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Stability and asymptotic behavior of difference equations[1]

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Abstract

The concept of h-stability is studied and compared with the classical stabilities. Basically, the h-stability is applied to obtain a uniform treatment for the concept of stability in difference equations.

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1. Introduction

In [8] the notion of h-stability was introduced in order to obtain results about stability for weakly stable difference systems under some perturbations (at least, for systems with stabilities weaker than those given by exponential stability and uniform Lipschitz stability). In Section 4 of this paper we obtain asymptotic formulae for these systems, which state new results about asymptotic behavior for perturbed systems under general hypotheses. Moreover, the corresponding results for linear difference systems give new insights about discrete Levinson's Theorem (see [2, 3]).

Consider the following systems of difference equations:

$$x(n+1) = A(n)x(n), \tag{1}$$

$$y(n+1) = A(n)y(n) + f(n, y(n)), \tag{2}$$

where $x(x), y(n) \in \mathbb{R}^m$, $f: N_0 \times \mathbb{R}^m \to \mathbb{R}^m$, $N_0 = \{n_0, n_0+1, \ldots, n_0+k, \ldots\}$ (n_0 a nonnegative integer, for $n_0 = 0, N_0 =: N$ the set of positive integers); $f(n, 0) = 0$ for all $n \in N_0$ and $A(n)$ is a discrete $m \times m$ matrix function defined for all $n \in N_0$.

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For $n_0 \in N_0$ and $x_0 \in \mathbb{R}^m$, let $x(n, n_0, x_0) \equiv x(n)$, $n \geqslant n_0$, denote the solution of (1) with $x(n_0, n_0, x_0) = x_0$. Then, $x(n, n_0, x_0) = \Phi(n, n_0)x_0$, where $\Phi(n, n_0)$ is the fundamental matrix of (1) defined as

$$\Phi(n, n_0) = A(n-1)A(n-2)\ldots A(n_0) = \prod_{i=n_0}^{n-1} A(i).$$

In Section 3 we give sufficient conditions in order to ensure the perturbed system (2) inherits the stability from the original system (1). We shall study the perturbed system (2) assuming that the original system (1) is h-stable, and the perturbation $f = f(n, y)$ satisfies

$$|f(n, y)| \leqslant \sum_{i=1}^{p} \lambda_i(n)\omega_i(|y|), \quad p \in N \tag{3}$$

with $\lambda_i : N_0 \to [0, \infty)$ $(1 \leqslant i \leqslant p)$ properly summable functions and $\omega_i : [0, \infty) \to [0, \infty)$ $(1 \leqslant i \leqslant p)$ suitable nondecreasing and positive functions on $(0, \infty)$. We shall prove that under certain conditions, the zero solution of (2) is also h-stable. In particular, if (1) is uniformly Lipschitz stable then (2) is.

An important class of admissible functions ω_i, is any polynomial system:

$$\omega_i(u) = u^{\gamma_i}, \quad \gamma_i \geqslant 1 \ (1 \leqslant i \leqslant p)$$

for which, if $\lambda_i(n)h^{\gamma_i}(n)h^{-1}(n+1) \in \ell_1(N_0)$ then (2) is also h-stable. Thus, the results in this paper extend many of the classical stabilities appeared in the literature (see [1, 4–6, 11]).

2. Preliminaries

Let us consider the difference system

$$x(n+1) = g(n, x(n)) \tag{4}$$

with $g(n, 0) = 0$ for all $n \in N_0$, and $g : N_0 \times \mathbb{R}^m \to \mathbb{R}^m$. Then,

Definition 2.1 (*Medina and Pinto* [8]). System (4) is called an h-system around the null solution, or more briefly an h-system, if there exist a positive function $h : N_0 \to R$ and a constant $c \geqslant 1$ such that

$$|x(n, n_0, x_0)| \leqslant c|x_0|h(n)h^{-1}(n_0), \quad n \geqslant n_0 \tag{5}$$

for $|x_0|$ small enough $(h^{-1}(n) =: 1/h(n))$.

The function h as well as the constant c depend only on g. If h is a bounded function, then an h-system allows the following types of stability:

Definition 2.2 (*Medina and Pinto* [8]). The zero solution of (4), or more briefly (4), is said to be (hS) h-stable if there exits $\delta > 0$ such that (4) is an h-system for $|x_0| \leqslant \delta$ (and h is bounded).

Definition 2.3 (*Medina and Pinto* [8]). The zero solution of (4), or more briefly (4), is said to be (GhS) globally h-stable if (4) is an h-system for every $x_0 \in D$; where $D \subseteq \mathbb{R}^m$ is a region which includes the origin (and h is bounded).

