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Cubature formulae of the seventh degree of accuracy for the hypersphere

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Abstract

Cubature formulae for evaluating integrals on the hypersphere in $\mathbb{R}^n$ for $n \geqslant 5$ are obtained, which are exact for any polynomial of degree not exceeding 7, and are invariant with respect to the group of transformations of the regular simplex.

Keywords: Cubature formula; Hypersphere; Regular simplex

AMS classification: 65D

1. Introduction

Let S_n denote the unit hypersphere in $\mathbb{R}^n$

$$S_n = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} x_i^2 \leqslant 1 \right\}. \tag{1.1}$$

Denoted by T_n the regular simplex in $\mathbb{R}^n$, with vertices at $n+1$ points in $\mathbb{R}^n$

$$a^{(r)} = (a_1^{(r)}, a_2^{(r)}, \dots, a_n^{(r)}), \quad r = 1, 2, \dots, n+1, \tag{1.2}$$

where

$$a_i^{(r)} = \begin{cases} -\sqrt{\dfrac{n+1}{n(n-i+2)(n-i+1)}}, & i<r, \\ \sqrt{\dfrac{(n+1)(n-r+1)}{n(n-r+2)}}, & i=r, \\ 0, & i>r. \end{cases}$$

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The center of the simplex T_n is the point $\theta=(0,0,\ldots,0)$. The vertices of the T_n lie on the surface of the unit hypersphere in $\mathbb{R}^n$

$$U_n=\left\{x\in\mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} x_i^2=1\right\}.$$

The projection onto U_n of the mid-point of the edge which connect vertices $a^{(1)}$ and $a^{(2)}$ will be denoted by

$$b^{(1)}=\left(\sqrt{\frac{n-1}{2n}},\sqrt{\frac{n+1}{2n}},0,\ldots,0\right). \tag{1.3}$$

The projection onto U_n of the center of the two-dimensional face of T_n, with vertices $a^{(1)},a^{(2)},a^{(3)}$ will be denoted by

$$c^{(1)}=\left(\sqrt{\frac{n-2}{3n}},\sqrt{\frac{(n+1)(n-2)}{3n(n-1)}},\sqrt{\frac{n+1}{3(n-1)}},0,\ldots,0\right). \tag{1.4}$$

The projection onto U_n of a t-point $(1-t)a^{(1)}+ta^{(2)}$, $0<t<1/2$, which lies on the edge connecting the vertices $a^{(1)}$ and $a^{(2)}$ at the T_n, will be denoted by

$$b^{(1)}(t)=(nT)^{-1/2}(n-t(n+1),t\sqrt{n^2-1},0,\ldots,0), \tag{1.5}$$

where

$$T=2(n+1)t^2-2(n+1)t+n.$$

Let G is the group of all transformations of the regular polyhedron in $\mathbb{R}^n$ (its center being point θ) into itself. The set of the points of the sort ga, where a is a fixed point in $\mathbb{R}^n$ and g involves all transformations of the group G, will be called an orbit, or G-orbit, which contains the point a and will be denoted by $G(a)$. The number of points in the G-orbit depends on the point a.

The group of all transformations of T_n into itself will be denoted by T_nG. It is known [4], that the order of the group T_nG is equal to $(n+1)!$.

It is known [4] that the basis of polynomials which are invariant with respect to the group T_nG consists of n polynomials

$$\pi_k(x)=\sum_{r=1}^{n+1} l_r^k(x), \quad k=2,3,\ldots,n+1, \tag{1.6}$$

where

$$l_r(x)=\sum_{i=1}^{n} a_i^{(r)}x_i, \quad r=1,2,\ldots,n+1.$$

This means that any polynomial invariant with respect to the group T_nG is a polynomial in the polynomials (1.6).

