



A Petrov–Galerkin method for solving the generalized equal width (GEW) equation

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ABSTRACT

The generalized equal width (GEW) equation is solved numerically by the Petrov–Galerkin method using a linear hat function as the test function and a quadratic *B*-spline function as the trial function. Product approximation has been used in this method. A linear stability analysis of the scheme shows it to be conditionally stable. Test problems including the single soliton and the interaction of solitons are used to validate the suggested method, which is found to be accurate and efficient. Finally, the Maxwellian initial condition pulse is studied.

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1. Introduction

The equal width (EW) equation of the form

$$u_t + \varepsilon uu_x - \delta u_{xxt} = 0 \quad (1)$$

where ε and δ are positive constants and the subscripts x and t denote space and time derivatives respectively is a model nonlinear partial differential equation used for the simulation of one-dimensional nonlinear waves propagating in dispersive media. Peregrine [1] was the first to derive the regularized long wave (RLW) equation for modeling the development of an undular bore. Later on, Benjamin et al. [2] proposed the use of the RLW equation as preferred alternative over the more classical Korteweg–de Vries (KdV) equation for modeling a large class of physical phenomena. The EW equation which is less well known and was proposed in [3] is a description alternative to the more usual KdV and RLW equations. The solutions of this equation are kinds of solitary waves known as solitons whose shapes are not affected by collision. The EW equation was solved numerically by various methods such as the Galerkin [4–6], least squares [7], and collocation methods [8,9] etc. Indeed, the EW equation is a special case of the generalized equal width (GEW) equation of the form

$$u_t + \varepsilon u^p u_x - \delta u_{xxt} = 0 \quad (2)$$

where p is a positive integer. It is the model equation governing the wave phenomena in which the long wave incorporates the competing effects of nonlinearity and dispersion. In fluid problems, u is related to the vertical displacement of the water surface, whereas in the plasma applications u is the negative of the electrostatic potential [10].

The GEW equation was studied in [10,11] by applying the collocation method with the use of quadratic and cubic *B*-spline functions. Comparatively little work has been done for this case. In this work, the Petrov–Galerkin method is developed for

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the GEW equation using a linear hat function as the test function and a quadratic B -spline as the trial function. Here the proposed method is shown to represent accurately the migration of single solitary waves for the cases $p = 2, 3$ and 4. The interaction of solitary waves and other properties are also studied.

2. The Petrov–Galerkin method (PGM)

For convenience, the GEW equation (2) is rewritten in the form

$$u_t + \frac{\varepsilon}{p+1} (u^{p+1})_x - \delta u_{xxt} = 0. \quad (3)$$

Periodic boundary conditions on the region $a \leq x \leq b$ are assumed in the form

$$u(a, t) = u(b, t) = 0. \quad (4)$$

The space interval $a \leq x \leq b$ is discretized with $(N+1)$ uniform grid points $x_j = a + jh$, where $j = 0, 1, 2, \dots, N$, and the grid spacing is given by $h = (b-a)/N$. Let $U_j(t)$ denote the approximation to the exact solution $u(x_j, t)$. Following the method used in [12] for solving the KdV equation, by the Petrov–Galerkin method we modify the method in order to solve Eq. (3). Using the Petrov–Galerkin method, we assume that the approximate solution of Eq. (3) is

$$u_h(x, t) = \sum_{j=0}^N U_j(t) \phi_j(x). \quad (5)$$

The product approximation technique [13] is used for treating the nonlinear term in the following manner:

$$u_h^{p+1}(x, t) = \sum_{j=0}^N U_j^{p+1}(t) \phi_j(x) \quad (6)$$

where $\phi_j(x)$, $j = 0, 1, 2, \dots, N$, are the usual piecewise linear “hat” functions given by

$$\phi_j(x) = \begin{cases} 1 + (x - jh)/h, & x \in [x_{j-1}, x_j] \\ 1 - (x - jh)/h, & x \in [x_j, x_{j+1}] \\ 0, & \text{otherwise.} \end{cases}$$