Lemma 2.4 (*Medina and Pinto* [8]). *The linear system*

$$x(n+1) = A(n)x(n), \quad x(n_0) = x_0, \tag{6}$$

where $A(n)$ is an $m \times m$ matrix is h-stable, if and only if, the following condition (A) *holds*:

(A) *There exist a constant $c \geqslant 1$ and a positive and bounded function h defined on N_0 such that for every $x_0 \in \mathbb{R}^m$,*

$$|\Phi(n, n_0)| \leqslant ch(n)h^{-1}(n_0), \quad n \geqslant n_0, \tag{7}$$

where $\Phi(n, n_0)$ is the fundamental matrix of system (6).

We shall use the following theorem, which gives an explicit pointwise estimate, independent of u, for a function $u = u(n)$ which satisfies the inequality

$$u(n) \leqslant c + \sum_{i=1}^{p} \left[\sum_{j=n_0}^{n-1} \lambda_i(j)\omega_i(u(j)) \right], \quad p \in N, \tag{8}$$

where

(I) the functions $\omega_i : [0, \infty) \to [0, \infty)$, $1 \leqslant i \leqslant p$, are continuous and nondecreasing, $\omega_i(u) > 0$ for $u > d$ and ω_{i+1}/ω_i $(1 \leqslant i \leqslant p-1)$ are nondecreasing on (d, ∞).

(II) $u : N \to [d, \infty)$ and $\lambda : N \to [d, \infty)$ are functions, c is a constant such that $c > d$.

We define the functions

(i) $W_i(u) = \int_{u_i}^{u} \mathrm{d}s/\omega_i(s)$, $u > 0$, $u_i > 0$ $(1 \leqslant i \leqslant p)$ and W_i^{-1} is their inverse function.

(ii) $\varphi_0(u) = u$ and

$$\varphi_i(u) = \phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_1, \quad 1 \leqslant i \leqslant p, \tag{9}$$

where $\phi_i(u) = W_i^{-1}[W_i(u) + \alpha_i]$, $\alpha_i \geqslant 0$ is a constant. Thus, we can establish the following theorem:

Theorem A (*Medina and Pinto* [9]). *Let $d \in R$ and assume* (I) *and* (II) *hold. Let $m \in N$ such that*

$$\alpha_i(m) =: \sum_{j=1}^{m} \lambda_i(j) \leqslant \int_{\varphi_{i-1}(c)}^{\infty} \frac{\mathrm{d}s}{\omega_i(s)} \quad (1 \leqslant i \leqslant p),$$

where the functions φ_i $(0 \leqslant i \leqslant p-1)$ are given in (9) *with $\alpha_i = \alpha_i(m)$. If the function u satisfies the inequality* (8), *then*

$$u(n) \leqslant W_p^{-1} \left[W_p(\varphi_{p-1}(c)) + \sum_{j=1}^{n-1} \lambda_p(j) \right]$$

for any $n \leqslant m$.

3. Main results

In this section we shall obtain some h-stability criteria for the perturbed system (2) satisfying (3).

Theorem 3.1. *Suppose that* (1) *is h-stable and the perturbation $f = f(n, y)$ is defined on $N \times \mathbb{R}^m$ satisfying* (3), *where*

(H_1) *The functions ω_i $(1 \leqslant i \leqslant p)$ satisfy conditions* (I) *and for any i, $1 \leqslant i \leqslant p$, there is a function r_i defined on $(0, \infty)$ such that*

$$\omega_i(\alpha u) \leqslant r_i(\alpha)\omega_i(u) \quad \text{for } \alpha > 0,\ u \geqslant 0, \tag{10}$$

and

(H_2) *the functions λ_i $(1 \leqslant i \leqslant p)$ are nonnegatives on N_0. Furthermore, suppose that*

(H_3) *there exists $\delta > 0$ such that*

$$\begin{aligned} K_i(\delta) &=: \sup\{\alpha_i(n_0, y_0)/n_0 \geqslant 0,\ 0 < |y_0| < \delta\} \\ &< c^{-1} \int_{\varphi_{i-1}(c)}^{\infty} \frac{\mathrm{d}s}{\omega_i(s)} \quad (1 \leqslant i \leqslant p), \end{aligned} \tag{11}$$

where c is the constant satisfying (7), *and*

$$\alpha_i(n_0, y_0) = \frac{h(n_0)}{|y_0|} \sum_{j=n_0}^{\infty} \lambda_i(j) h^{-1}(j+1) r_i(|y_0| h(j) h^{-1}(n_0)) \quad (1 \leqslant i \leqslant p),$$

and

$$\varphi_i = \psi_i \circ \psi_{i-1} \circ \cdots \circ \psi_1, \qquad \psi_i(u) = W_i^{-1}[W_i(u) + cK_i(\delta)].$$