The group which is derived from the group T_nG by adding central symmetry transformation under the point θ, will be denoted by T_nG^*. The set of the polynomials which are invariant with respect to the group T_nG^* coincides with the set of the even invariant polynomials of T_nG group. It is known [4] that the order of the group T_nG^* is equal to $2(n+1)!$.

Cubature formulae for S_n, which are invariant with respect to the group T_nG^* and are exact for all polynomials of degree not exceeding 5 are obtained in [3, 5]. Cubature formulae for S_3, which are invariant with respect to the group T_3G^* and are exact for all polynomials of degree not exceeding 7, are obtained in [7].

In this paper, Sobolev's theorem [6] is used to construct cubature formulae for integrals of the hypersphere (1.1) for $n \geqslant 5$, which are exact for all polynomials of degree not exceeding 7, and invariant with respect to the group T_nG^*.

In Section 2 we derive the parameters of the cubature formulae. Numerical results are presented in Section 3.

Let G_n denote the hyperoctahedron in $\mathbb{R}^n$

$$G_n = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} |x_i| \leqslant 1 \right\}.$$

The group of all transformations of G_n into itself will be denoted by G_nG.

Cubature formulae for S_n, which are invariant with respect to the group G_nG are obtained in [2, 8]. A bibliography of references to cubature formulae for S_n is presented in [1].

The cubature formulae obtained in this paper, which are exact for all polynomials of degree not exceeding 7 and are invariant with respect to the group T_nG^* have less number of nodes than the number of nodes of the cubature formula in [2] and the cubature formula $S_n : 7-3$ in [8, pp. 272–273], which are exact for all polynomials of degree not exceeding 7 and are invarint with perspect to the group G_nG.

2. Cubature formula for $n \geqslant 5$

Since the cubature formula must be exact for all polynomials of degree not exceeding 7, for $n \geqslant 5$ it must be exact for 8 invariant polynomials

$$1, \quad \pi_2(x), \quad \pi_2^2(x), \quad \pi_4(x), \quad \pi_2^3(x), \quad \pi_2(x)\pi_4(x), \quad \pi_3^2(x), \quad \pi_6(x), \tag{2.1}$$

where the polynomials $\pi_k(x)$, $k = 2, 3, 4, 6$ are defined by (1.6).

Accordingly, the nodes of the cubature formula are selected such that the cubature sum depends on 8 parameters at least.

The nodes of the cubature formula are taken as the following orbits: (1) $T_nG^*(\lambda a^{(1)})$, (2) $T_nG^*(\beta b^{(1)})$, (3) $T_nG^*(\gamma c^{(1)})$, (4) $T_nG^*(\delta b^{(1)}(1/4))$, (5) $T_nG^*(\theta)$, where λ, β, γ, are unknown parameters; parameter $\delta \neq 0$ is assigned arbitrary; $a^{(1)}$, $b^{(1)}$, $c^{(1)}$, are defined by (1.2), (1.3), (1.4), respectively; $b^{(1)}(1/4)$ is defined by (1.5) for $t = 1/4$; θ is the center of the simplex T_n. The first orbit consists of the points $\pm\lambda a^{(j)}$, where $a^{(j)}$ are the vertices of the simplex T_n. The second orbit consists of the points $\pm\beta b^{(j)}$, where $b^{(j)}$ are the projections onto U_n of the mid-points of the edges of the T_n. The third orbit consists of the points $\pm\gamma c^{(j)}$, where $c^{(j)}$ are the projections onto U_n of the centers of the two-dimensional faces of the T_n. The fourth orbit consists of the points $\pm\delta b^{(j)}(\frac{1}{4})$, where $b^{(j)}(\frac{1}{4})$ are the projections onto U_n of the t-points for $t = \frac{1}{4}$, which lie on the edges of the T_n. The fifth orbit consists of the point θ.

