratic equation. It would be nice to have a simple simulation scheme, analogous to that defined in the one-dimensional case in (5.8) and (5.9), for this k-dimensional distribution.

6. Related processes

(a) Higher-order autoregressive logistic processes can be developed. See [16] for details in the univariate case. The discussion is in the context of Pareto processes but only a minor modification is needed, see (b).

(b) If $\{\underline{X}_n\}$ is a k-dimensional process and if we define $\{\underline{\tilde{X}}_n\}$ by coordinatewise exponentiation, i.e.,

$$\tilde{X}_n(l) = \mathrm{e}^{X_n(l)},$$

then we have a k-dimensional Pareto process. Several of the concepts in the current paper were first discussed in such a context (usually in one dimension).

(c) Analogous exponential and Linnik processes may be obtained by replacing geometric minimization by geometric summation throughout (see [1]).

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