Then, for all $n_0 \geqslant 0$ and $|y_0|$ small enough any solution $y(n) = y(n, n_0, y_0)$ of (2) *satisfies*

$$|y(n, n_0, y_0)| \leqslant \varphi_p(c)|y_0| h(n) h^{-1}(n_0), \quad n \geqslant n_0.$$

Proof. By variation of parameters formula, the solution $y(n) = y(n, n_0, y_0)$ of (2) satisfies

$$y(n) = \Phi(n, n_0) y_0 + \sum_{j=n_0}^{n-1} \Phi(n, j+1) f(j, y(j)) \quad \text{for } n \geqslant n_0.$$

Thus, using (3) and (7) we have

$$|y(n)| \leqslant c|y_0| h(n) h^{-1}(n_0) + \sum_{j=n_0}^{n-1} c h(n) h^{-1}(j+1) \sum_{i=1}^{p} \lambda_i(j) \omega_i(|y(j)|), \tag{12}$$

or, denoting $\ell(n, n_0, y_0) = |y_0| h(n) h^{-1}(n_0)$ for $0 < |y_0| < \delta$, we have that $u(n) = |y(n)|/\ell(n, n_0, y_0)$ satisfies for $0 \neq |y_0|$:

$$u(n) \leqslant c + c \sum_{i=1}^{p} \sum_{j=n_0}^{n-1} \frac{\lambda_i(j) r_i(\ell(j, n_0, y_0))}{\ell(j+1, n_0, y_0)} \omega_i(u(j)).$$

So, by Theorem A it follows that

$$u(n) \leqslant W_p^{-1}\left[W_p(\varphi_{p-1}(c)) + c\sum_{j=n_0}^{n-1}\frac{\lambda_p(j)r_p(\ell(j,n_0,y_0))}{\ell(j+1,n_0,y_0)}\right],$$

for $n \geqslant n_0$, where φ_i are the functions defined in (H_3). The inequalities (11) show that this estimation is valid for every $n \geqslant n_0$ and that the function in the right member is bounded by

$$W_p^{-1}\left[W_p(\varphi_{p-1}(c)) + c\sum_{j=n_0}^{\infty}\frac{\lambda_p(j)r_p(\ell(j,n_0,y_0))}{\ell(j+1,n_0,y_0)}\right] \leqslant \varphi_p(c).$$

Hence, $u(n) \leqslant \varphi_p(c)$ for $n \geqslant n_0$, that is, for $|y_0|$ small enough,

$$|y(n,n_0,y_0)| \leqslant \varphi_p(c)|y_0|h(n)h^{-1}(n_0),\ n \geqslant n_0.$$

Therefore, the perturbed system (2) is h-stable because, from [9, Corollary 2], we have $\varphi_p(c) \geqslant c > 1$. □

Remark 3.2. If in (11) the inequalities are not strict, then in general the conclusion of Theorem 3.1 is not true.

By considering $A \equiv 0$ in (1), we obtain an useful criteria for the Lipschitz stability.

Corollary 3.3. *Suppose that for* $(n,x) \in N_0 \times \mathbb{R}^m$,

$$|f(n,x)| \leqslant \sum_{i=1}^{p}\lambda_i(n)\omega_i(|x|),$$

where ω_i $(1 \leqslant i \leqslant p)$ *satisfy* (I), *and* λ_i $(1 \leqslant i \leqslant p)$ *are nonnegative and* $\lambda_i \in \ell_1(N_0)$. *Further, assume that for* $0 < |x_0| < \delta$,

$$q_i(\delta) =: sup\left\{\frac{r_i(|x_0|)}{|x_0|}\,\middle|\, 0 < |x_0| < \delta\right\}$$

is finite and verifies

$$K_i(\delta) =: q_i(\delta)\sum_{j=n_0}^{\infty}\lambda_i(j) < \int_{\varphi_{i-1}(1)}^{\infty}\frac{ds}{\omega_i(s)} \quad (1 \leqslant i \leqslant p), \tag{13}$$

where φ_i *is the same of* (H_3), *Theorem* 3.1.

Then, for all $n_0 \geqslant 0$ *and* $|x_0|$ *small enough, any solution* $x(n,n_0,x_0)$ *of system*

$$x(n+1) = f(n,x(n))$$

is Lipschitz stable, that is, there exists a constant $c \geqslant 1$ *such that*

$$|x(n,n_0,x_0)| \leqslant c|x_0|, \quad n \geqslant n_0, \tag{14}$$

for $|x_0|$ *small enough (respectively globally Lipschitz stable if* $|x_0| < \infty$).

Remark 3.4. Corollary 3.3 extends Theorem 2.13 [3] to difference equations with several nonlinearities.