The cubature formula can be written in the form

$$\frac{1}{\mu(S_n)}\int_{S_n} f(x)\,\mathrm{d}x \simeq A\sum_{1}^{N_1} f(\lambda a^{(1)}) + B\sum_{1}^{N_2} f(\beta b^{(1)}) + C\sum_{1}^{N_3} f(\gamma c^{(1)}) + E\sum_{1}^{N_4} f(\delta b^{(1)}(1/4)) + Df(\theta), \tag{2.2}$$

where the sum is accomplished for all points of the corresponding orbit; $\mu(S_n)=2\pi^{n/2}/[n\Gamma(n/2)]$ – the volume of the hypersphere (1.1). The number of nodes is $N=N_1+N_2+N_3+N_4+1$, for $n\geqslant 5$, where $N_1=2(n+1)$, $N_2=n(n+1)$, $N_4=2n(n+1)$ for $n\geqslant 5$; $N_3=2C_{n+1}^3$ for $n\geqslant 6$; $N_3=C_{n+1}^3$ for $n=5$, since the centers of two-dimensional faces form a centrally symmetric set when $n=5$. Hence, for $n=5$ the number of nodes is $N=123$; for $n\geqslant 6$ the number of nodes is $N=(n^3+9n^2+14n+9)/3$.

The cubature sum depends on 9 parameters. The parameter $\delta\neq 0$ is assigned arbitrary. The rest 8 parameters $D,A,B,C,E,\lambda,\beta,\gamma$ are calculated.

The requirement that formula (2.2) is exact for polynomials (2.1) yields the non-linear system of 8 equations with 8 unknowns $D,A,B,C,E,\lambda,\beta,\gamma$

$$\begin{aligned}
&(1)\colon\ D+N_1A+N_2B+N_3C+N_4E=1,\\
&(\pi_2)\colon\ r_5N_1A\lambda^2+r_5N_2B\beta^2+r_5N_3C\gamma^2+r_1N_4E\delta^2/r_4=(n+1)/(n+2),\\
&(\pi_2^2)\colon\ r_5^2N_1A\lambda^4+r_5^2N_2B\beta^4+r_5^2N_3C\gamma^4+r_1^2N_4E\delta^4/r_4^2=r_5(n+1)/(n+4),\\
&(\pi_4)\colon\ r_8N_1A\lambda^4+r_7N_2B\beta^4+r_6N_3C\gamma^4+r_2N_4E\delta^4/r_4^2=3(n+1)/[(n+2)(n+4)],\\
&(\pi_2^3)\colon\ r_5^3N_1A\lambda^6+r_5^3N_2B\beta^6+r_5^3N_3C\gamma^6+r_1^3N_4E\delta^6/r_4^3=r_5^2(n+1)/(n+6),\\
&(\pi_2\pi_4)\colon\ r_5r_8N_1A\lambda^6+r_5r_7N_2B\beta^6+r_5r_6N_3C\gamma^6+r_1r_2N_4E\delta^6/r_4^3\\
&\qquad =3r_5(n+1)/[(n+2)(n+6)],\\
&(\pi_3^2)\colon\ r_9N_1A\lambda^6/n^4+r_{10}N_2B\beta^6+r_{11}N_3C\gamma^6+r_3N_4E\delta^6/r_4^3\\
&\qquad =6r_5^2(n-1)/[(n+2)(n+4)(n+6)],\\
&(\sigma_6)\colon\ S_4'N_1A\lambda^6/n^5+S_2'N_2B\beta^6/[4n^3(n-1)^2]+S_1'N_3C\gamma^6/r_{12}+S_3'N_4E\delta^6/r_4^3\\
&\qquad =15(n+1)/[(n+2)(n+4)(n+6)],
\end{aligned}\tag{2.3}$$

