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A note on the nullity theorem☆

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Abstract

In this paper we take a closer look at the nullity theorem as formulated by Markham and Fiedler in 1986. The theorem is a valuable tool in the computations with structured rank matrices: it connects ranks of subblocks of an invertible matrix A with ranks of other subblocks in his inverse A^{-1}. A little earlier, Barrett and Feinsilver, 1981, proved a theorem very close to the nullity theorem, but restricted to semiseparable and tridiagonal matrices, which are each others inverses. We will adapt the ideas of Barrett and Feinsilver to come to a new, alternative proof of the nullity theorem, based on determinantal formulas.

In the second part of the paper, we extend the nullity theorem to make it suitable for two types of decompositions, namely the *LU* and the *QR*-decomposition. These theorems relate the ranks of subblocks of the factors L, U and Q to the ranks of subblocks of the factored matrix. It is shown, that a combination of the nullity theorem and his extended versions is suitable to predict in an easy manner the structure of decompositions and/or of inverses of structured rank matrices, e.g., higher-order band, higher-order semiseparable, Hessenberg, and many other types of matrices.

As examples, to show the power of the nullity theorem and the related theorems, we apply them to semiseparable and related matrices.

Keywords: Nullity theorem; Structured rank matrices; Inverses; *LU* and *QR*-decompositions of structured rank matrices; $\{p, q\}$-semiseparable matrices

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1. Introduction

The nullity theorem as formulated by Fiedler and Markham [13], is in fact a special case of a theorem proved by Gustafson [17] in 1984. This original theorem was formulated for general principal ideal domains. Markham and Fiedler translated this abstract formulation to matrices over a field. This makes it applicable to real and complex matrices. Based on this nullity theorem Fiedler predicted structures of the inverses of different types of structured rank matrices, including for example tridiagonal and semiseparable matrices (see e.g. [8–10,12,13]). One should be aware that the nullity theorem is a structure predicting theorem, this means that it does not provide inversion formulas, it just predicts the ranks of subblocks in the inverse.

Around the same time people in different types of fields were interested in inverting either tridiagonal and/or semiseparable matrices (see e.g. [14–16,19,20]). Also Barrett and Feinsilver were intrigued by this problem of inverting tridiagonal and semiseparable matrices. Moreover they wanted to omit the restriction of working with irreducible tridiagonal matrices, as all the inversion formulas of that time were highly based on the irreducibility of the tridiagonal matrices. As this problem is not so simple as it might seem, they produced three papers on this topic [2–4], where the last paper covers the most general case. The final version related the structure of the inverse, of a not necessarily irreducible, tridiagonal matrix to the vanishing of certain determinants in the original tridiagonal matrix. The vanishing of determinants corresponds in natural way to ranks of blocks, which is related to their nullity, which is the dimension of the right null-space. Of course one can only evaluate determinants of square matrices, whereas the nullity can also be defined for rectangular matrices. The only missing link between the theorem of Barrett and Feinsilver and the nullity theorem of Fiedler and Markham, is a simple lemma, which we will provide in this paper. This leads to an alternative proof for the nullity theorem.

Moreover, we will provide in this paper a simple extension of the nullity theorem towards the decomposition of structured rank matrices. Firstly, we will prove a relation between the structured rank of the L and U factor of the LU-decomposition of an invertible matrix A and the structured rank of the lower and upper part of the matrix A, respectively. The same is achieved for the QR-decomposition of an arbitrary invertible matrix A. We predict the structure of the orthogonal matrix Q by looking at the structured rank of the matrix A. As currently a lot of attention is being paid to recursively semiseparable matrices, $\mathscr{H}$-matrices, rank k plus diagonal matrices (see e.g. [5,6,18] and the references therein), these theorems can provide a valuable tool for predicting the structure of the LU and QR-decompositions and/or the inverse of these matrices. Moreover, more information connected to the nullity theorem and semiseparable matrices can be found in the recent paper by Strang and Nguyen [21].

As examples, to show the power of the different theorems provided, we apply them to the classes of semiseparable and closely related matrices. Structures of the inverses of banded, $\{p, q\}$-semiseparable, higher-order Hessenberg and other matrices are derived.