Similarly, if (1) is exponentially asymptotically stable, we have:

Corollary 3.5. *Assume that* (1) *is exponentially asymptotically stable and that* (3) *holds, where the functions* ω_i $(1 \leqslant i \leqslant p)$ *satisfy* (H_1) *such that*

$$r_i(\alpha u) \leqslant \alpha r_i(u) \leqslant M\alpha u \quad (1 \leqslant i \leqslant p)$$

for $0 < \alpha \leqslant 1$, $0 < u < \delta$, *where* $M > 0$ *is a constant, and the functions* λ_i $(1 \leqslant i \leqslant p)$ *are nonnegatives and* $\lambda_i \in \ell_1(N_0)$. *Further, suppose that for some* $\delta > 0$, $K_i(\delta)$ *(defined in* (13)) *satisfies*

$$cK_i(\delta) < \int_{\varphi_{i-1}(c)}^{\infty} \frac{\mathrm{d}s}{\omega_i(s)} \quad (1 \leqslant i \leqslant p),$$

where φ_i *is shown in* (H_3) *of Theorem* 3.1 *and* c *is the constant in* (7) *for* $h(n) = \mathrm{e}^{-\alpha n}$. *Then, the perturbed system* (2) *is exponentially asymptotically stable.*

The stabilities considered in Corollaries 3.3 and 3.5 are Lipschitz stabilities, that is, for which (14) holds. In the next corollary we shall study h-stability and Lipschitz stability simultaneously, assuming that the perturbations is "polynomial".

Corollary 3.6. *Assume that* (1) *is uniformly h-stable, that is, h-stable such that* $h(n)h^{-1}(n_0) \leqslant M$ *for* $n \geqslant n_0$ *and* $M \geqslant 1$ *a constant. Furthermore, the perturbation* $f(n, y)$ *satisfies condition* (3) *for* $\omega_i(u) = u^{\gamma_i}$, $\gamma_i \geqslant 1$, $1 \leqslant i \leqslant p$, *where* $\lambda_i(n) \in \ell_1(N_0)$.

Then, the statements of Theorem 3.1 remain valid.

Proof. In fact, for $\omega_i(u) = u^{\gamma_i}$, ω_{i+1}/ω_i is nondecreasing, if and only if, $\gamma_i \leqslant \gamma_{i+1}$ and $r_i = \omega_i$. Thus, if $\gamma_i \geqslant 1$, (11) follows provided that

$$K_i(\delta) = (\delta M)^{\gamma_i - 1}\alpha_i < c^{-1} \int_{\varphi_{i-1}(c)}^{\infty} s^{-\gamma_i}\,\mathrm{d}s, \tag{15}$$

where

$$\alpha_i = \sum_{j=n_0}^{\infty} \lambda_i(j) \quad (1 \leqslant i \leqslant p).$$

The existence of such δ will follow from the fact that for $\gamma_i > 1$, $K_i(0^+) = 0$.

Since

$$\int_{\varphi_{i-1}(c)}^{\infty} \frac{\mathrm{d}s}{s^{\gamma_i}} = \begin{cases} \infty & \text{if } \gamma_i = 1, \\ c^{-1}(\gamma_i - 1)^{-1}(\varphi_{i-1}(c))^{1-\gamma_i} & \text{if } \gamma_i > 1, \end{cases}$$

we must study only the case $\gamma_i > 1$. So, let $\delta_1 > 0$ satisfying (15) for $i = 1$:

$$K_1(\delta_1) = (\delta_1 M)^{\gamma_1 - 1}\alpha_1 < c^{-1}/(\gamma_1 - 1), \quad \alpha_1 = \sum_{j=n_0}^{\infty} \lambda_1(j).$$

Moreover, also there exists $\delta_2 \leqslant \delta_1$, satisfying (15) for $i = 2$:

$$\begin{aligned} K_2(\delta_2) &= (\delta_2 M)^{\gamma_2 - 1}\alpha_2 < c^{-1}(\tilde{\varphi}_1(c))^{1-\gamma_2}/(\gamma_2 - 1) \\ &= c^{-1}(\gamma_2 - 1)^{-1}(W_1^{-1}[W_1(c) + cK_1(\delta_1)])^{1-\gamma_2}, \end{aligned}$$

because $K_2(0^+) = 0$. Thus, we find $\delta_1 \geqslant \delta_2 \geqslant \cdots \geqslant \delta_p$ such that

$$K_i(\delta_i) < c^{-1}(\gamma_i - 1)^{-1}(\tilde{\varphi}_{i-1}(c))^{1-\gamma_i},$$

where

$$\tilde{\varphi}_i = \tilde{\psi}_i \circ \tilde{\psi}_{i-1} \circ \cdots \circ \tilde{\psi}_1, \quad \tilde{\psi}_i(u) = W_i^{-1}[W_i(u) + cK_i(\delta_i)], \; 1 \leqslant i \leqslant p.$$