where

$r_1=(3n-1)^2+(n-3)^2+16(n-1)$, $r_2=(3n-1)^4+(n-3)^4+256(n-1)$,

$r_3=[(3n-1)^3+(n-3)^3-64(n-1)]^2$, $r_4=2n(5n-3)$,

$r_5=(n+1)/n$, $r_6=[(n-2)^3+27]/[3n^2(n-2)]$,

$r_7=[(n-1)^3+8]/[2n^2(n-1)]$, $r_8=(n^3+1)/n^3$,

$r_9=(n^2-1)^2$, $r_{10}=[(n-1)^2-4]^2/[2n^3(n-1)]$,

$r_{11}=[(n-2)^2-9]^2/[3n^3(n-2)]$, $r_{12}=9n^3(n-2)^2$,

$$S'_1 = (n-2)^5 + 243, \qquad S'_2 = (n-1)^5 + 32, \tag{2.4a}$$

$$S'_3 = (3n-1)^6 + (n-3)^6 + 4096(n-1), \qquad S'_4 = n^5 + 1. \tag{2.4b}$$

The system (2.3) can be solved as follows.

Introducing the notations

$$A_1 = N_1 A\lambda^6, \qquad B_1 = N_2 B\beta^6, \qquad C_1 = N_3 C\gamma^6, \qquad E_1 = N_4 E\delta^6, \tag{2.5}$$

from the equations (π_2^3), $(\pi_2\pi_4)$, (π_3^2) and (π_6) we obtain a linear system of four equations with four unknowns A_1, B_1, C_1, E_1 and we solve this system.

Then, from the equations (π_2), (π_2^2) and (π_4) we obtain a non-linear system of three equations and we find unknowns $\lambda^2, \beta^2, \gamma^2$.

Afterwards, using (2.5) we find the coefficients A, B, C, E. From the first equation of the system (2.3) we find the coefficient D.

The solution of the system (2.3) for $n \geqslant 5$ is

$\delta \neq 0$ being assigned arbitrary,

$S'_1 - S'_4$ are found from (2.4a)–(2.4b), $S'_5 = n^2 + 4n - 8$,

$S'_6 = (n+1)(n+2)(n+4)$, $S'_7 = n(n-1)$, $S'_8 = n(n-2)$,

$S'_9 = (n-1)(n-2)$, $S'_{10} = (n+1)(n+6)S'_6$, $S'_{11} = (5n-3)^3$,

$S'_{12} = (n-1)(n-3)$, $S'_{13} = n^2 - 7n + 19$, $S'_{14} = n^2 - n + 1$,

$S'_{15} = 2(n-3)(n+1)^2$, $S'_{16} = n^3 - 9n^2 + 33n - 38$, $S'_{17} = 3(n-2)$,

$$q_1 = 9(n+1)[10n^5(n+1)^2 S'_{17} - n^3 S'_2 S'_5 + 4(n-1)S'_4 S'_5 S'_7 - 2S'_4 S'_6 S'_8] - n^2 S'_3 S'_8 + 8S'_4 S'_8 S'_{11} + 108nS'_2 S'^2_7 - 432(n-1)^3 S'_4 S'_7,$$

$q_2 = nS'_1 S'_7 - 3S'_4 S'_9 S'_{17} + 12(n-1)S'_4 S'_{12} - 3n^2(n-3)S'_2$,

$B_1 = 2(n-1)[3(n+1)S'_5 S'_7 q_2 - S'_{12} q_1 - 36(n-1)^2 S'_7 q_2]/(S'_{10} S'_{17} q_2)$,

$C_1 = S'_9 q_1/(2S'_{10} q_2)$, $E_1 = 4nS'_{11}/(9S'_{10})$, $A_1 = n/(n+6) - B_1 - C_1 - E_1$,

$Y_1 = n/(n+2) - E_1/\delta^4$, $Y_2 = n/(n+4) - E_1/\delta^2$,

$Y_3 = 3n^2/[(n+2)(n+4)] - E_1(41n^3 - 101n^2 + 155n - 87)/[2(5n-3)^2\delta^2]$,

$w_0 = S'_{17}[Y_3(S'_{15} - S'_{14} S'_{17}) + S'_{13} S'_{14} Y_2]/(S'_{13} S'_{15} C_1)$,

$u_0 = n(S'_{17} Y_3 - S'_{13} Y_2)/(S'_{15} A_1)$, $p_1 = n(n-4)B_1/(4S'_{12} A_1)$,