The paper is organized as follows. In Section 2 we define the concept of a structured rank matrix, and use this concept to define in an easy manner semiseparable and other related matrices. In the third section we formulate the nullity theorem, incorporate the proof of Markham and Fiedler and provide an alternative way to prove this nullity theorem based on observations made by Barrett and Feinsilver. In the fourth section several examples are included showing the power of the nullity theorem. In Section 5 we provide two generalizations towards the prediction of structured rank blocks in the QR and LU-decomposition of structured rank matrices. We apply the theorems to the class of semiseparable matrices.

2. Structured rank matrices

Semiseparable matrices are structured rank matrices, i.e. all submatrices corresponding to a structure satisfy certain rank properties. Structure and structured ranks are defined as follows. The definitions and results in this section are based on [8,9,11,13].

Definition 1. Let A be an $m \times n$ matrix. Denote with M the set of numbers $\{1, 2, \dots, m\}$ and with N the set of numbers $\{1, 2, \dots, n\}$. Let α and β be nonempty subsets of M and N, respectively. Then, we denote with the matrix $A(\alpha; \beta)$ the submatrix of A with row indices in α and column indices in β. A structure Σ is defined as a nonempty subset of $M \times N$. Based on a structure, the structured rank $\mathrm{r}(\Sigma; A)$ is defined as (where $\alpha \times \beta$ denotes the set $\{(i, j)|i \in \alpha, j \in \beta\}$):

$$\mathrm{r}(\Sigma; A) = \max_{\alpha,\beta} \{\mathrm{rank}(A(\alpha; \beta))|\alpha \times \beta \subseteq \Sigma\}.$$

Before giving the definition of a semiseparable matrix we have to specify the corresponding structure. In the papers [8,9,11,13] more structures are given and investigated.

Definition 2. For $M = \{1, \dots, m\}$ and $N = \{1, \dots, n\}$ we define the following structures:

- The subset

  $$\Sigma_l = \{(i, j)|i \geqslant j, i \in M, j \in N\}$$

  is called the lower triangular structure; in fact the elements of the structure correspond to the indices from the lower triangular part of the matrix.
- The subset

  $$\Sigma_{wl} = \{(i, j)|i > j, i \in M, j \in N\}$$

  is called the weakly lower triangular structure.
- The subset

  $$\Sigma_l^{(p)} = \{(i, j)|i > j - p, i \in M, j \in N\}$$

  is called the p-lower triangular structure and corresponds with all the indices of the matrix A, below the pth diagonal. The 0th diagonal corresponds to the main diagonal, while the pth diagonal refers to the pth superdiagonal (for $p > 0$) and the $-p$th diagonal refers to the pth subdiagonal (for $p > 0$).

Note that $\Sigma_l^{(1)} = \Sigma_l$, $\Sigma_l^{(0)} = \Sigma_{wl}$ and $\Sigma_{wl} \subsetneq \Sigma_l$. Note that the structure $\Sigma_l^{(p)}$ for $p > 1$ contains all the indices from the lower triangular part, but also contains some superdiagonals of the strictly upper triangular part of the matrix. The weakly lower triangular structure is sometimes also called the strictly lower triangular structure or the subdiagonal structure. (We remark that the structures as defined here are slightly different from the ones in [8,9,11,13].) For the upper triangular part of the matrix, the structures Σ_u, Σ_{wu} and $\Sigma_u^{(p)}$ are defined similarly, and are called the upper triangular structure, the weakly upper triangular structure and the p-upper triangular structure, respectively. The structured rank connected to the lower triangular structure, is called the lower triangular rank. Similar definitions are assumed for the other structures.

With the above defined structures we can define semiseparable and closely related matrices.

Definition 3. An $n \times n$ matrix S is called a $\{p, q\}$-semiseparable matrix, with $p \geqslant 0$ and $q \geqslant 0$, if the following two properties are satisfied:

$$\mathrm{r}(\Sigma_l^{(p)}; S) \leqslant p$$
$$\mathrm{r}(\Sigma_u^{(-q)}; S) \leqslant q.$$

This means that the p-lower triangular rank is less than or equal to p and the $(-q)$-upper triangular rank is less than or equal to q.