Since K_i $(1 \leqslant i \leqslant p)$ are nondecreasing functions, for $\delta = \delta_p$ it verifies

$$\begin{aligned} K_p(\delta) \leqslant K_i(\delta_i) &< c^{-1}(\gamma_i - 1)^{-1}(\tilde{\varphi}_{i-1}(c))^{1-\gamma_i} \\ &\leqslant c^{-1}(\gamma_i - 1)^{-1}(\varphi_{i-1}(c))^{1-\gamma_i} \end{aligned}$$

because $\varphi_i(c) \leqslant \tilde{\varphi}_i(c)$. Thus, $\delta = \delta_p$ satisfies (15). □

Remark 3.7. The method proposed in Corollary 3.6 to compute δ, which satisfies (11) is not only exclusive of $\omega_i(u) = u^{\gamma_i}$, $\gamma_i \geqslant 1$. It is rather proper of the situation: K_i $(1 \leqslant i \leqslant p)$ are nondecreasing functions such that $K_i(0^+) = 0$. For $\omega_i(u) = u^{\gamma_i}$, $\gamma_i < 1$ this last assertion does not satisfy the conclusion of Theorem 3.1.

4. Asymptotic formulae

Our objective is to obtain some results about asymptotic summation of systems resulting from the perturbation of an h-system.

Definition 4.1. A function $\omega : [0, \infty) \to [0, \infty)$ is said to be of the class $\mathscr{F}$ if

(i) $\omega(u)$ is nondecreasing and continuous for $u \geqslant 0$ and positive for $u > 0$, and

(ii) there exists a nonnegative function r (multiplier function) defined on $(0, \infty)$ such that

$$\omega(\alpha u) \leqslant r(\alpha)\omega(u) \quad \text{for } \alpha > 0, \; u \geqslant 0.$$

Theorem 4.2. *Assume that*

(F) *The system* (1) *is an h-system for x_0 small enough.*

(G) *The perturbation $f = f(n, y)$ is defined on $N_0 \times \mathbb{R}^m$ such that*

$$|f(n, y)| \leqslant \lambda(n)\omega(|y|),$$

where

(G.1) *The function* $\omega \in \mathscr{F}$, *with corresponding multiplier function* r, *and*

(G.2) *the function* $\lambda : N_0 \to [0, \infty)$ *satisfies*

$$\lambda(n) r(h(n) h^{-1}(n_0)) h^{-1}(n+1) \in \ell_1(N_0).$$

Then,

(T_1) *for* $n_0 \geqslant 0$ *and* y_0 *sufficiently small, every solution* $y(n) = y(n, n_0, y_0)$ *of* (2) *satisfies*

$$|y(n, n_0, y_0)| \leqslant K h(n) h^{-1}(n_0),$$

where K *is a positive constant.*

(T_2) *for each one of these solutions* $y(n)$ *of* (2), *there is* $z_0 \in \mathbb{R}^m$ *such that*

$$y(n) = \Phi(n, n_0) z_0 + h(n) \tilde{o}(1) \quad \text{as } n \to \infty, \tag{16}$$

where $\tilde{o}(1)$ *represents a function which has a limit when* n *approaches to infinity.*

Proof. By variation of parameters formula, $y(n) = y(n, n_0, y_0)$ satisfies

$$y(n) = \Phi(n, n_0) y_0 + \sum_{j=n_0}^{n-1} \Phi(n, j+1) f(j, y(j)).$$

Hence, by (F) and (G) it follows that

$$|y(n)| \leqslant c |y_0| h(n) h^{-1}(n_0) + \sum_{j=n_0}^{n-1} c h(n) h^{-1}(j+1) \lambda(j) \omega(|y(j)|);$$

thus,

$$\frac{|y(n)|}{h(n) h^{-1}(n_0)} \leqslant c |y_0| + \sum_{j=n_0}^{n-1} \frac{c \lambda(j) \omega(|y(j)|)}{h(j+1) h^{-1}(n_0)}.$$

So, by (G.1) and (G.2) one obtains

$$\frac{|y(n)|}{\beta(n, n_0)} \leqslant c |y_0| + \sum_{j=n_0}^{n-1} \frac{c \lambda(j) r(\beta(j, n_0))}{\beta(j+1, n_0)} \omega\left(\frac{|y(j)|}{\beta(j, n_0)}\right),$$

where $\beta(n, n_0) =: h(n) h^{-1}(n_0)$, $n \geqslant n_0$.