$p_2 = S'_{16} S'_{17} B_1/(4S'_{12} S'_{13} C_1)$, $d_1 = A_1 p_1^2 + C_1 p_2^2 + B_1$,

$d_2 = A_1 p_1 u_0 + C_1 p_2 w_0$, $d_3 = A_1 u_0^2 + C_1 w_0^2 - Y_1$, $d_0 = d_2^2 - d_1 d_3$,

$v = (d_2 \pm \sqrt{d_0})/d_1$, $u = u_0 - p_1 v$, $w = w_0 - p_2 v$,

$\lambda^2 = 1/u$, $\beta^2 = 1/v$, $\gamma^2 = 1/w$,

$$A = A_1/(N_1\lambda^6), \quad B = B_1/(N_2\beta^6), \quad C = C_1/(N_3\gamma^6),$$
$$E = E_1/(N_4\delta^6), \quad D = 1 - N_1A - N_2B - N_3C - N_4E.$$

3. Numerical results for $n \geqslant 5$

A FORTRAN program written to compute the parameters of the formula (2.2) can be used for any $n \geqslant 5$ if the formula exists, or to establish that the formula does not exist and why.

The program can verify whether the nodes are inside S_n. Since $\delta \neq 0$ is assigned arbitrary, we can derive an infinite set of cubature formulae and one may seek such values for δ for which the derived nodes are inside S_n. Since two values have been obtained for v, we can derive two infinite sets of cubature formulae. With this program computations are conducted for $n = 5(1)40$, and it is established that the formula (2.2) exists.

For $v = (d_2 + \sqrt{d_0})/d_1$ the following results are obtained: when $n = 5(1)12$, part of the nodes are outside S_n; when $n = 13(1)40$, the nodes are inside S_n for $\delta = 1$. The results for $n = 5(1)9$ are given in Table 1.

For $v = (d_2 - \sqrt{d_0})/d_1$ the following results are obtained: when $n = 5(1)7$, part of the nodes are outside S_n; when $n = 8(1)40$, the nodes are inside S_n for $\delta = 0.8$. The results for $n = 5(1)9$ are given in Table 2.

Table 1

n	5	6	7	8	9
D	0.0078205	0.0065905	0.0068249	0.0066561	0.0088018
A	−0.0386067	−0.0380236	−0.0359039	−0.0332999	−0.0298446
B	−0.0092371	−0.0106772	−0.0100358	−0.0092346	−0.0071381
C	0.0360897	0.0127323	0.0090270	0.0066849	0.0047443
E	0.0168463	0.0128919	0.0099876	0.0078790	0.0060661
λ	1.0258720	1.0395800	1.0490710	1.0556250	1.0648150
β	0.7549520	0.8109539	0.8554713	0.8864607	0.9351504
γ	0.7161868	0.7685523	0.8106032	0.8419303	0.8770112
δ	0.98945	1.005	1.017	1.026	1.04

Table 2

n	5	6	7	8	9
D	0.0064428	0.0408036	0.0263534	0.0167648	0.0088672
A	−0.0385480	−0.1832419	−0.1903818	−0.1944091	−0.1804138
B	−0.0102778	−0.0073206	−0.0108513	−0.0137475	0.0144551
C	0.0376321	0.0015358	0.0021891	0.0025222	0.0026250
E	0.0168637	0.0443398	0.0391272	0.0350603	0.0292798
λ	1.0261320	0.7998886	0.7944336	0.7866791	0.7889329
β	0.7416388	0.8636032	0.8444049	0.8295807	0.8313973
γ	0.7112085	1.0933710	1.0264880	0.9904379	0.9679320
δ	0.98928	0.818	0.81	0.8	0.8

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