The above definition says that the maximum rank of all subblocks which one can take out of the matrix below the pth superdiagonal is less than or equal to p and the maximum rank of all subblocks which one can take above the qth subdiagonal is less than or equal to q. When speaking about a $\{p\}$-semiseparable matrix or a semiseparable matrix of semiseparability rank p, we mean a $\{p, p\}$-semiseparable matrix. When briefly speaking about a semiseparable matrix, we refer to a semiseparable matrix of semiseparability rank 1.

It is not necessary for a structured rank matrix to take the structure from the upper as well as from the lower triangular part of the matrix.

Definition 4. A matrix Z is called an upper $\{p\}$-Hessenberg-like matrix if the p-lower triangular rank of Z is less than or equal to p:

$$\mathrm{r}(\Sigma_l^{(p)}; Z) \leqslant p.$$

A lower $\{q\}$-Hessenberg-like matrix is defined in a similar way.

Like in the above case, when speaking about a Hessenberg-like, a $\{1\}$-Hessenberg-like matrix is meant. When it is clear from the context, we omit the notation "upper".

In the next two sections we will derive that the inverse of an invertible tridiagonal matrix is an invertible semiseparable one and vice versa. We will prove even more, namely that the inverse of an invertible $\{p, q\}$-semiseparable matrix is a $\{p, q\}$-band matrix (this is a matrix with p subdiagonals, and q-superdiagonals). Using the following definition, we will prove that the inverse of an invertible $\{p\}$-generalized Hessenberg matrix (see the definition here below) is an invertible $\{p\}$-Hessenberg-like matrix, and vice versa.

Definition 5. A matrix H is defined as a $\{p\}$-generalized Hessenberg matrix if and only if all the elements below the pth subdiagonal are equal to zero.

In order to prove all the above-mentioned properties of semiseparable and Hessenberg-like matrices, in Section 3 we will first prove a very powerful theorem, namely the nullity theorem. Based on this theorem, we will deduce properties of different classes of structured rank matrices in Section 4.

3. The nullity theorem

In this section we will prove the nullity theorem in two different ways. Although this theorem is not so widely spread, it can easily be used to derive several interesting results about structured rank matrices and their inverses. It was formulated for the first time by Gustafson [17] for matrices over principal ideal domains. In [11], Fiedler and Markham translated this abstract formulation to matrices over a field. Barrett and Feinsilver formulated theorems close to the nullity theorem in [2,4]. Based on their observations we will provide an alternative proof of this theorem. The lemma will be followed by some small corollaries. In the following section we will apply these corollaries, to classes of structured rank matrices predicting thereby the structured rank of their inverses.

Definition 6. Suppose a matrix $A \in \mathbb{R}^{m\times n}$ is given. The nullity $\mathrm{n}(A)$ is defined as the dimension of the right null space of A.

Theorem 7 (*The nullity theorem*). *Suppose we have the following invertible matrix $A \in \mathbb{R}^{n\times n}$ partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

with A_{11} of size $p \times q$. The inverse B of A is partitioned as

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

with B_{11} of size $q \times p$. Then the nullities $\mathrm{n}(A_{11})$ and $\mathrm{n}(B_{22})$ are equal.

Proof (*Fiedler and Markham [11]*). Suppose $\mathrm{n}(A_{11}) \leqslant \mathrm{n}(B_{22})$. If this is not true, we can prove the theorem for the matrices

$$\begin{pmatrix} A_{22} & A_{21} \\ A_{12} & A_{11} \end{pmatrix} \quad \begin{pmatrix} B_{22} & B_{21} \\ B_{12} & B_{11} \end{pmatrix}$$

which are also each others inverse. Suppose $\mathrm{n}(B_{22}) > 0$ otherwise $\mathrm{n}(A_{11}) = 0$ and the theorem is proved.