Now, we apply Theorem A to $v(n) = |y(n)| / \beta(n, n_0)$, thus establishing that

$$|y(n)| \leqslant h(n) h^{-1}(n_0) W^{-1}\left[W(c|y_0|) + \sum_{j=n_0}^{n-1} \frac{c \lambda(j) r(\beta(j, n_0))}{\beta(j+1, n_0)}\right] \tag{17}$$

and then for y_0 small enough, there is a positive constant K such that

$$|y(n, n_0, y_0)| \leqslant K h(n) h^{-1}(n_0), \quad n \geqslant n_0.$$

Now, for these solutions $y(n)$ of system (2), we have

$$\begin{aligned}\left|\sum_{j=n_0}^{n-1} \Phi(n,j+1)f(j,y(j))\right| &\leqslant \sum_{j=n_0}^{n-1} ch(n)h^{-1}(j+1)\lambda(j)\omega(|y(j)|)\\ &\leqslant c\beta(n,n_0)\sum_{j=n_0}^{n-1}\frac{c\lambda(j)r(\beta(j,n_0))}{\beta(j+1,n_0)}\omega\left(\frac{|y(j)|}{\beta(j,n_0)}\right)\\ &\leqslant c\omega(K)\beta(n,n_0)\sum_{j=n_0}^{n-1}\frac{\lambda(j)r(\beta(j,n_0))}{\beta(j+1,n_0)}.\end{aligned}$$

Then, for every solution y of system (2) the solution x of system (1), given by

$$x(n) = y(n) - \sum_{j=n_0}^{n-1} \Phi(n,j+1)f(j,y(j))$$

has the property

$$y(n) = x(n) + h(n)\tilde{o}(1) \quad \text{as } n \to \infty.$$

On the other hand, if x is solution of system (1) then there is $z_0 \in \mathbb{R}^m$ such that $x(n) = \Phi(n,n_0)z_0$, where $\Phi(n,n_0)$ is the fundamental matrix of (1).
Therefore,

$$y(n) = \phi(n,n_0)z_0 + h(n)\tilde{o}(1) \quad \text{as } n \to \infty. \qquad \square$$

Remark 4.3. In (17), we are assuming that $|y_0|$ is small enough and that

$$W(0^+) = -\infty, \tag{18}$$

where

$$W(u) = \int_{u_0}^{u} \frac{\mathrm{d}s}{\omega(s)}, \quad u > 0, \quad u_0 > 0,$$

in order that W^{-1} has meaning. That is, the inverse function $W^{-1}(v)$ is defined for $v \in (0, \delta_0)$, where $\delta_0 > 0$ is small enough.

Remark 4.4. We remark that under the conditions of Theorem 4.2, the error given by the asymptotic formula (16) is always dominated by h. However, if the function h is not a good majorant then all information can be added to the error. Also, a difficulty like this can be solved assuming that the function h satisfies the condition $\lim_{n\to\infty} h(n) < \infty$ exists, and the proof Theorem 4.2 remains valid without modifications.

Corollary 4.5. *Assume that*

(H_0) *there is a positive constant c such that $|\Phi(n,n_0)| \leqslant c$, for $n \geqslant n_0$; and*

(H_3) *there exist a nonnegative function* λ *such that for* $(n,y)\in N_0\times\mathbb{R}^m$,

$$|f(n,y)|\leqslant\lambda(n)\omega(|y|),\quad \lambda\in\ell_1(N_0),$$

where ω *is continuous, positive and nondecreasing function which satisfies* (18).

Then,

(T_1) *for* y_0 *sufficiently small, every solution* $y(n)=y(n,n_0,y_0)$ *of* (2) *satisfies* $|y(n,n_0,y_0)|\leqslant K$, $K>0$ *a constant,*

(T_2) *for each one of these solutions* y *of* (2), *there is a solution* x *of* (1) *such that*

$$y(n)=x(n)+\tilde{o}(1)\quad \text{as } n\to\infty.$$

A more precise asymptotic formula is given in the following theorem:

Theorem 4.6. *Let* ω *be a continuous, positive and nondecreasing function on* $[0,\infty)$ *such that* ω *satisfies* (18). *Assume that for the fundamental matrix* $\Phi(n,n_0)$ *of* (1) *we have*

$$|\Phi^{-1}(n+1,n_0)f(n,\Phi(n,n_0)z)|\leqslant\lambda(n)\omega(|z|),\lambda\in\ell_1(N_0)\quad \text{for } n\in N_0 \text{ and } z\in\mathbb{R}^m.$$

Then, for every solution $y(n)=y(n,n_0,y_0)$ *of* (2), *with* $|y_0|$ *sufficiently small, there is* $z_0\in\mathbb{R}^m$ *such that*

$$y(n)=\Phi(n,n_0)\left[z_0+O\left(\sum_{j=n}^{\infty}\lambda(j)\right)\right]\quad \text{as } n\to\infty. \tag{19}$$

Proof. Making $y(n)=\Phi(n,n_0)z(n)$ in system (2), we obtain

$$z(n)=y_0+\sum_{j=n_0}^{n-1}\Phi^{-1}(j+1,n_0)f(j,\Phi(j,n_0)z(j)).$$

Thus,

$$|z(n)|\leqslant|y_0|+\sum_{j=n_0}^{n-1}\lambda(j)\omega(|z(j)|)$$

and by Theorem A, for $p=1$,

$$|z(n)|\leqslant W^{-1}\left[W(|y_0|)+\sum_{j=n_0}^{n-1}\lambda(j)\right].$$

Moreover, for y_0 sufficiently small, there exists a positive constant K such that $|z(n)|\leqslant K$ for all $n\geqslant n_0$. Thus, for $n\geqslant n_0$

$$|y(n,n_0,y_0)|\leqslant K|\Phi(n,n_0)|.$$

Moreover, $\Phi^{-1}(n+1,n_0)f(n,y(n))\in\ell_1(N_0)$ and

$$\left|\sum_{j=n}^{\infty}\Phi^{-1}(j+1,n_0)f(j,y(j))\right| = \left|\sum_{j=n}^{\infty}\Phi^{-1}(j+1,n_0)f(j,\Phi(j,n_0)z(j))\right|$$
$$\leqslant \omega(K)\sum_{j=n}^{\infty}\lambda(j).$$

So, for each one of these solutions y of system (2), the function

$$x(n) = y(n) + \sum_{j=n}^{\infty}\Phi(n,n_0)\Phi^{-1}(j+1,n_0)f(j,y(j))$$

is a solution of system (1) and satisfies

$$\Phi^{-1}(n,n_0)(x(n)-y(n)) = \mathrm{O}\left(\sum_{j=n}^{\infty}\Phi^{-1}(j+1,n_0)f(j,y(j))\right),$$

and then there is $z_0\in\mathbb{R}^m$ such that

$$y(n) = \Phi(n,n_0)\left[z_0 + \mathrm{O}\left(\sum_{j=n}^{\infty}\lambda(j)\right)\right] \quad \text{as } n\to\infty. \qquad \square$$

Notice that in Theorem 4.2 the hypothesis over the system (1) is independent of the perturbation f, but in Theorem 4.6 that dependency really exists. Also, these two theorems differ by the error given in their respective asymptotic formulae: In (16) we get only $h\tilde{o}(1)$, however in (19) we get an error of order $\Phi o(1)$. These difference are notorious when the perturbed system (2) is linear:

Corollary 4.7. *Assume that (F) of Theorem 4.2 is satisfied. Then, for every fundamental matrix Ψ of*

$$y(n+1) = (A(n)+B(n))y(n), \quad B\in\ell_1(N_0), \tag{20}$$

there exists a constant and invertible matrix C such that

$$\Psi(n) = \Phi(n,n_0)C + h(n)\tilde{o}(1) \quad \textit{as } n\to\infty,$$

where $\Phi(n,n_0)$ is the fundamental matrix of system (1).

Corollary 4.8. *Assume that for the fundamental matrix $\Phi(n,n_0)$ of (1) we have*

$$\Phi^{-1}(n+1,n_0)B(n)\Phi(n,n_0)\in\ell_1(N_0).$$

Then, for each fundamental matrix Ψ of system (20) there exists an invertible and constant matrix C such that

$$\Psi(n) = \Phi(n,n_0)[C+o(1)] \quad \textit{as } n\to\infty.$$

The last two corollaries are different versions of [2, Lemma 2.1], which do not require that $A(n)$ be diagonalizable. Notice that Lemma 2.1 of [2] is the discrete analogue of Levinson's Theorem [7, p. 92] about asymptotic representation of solutions of differential systems.

Corollary 4.9. *If all the solutions of the linear system with constant coefficients*

$$x(n+1) = Ax(n),$$

where A is a constant matrix, are bounded, then for each fundamental matrix $\Psi(n)$ of the perturbed system

$$y(n+1) = (A + B(n))y(n), \quad B \in \ell_1(N_0),$$

there exists an $m \times m$ constant and invertible matrix C such that

$$\Psi(n) = A^{n-n_0}C + \tilde{o}(1) \quad \text{as } n \to \infty.$$

More delicate arguments ensure that in this case $\tilde{o}(1)$, the error, is really an $o(1)$. (see [10]).

5. Examples and applications

Example 5.1. Consider the Emden–Fowler difference equation

$$\Delta^2 y(n) = p(n)y^{\gamma}(n), \tag{21}$$

where $\{p(n)\}_1^{\infty}$ is a sequence of real numbers, γ ($\neq 0, 1$) is a real number and Δ is the forward difference operator with unit spacing, i.e., $\Delta u(i) = u(i+1) - u(i)$, and $\Delta^2 u(n) = \Delta(\Delta u(n))$.