When $\mathrm{n}(B_{22}) = c > 0$, then there exists a matrix F with c linearly independent columns, such that $B_{22}F = 0$. Hence, multiplying the following equation to the right by F:

$$A_{11}B_{12} + A_{12}B_{22} = 0,$$

we get

$$A_{11}B_{12}F = 0. \tag{1}$$

Applying the same operation to the relation

$$A_{21}B_{12} + A_{22}B_{22} = I,$$

it follows that $A_{21}B_{12}F = F$, and therefore $\mathrm{rank}(B_{12}F) \geqslant c$. Using this last statement together with Eq. (1), we derive

$$\mathrm{n}(A_{11}) \geqslant \mathrm{rank}(B_{12}F) \geqslant c = \mathrm{n}(B_{22}).$$

Together with our assumption $\mathrm{n}(A_{11}) \leqslant \mathrm{n}(B_{22})$, this proves the theorem. □

This provides us the first proof of the theorem. The alternative proof is based on some lemmas, and makes use of determinantal formulas. Let us denote with $|\alpha|$ the cardinality of the corresponding set α.

Lemma 8 (*Gantmacher and Krein [15, p. 13]*). *Suppose A is an $n \times n$ invertible matrix and α and β two nonempty sets of indices in $N = \{1, 2, \ldots, n\}$, such that $|\alpha| = |\beta| < n$. Then, the determinant of any square submatrix of the inverse matrix $B = A^{-1}$ satisfies the following equation*:

$$|\det B(\alpha; \beta)| = \frac{1}{|\det(A)|} |\det A(N\backslash\beta; N\backslash\alpha)|.$$

With $N\backslash\beta$ the difference between the sets N and β is meant (N *minus* β).

The theorem can be seen as an extension of the standard formula for calculating the inverse of a matrix, for which each element is determined by a minor in the original matrix. This lemma already implies the nullity theorem for square subblocks and for nullities equal to 1, since this case is equivalent with the vanishing of a determinant. The following lemma shows that we can extend this argument also to the general case, i.e. every rank condition can be expressed in terms of the vanishing of certain determinants.

Lemma 9. *Suppose $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix and $n \geqslant |\alpha| \geqslant |\beta|$. The following three statements are equivalent*:[1]

(1) $\mathrm{n}(A(\alpha; \beta)) \geqslant d$.
(2) $\det A(\alpha'; \beta') = 0$ *for all* $\alpha' \subseteq \alpha$ *and* $\beta' \subseteq \beta$ *and* $|\alpha'| = |\beta'| = |\beta| - d + 1$.
(3) $\det A(\alpha'; \beta') = 0$ *for all* $\alpha \subseteq \alpha'$ *and* $\beta \subseteq \beta'$ *and* $|\alpha'| = |\beta'| = |\alpha| + d - 1$.

Proof. The arrows (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) are straightforward. The arrow (3) $\Rightarrow$ (1) makes use of the nonsingularity of the matrix A. Suppose the nullity of $A(\alpha; \beta)$ to be less than d. This would mean that there exist $|\beta| - d + 1$ linearly independent columns in the block $A(\alpha; \beta)$. Therefore $A(\alpha; N)$ has rank less then $|\alpha|$, implying the singularity of the matrix A. □

An alternative proof of the nullity theorem can be derived easily combining the previous two lemmas. In [21], Strang proves a related result and comments on different ways to prove the nullity theorem.

The following corollary is a straightforward consequence of the nullity theorem.

Corollary 10 (*Fiedler and Markham [11, Corollary 3]*). *Suppose $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix, and α, β to be nonempty subsets of N with $|\alpha| < n$ and $|\beta| < n$. Then*

$$\mathrm{rank}(A^{-1}(\alpha; \beta)) = \mathrm{rank}(A(N\backslash\beta; N\backslash\alpha)) + |\alpha| + |\beta| - n.$$

Proof. By permuting the rows and columns of the matrix A, we can always move the submatrix $A(N\backslash\beta; N\backslash\alpha)$ into the upper left part A_{11}. Correspondingly, the submatrix $B(\alpha; \beta)$ of the matrix $B = A^{-1}$ moves into the lower right part B_{22}. We have

$$\mathrm{n}(A_{11}) = n - |\alpha| - \mathrm{rank}(A_{11})$$
$$\mathrm{n}(B_{22}) = |\beta| - \mathrm{rank}(B_{22})$$

and because $\mathrm{n}(A_{11}) = \mathrm{n}(B_{22})$, this proves the corollary. □

When choosing $\alpha = N\backslash\beta$, we get

[1] The authors thank Steven Delvaux for formulating and proving this lemma.

Corollary 11. *For a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ and $\alpha \subseteq N$, we have*

$$\operatorname{rank}(A^{-1}(\alpha; N\backslash\alpha)) = \operatorname{rank}(A(\alpha; N\backslash\alpha)).$$

In the next section we will use the previously obtained results about the ranks of complementary blocks of a matrix and its inverse to prove the rank properties of the inverse for some classes of structured rank matrices.

4. Applications of the nullity theorem

Here we will briefly formulate some results by applying the previously mentioned nullity theorem. Proofs are not included as they can be reconstructed rather easily by taking the correct subblocks and afterwards applying the nullity theorem (see also [1,2,4,7–9,11–13,21]).

Using Corollaries 10 and 11, one can easily prove the well-known result that the inverse of a lower triangular matrix is again a lower triangular matrix. Moreover, one can also easily prove that the weakly lower triangular rank is maintained.

Considering the structure

$$\Sigma_\sigma = (N \times N)\backslash\{(1, 1), (2, 2), \ldots, (n, n)\},$$

which is called the off-diagonal structure, one can easily prove the following theorem.

Theorem 12 (*Fiedler and Markham [12, Theorem 2.2]*). *Assume A is a nonsingular matrix. Then the off-diagonal rank of A equals the off-diagonal rank of A^{-1}.*

$$\mathrm{r}(\Sigma_\sigma; A) = \mathrm{r}(\Sigma_\sigma; A^{-1}).$$

Using Theorem 12, we can see that the inverse of an invertible rank k matrix plus a diagonal is again a rank k matrix plus a diagonal.

A partition of a set N is a decomposition of $N = N_1 \dot{\cup} N_2 \dot{\cup} \cdots \dot{\cup} N_p$, where $\dot{\cup}$ denotes the disjunct union: this means that $N_i \cap N_j$ is the empty set, $\forall i, j$ with $i \neq j$. A generalization of Theorem 12, from diagonal to block diagonal is as follows

Theorem 13 (*Fiedler [8, Theorem 2]*). *Let $N = N_1 \dot{\cup} N_2 \dot{\cup} \cdots \dot{\cup} N_p$ a partition of N with $N = \{1, 2, \ldots, n\}$. Let*

$$\Sigma_{\sigma b} = (N \times N) \Big\backslash \bigcup_{i=1}^{p} (N_i \times N_i).$$

Then, for every nonsingular $n \times n$ matrix A we have

$$\mathrm{r}(\Sigma_{\sigma b}; A^{-1}) = \mathrm{r}(\Sigma_{\sigma b}; A).$$

This means that the inverse of a rank k matrix plus a block diagonal matrix is again a rank k matrix plus a block diagonal matrix for which the sizes of the blocks of the first and the latter diagonal are the same. More general extensions to blocks and different structures put on matrices can be found in the above references. Let us now formulate briefly an interesting, known, result, which is in fact a straightforward consequence of the nullity theorem. We remark once more that this theorem covers the general case of invertible band and tridiagonal matrices, whereas a lot of papers put the restriction of irreducibility on the matrices.

Theorem 14. *The inverse of an invertible*

- *tridiagonal matrix is a semiseparable matrix*;
- $\{p, q\}$*-band matrix is a* $\{p, q\}$*-semiseparable matrix*;
- $\{p\}$*-generalized Hessenberg matrix is a* $\{p\}$*-Hessenberg-like matrix.*

We already deduced very useful properties from the nullity theorem. Nevertheless, we can adapt the theorem a little bit and obtain immediate information about decompositions of structured rank matrices.