If we define $u_i(n) = y(n+i-1)$, $1 \leqslant i \leqslant 2$ then Eq. (21) can be written as

$$u(n+1) = Au(n) + g(n, u(n)), \tag{22}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \qquad g(n, u(n)) = p(n)u_1^{\gamma}(n)\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and $u(n) = (u_1(n), u_2(n)) = (y(n), y(n+1))$.

A fundamental matrix solution of equation

$$v(n+1) = Av(n) \tag{23}$$

is given by

$$\Phi(n, 0) = \begin{bmatrix} n & 1 \\ n+1 & 1 \end{bmatrix}.$$

Moreover, we have

$$\begin{aligned}|g(n,u(n))| &\leqslant |p(n)|\,|u_1(n)|^{\gamma}\\ &\leqslant |p(n)|\,|u(n)|^{\gamma}=\lambda(n)\omega(|u(n)|),\end{aligned}$$

where $\lambda(n)=|p(n)|$ and $\omega(v)=v^{\gamma}$.

On the other hand, Eq. (23) is an h-system, because

$$|\Phi(n,0)|\leqslant ch(n)h^{-1}(0) \quad \text{where } h(n)=2n+3 \text{ and } c\geqslant 3, \text{ for } n\geqslant n_0=0.$$

Then, for $p(n)$ and γ such that $((2n+3)^{\gamma}/(2n+5))p(n)\in\ell_1(N_0)$, we can apply Theorem 4.2 and therefore, for each solution $u(n)$ of (22) with small initial conditions, there exists $z_0=(z_0^1,z_0^2)\in\mathbb{R}^2$ such that

$$u(n)=\Phi(n,0)z_0+h(n)\tilde{o}(1) \quad \text{as } n\to\infty.$$

Thus, it follows that

$$y(n)=nz_0^1+z_0^2+n\tilde{o}(1) \quad \text{as } n\to\infty.$$

Example 5.2. Consider the second-order linear difference equation

$$y(n+2)+(a(n)+b(n))y(n)=0, \qquad b\in\ell_1(N_0). \tag{24}$$

If the solutions of the nonperturbed equation

$$x(n+2)+a(n)x(n)=0 \tag{25}$$

are all bounded, then by Corollary 4.5, for each solution $y(n)$ of (24) there exists a solution $x(n)$ of (25) such that

$$y(n)=x(n)(1+o(1)) \quad \text{as } n\to\infty. \tag{26}$$

We remark that formula (26) is a precise asymptotic formula.

Example 5.3. Consider the scalar difference equation

$$y(n+1)=\frac{n}{n+1}y(n)+\frac{y^7(n)e^{-n}}{1+4y^6(n)}\sin(ny(n)), \quad n\geqslant n_0\geqslant 1. \tag{27}$$

The nonperturbed difference equation

$$x(n+1)=\frac{n}{n+1}x(n)$$

possesses the solution

$$x(n,n_0,x_0)=\frac{n_0}{n}x_0 \quad \text{for } n_0\geqslant 1 \text{ and } x_0\in\mathbb{R}.$$

So, by Corollary 4.5, Eq. (27) has solutions $y(n,n_0,y_0)$ for $n_0 \geqslant 1$ and y_0 small enough such that

$$y(n,n_0,y_0) = \frac{n_0}{n} y_0 + \tilde{o}(1) \quad \text{as } n \to \infty,$$

where $\tilde{o}(1)$ represents a function which has limit when n approaches to infinity.

Example 5.4. For a more special example, consider the system

$$y(n+1) = (\Lambda(n) + B(n))y(n), \qquad n \geqslant n_0, \tag{28}$$

where

$$\Lambda(n) = diag\{\lambda_1(n), \dots, \lambda_m(n)\},$$

and $B(n)$ is an $m \times m$ matrix.

Assume that $B \in \ell_1(N_0)$ and $\Lambda(n)$ satisfies the following condition:

For every i, $1 \leqslant i \leqslant m$,

$$\prod_{\ell=n_0}^{n-1} \left| \frac{\lambda_i(\ell)}{h(\ell)} \right| \leqslant K \quad \text{for } n \geqslant n_0, \text{ and } \quad K \text{ a positive constant.}$$

Then, by Corollary 4.7, for every fundamental matrix Ψ of system (28), there exists a constant and invertible matrix C such that

$$\Psi(n) = \prod_{\ell=n_0}^{n-1} \Lambda(\ell)C + h(n)\tilde{o}(1) \quad \text{as } n \to \infty.$$

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