5. Extensions to the *QR* and *LU* decompositions

Based on the proof of the nullity theorem by Fiedler, it is very easy to generalize the nullity theorem and apply it to decompositions of structured rank matrices.

Theorem 15. *Suppose we have an invertible matrix A, with an LU factorization of the following form*:

$$A = LU.$$

Suppose A to be partitioned in the following form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

with A_{11} *of dimension* $p \times q$. *The inverse B of U is partitioned as*

$$U^{-1} = B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix},$$

with B_{11} *of dimension* $q \times p$. *Then the nullities* $\mathrm{n}(A_{12})$ *and* $\mathrm{n}(B_{12})$ *are equal.*

It is enough to have the structure in terms of ranks of the matrix U^{-1}, because using the nullity theorem one can easily deduce the structured rank of the matrix U. The proof is very similar to the one of the nullity theorem.

Proof. First, we will prove that $\mathrm{n}(A_{12}) \geqslant \mathrm{n}(B_{12})$, by using the relation $AU^{-1} = L$. Suppose that the nullity of B_{12} equals c. Then there exists a matrix F with c linearly independent columns such that $B_{12}F = 0$. Partitioning L in the following way

$$L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix},$$

with L_{11} of dimension $p \times p$, we can write down the following equations:

$$A_{11}B_{12} + A_{12}B_{22} = 0,$$
$$A_{12}B_{22}F = 0,$$

and

$$A_{21}B_{12} + A_{22}B_{22} = L_{22},$$
$$A_{22}B_{22}F = L_{22}F.$$

Therefore $\operatorname{rank}(B_{22}F) \geqslant c$, because L_{22} is of full rank. This leads us to the result:

$$\mathrm{n}(A_{12}) \geqslant \operatorname{rank}(B_{22}F) \geqslant c = \mathrm{n}(B_{12}).$$

This proves already one direction of the proof. For the other direction, $\mathrm{n}(A_{12}) \leqslant \mathrm{n}(B_{12})$, we use a partitioning for the inverse of A and the matrix U, such that the upper left block of $C = A^{-1}$, denoted as C_{11} has size $q \times p$ and the upper left block of U denoted as U_{11} has size $p \times q$. Using the equation $UA^{-1} = L^{-1}$ we can prove in a similar way as above that

$$\mathrm{n}(U_{12}) \geqslant \mathrm{n}(C_{12}).$$

Using the nullity theorem gives us

$$\mathrm{n}(B_{12}) = \mathrm{n}(U_{12}) \geqslant \mathrm{n}(C_{12}) = \mathrm{n}(A_{12}).$$

This proves the theorem. □

An analogous theorem can be formulated for the lower triangular matrix L. This theorem is very useful because the structured rank of both of the factors U and L can be determined now in terms of the structured rank of the original matrix A. This can be generalized in a straightforward manner to the Cholesky decomposition, which is a special case of the LU-decomposition, for positive definite structured rank matrices. We will give as an example here the LU factorization of semiseparable matrices. We derive the structure of L and U in two different ways: based on the generalized nullity Theorem 15 and based on the structure of the LU factorization of the inverse.

Example 16. We will prove here that the inverse U^{-1} of the matrix U in the LU-decomposition of an invertible semiseparable matrix S is an upper bidiagonal matrix. Hence, the factor U is an upper triangular semiseparable matrix. (One can deduce similar properties for the lower triangular matrix L.) Suppose our semiseparable matrix S is of size $n \times n$. Use the set of indices:

$$\alpha = \{1, \ldots, k\} \quad \text{and} \quad N \backslash \alpha = \{k+1, \ldots, n\},$$
$$\beta = \{k, \ldots, n\} \quad \text{and} \quad N \backslash \beta = \{1, \ldots, k-1\}.$$

We have

$$\mathrm{n}(S(\alpha; \beta)) = n - k + 1 - \operatorname{rank}(S(\alpha; \beta)),$$
$$\mathrm{n}(U^{-1}(N\backslash\beta; N\backslash\alpha)) = n - k - \operatorname{rank}(U^{-1}(N\backslash\beta; N\backslash\alpha)).$$

Rewriting these equations and using Theorem 15, we get:

$$\operatorname{rank}(S(\alpha;\beta)) = \operatorname{rank}(U^{-1}(N\backslash\beta; N\backslash\alpha)) + 1.$$

This means that for a semiseparable matrix of semiseparability rank 1 all elements above the super diagonal in the matrix U^{-1} have to be zero. Therefore the inverse U will be an upper triangular semiseparable matrix.

An alternative approach is to look at the inverse of the matrix S, namely T, which is tridiagonal. Let $T = U_T L_T$ be a UL decomposition of the tridiagonal matrix T, where L_T is a lower bidiagonal matrix, and U_T is an upper bidiagonal matrix. This means that the LU decomposition of S has the following form:

$$\begin{aligned} S &= T^{-1} \\ &= L_T^{-1} U_T^{-1} \\ &= LU, \end{aligned}$$

for which L is lower triangular semiseparable and U is upper triangular semiseparable.

Note that for the more general LU factorization

$$PA = LU,$$

with P a nontrivial permutation matrix the two factors L and U are not necessarily semiseparable. Take for example the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then, $PA = LU$ with

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, it is clear that L is not semiseparable.

Similar theorems as Theorem 15 can be deduced for other types of decompositions. We prove a similar theorem for the QR-decomposition, which will give us information about the structured rank of the factor Q.

Theorem 17. *Suppose we have an invertible matrix A, with a QR-factorization $A = QR$. Suppose A to be partitioned in the following form*:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

with A_{11} of dimension $p \times q$. The inverse of Q, the matrix B, is partitioned as

$$Q^{-1} = B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

with B_{11} of dimension $q \times p$. Then the nullities $\mathrm{n}(A_{21})$ and $\mathrm{n}(B_{21})$ are equal.

Proof. Similar to the one from Theorem 15. □

Using this theorem one can deduce the structure of the lower triangular part of the orthogonal matrix Q. Even more, one can deduce the structure of the Q factor of decompositions of for example rank k matrices plus a diagonal or of $\{p, q\}$-semiseparable matrices plus a diagonal. As an example we will investigate as in the previous example the QR-decomposition of semiseparable matrices. A more elaborate study of the QR-factorization of semiseparable matrices can be found in for example [22].

Theorem 18. *Suppose S is an invertible semiseparable matrix. Suppose $S = QR$ is a QR-decomposition of the semiseparable matrix S. Then we have that R has upper triangular rank* 2. *Moreover, Q is a lower Hessenberg matrix for which the lower triangular rank is* 1.

Proof. The structured rank properties of the orthogonal matrix Q can easily be derived from Theorem 17. However, one can also work via the inverse $T = S^{-1}$. Denote with $T = R_T Q_T$ an RQ decomposition of the matrix T. R_T denotes an upper triangular matrix and Q_T an orthogonal matrix. Because T is a tridiagonal matrix, we know that R_T is an upper triangular matrix with only the diagonal and the next two superdiagonals different from zero. Moreover, the orthogonal matrix Q_T is an upper Hessenberg matrix, for which the upper triangular rank is 1.

Translating the above equations towards S we get:

$$\begin{aligned} S &= T^{-1} \\ &= Q_T^{-1} R_T^{-1} \\ &= QR. \end{aligned}$$

Because of Theorem 14 we know that R has upper triangular rank 2. The matrix Q is a lower Hessenberg matrix for which the lower triangular rank is 1. □

6. Conclusions and future work

In this paper we provided an alternative proof of the nullity theorem, based on the observations by Barrett and Feinsilver. Moreover, we expanded the nullity theorem in its current form to be suitable for predicting structures in the QR and LU decomposition of structured rank matrices. Examples, connected to semiseparable and closely related matrices were included proving the power of these theorems, in predicting rank structures in matrices.

A clear and rather strong restriction to the decomposition theorems is the demand of invertibility. In our current research we try to omit this restriction, such that we can formulate these theorems also for noninvertible matrices, thereby predicting the structures of the factors.

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