The unknown functions $U_j(t)$, $j = 0, 1, 2, \dots, N$, are determined from the variational formulation

$$\left((u_h)_t, \psi_j \right) + \frac{\varepsilon}{p+1} \left((u_h^{p+1})_x, \psi_j \right) - \delta \left((u_h)_{xxt}, \psi_j \right) = 0 \quad (7)$$

where ψ_j , $j = 0, 1, 2, \dots, N$, are test functions, which are taken to be quadratic B -splines given by

$$\psi_j(x) = \frac{1}{h^2} \begin{cases} (x - x_{j-1})^2, & x_{j-1} \leq x \leq x_j \\ 2h^2 - (x_{j+1} - x)^2 - (x - x_j)^2, & x_j \leq x \leq x_{j+1} \\ (x_{j+2} - x)^2, & x_{j+1} \leq x \leq x_{j+2} \\ 0, & \text{otherwise} \end{cases}$$

and $(\cdot, \cdot)$ denotes the usual inner product:

$$(f, g) = \int_a^b f(x)g(x)dx.$$

Integrating by parts and using the fact that $\psi(a) = \psi(b) = 0$, Eq. (7) leads to the formulation

$$\left((u_h)_t, \psi_j \right) + \frac{\varepsilon}{p+1} \left((u_h^{p+1})_x, \psi_j \right) + \delta \left((u_h)_{xt}, (\psi_j)_x \right) = 0. \quad (8)$$

Performing the integrations on (8) will give the following system of ordinary differential equations (ODEs):

$$\alpha \dot{U}_{j-1} + \beta \dot{U}_j + \beta \dot{U}_{j+1} + \alpha \dot{U}_{j+2} = \frac{2\varepsilon}{9(p+1)h} \left[(U_{j-1})^{p+1} + 3(U_j)^{p+1} - 3(U_{j+1})^{p+1} - (U_{j+2})^{p+1} \right] \quad (9)$$

where $\alpha = \frac{1}{18} \left(1 - \frac{12\delta}{h^2} \right)$, $\beta = \frac{1}{18} \left(11 + \frac{12\delta}{h^2} \right)$ and ‘ $\cdot$ ’ means the derivative with respect to time t .

Now to solve the ODEs, we assume U_j^n to be a fully discrete approximation to the exact solution $u(x_j, t_n)$, where $t_n = n\Delta t$ and Δt is the time step size. Using the central difference scheme for the time derivative, $\dot{U} = \frac{U^{n+1} - U^{n-1}}{2\Delta t}$, Eq. (9) is reduced to the system of equations

$$\begin{aligned} \alpha U_{j-1}^{n+1} + \beta U_j^{n+1} + \beta U_{j+1}^{n+1} + \alpha U_{j+2}^{n+1} &= \alpha U_{j-1}^{n-1} + \beta U_j^{n-1} + \beta U_{j+1}^{n-1} + \alpha U_{j+2}^{n-1} \\ &+ \frac{4\Delta t \varepsilon}{9(p+1)h} \left[(U_{j-1}^n)^{p+1} + 3(U_j^n)^{p+1} - 3(U_{j+1}^n)^{p+1} - (U_{j+2}^n)^{p+1} \right] \end{aligned} \quad (10)$$

where $j = 1, 2, 3, \dots, N-2$ and $n = 1, 2, 3, \dots$

The system (10) required two initial time levels and in our practical calculations; the exact values at time equals zero and time equals Δt are used to obtain the required initial conditions. However, in general the exact solution at time Δt is not known. In such cases, the initial solution at time zero is assumed to be the solution at time Δt .

3. Stability analysis

To investigate the stability of the scheme Eq. (10) we apply the Von Neumann stability analysis. Assuming u in the nonlinear term $u^p u_x$ of the GEW equation as locally constant, $\tilde{u}$, the linearized scheme is

$$\alpha U_{j-1}^{n+1} + \beta U_j^{n+1} + \beta U_{j+1}^{n+1} + \alpha U_{j+2}^{n+1} = \alpha U_{j-1}^{n-1} + \beta U_j^{n-1} + \beta U_{j+1}^{n-1} + \alpha U_{j+2}^{n-1} + \frac{4\varepsilon \Delta t \tilde{u}^p}{9h} [U_{j-1}^n + 3U_j^n - 3U_{j+1}^n - U_{j+2}^n]. \quad (11)$$

For convenience, each element $[x_j, x_{j+1}]$ is transformed into the element $[x_{j-1/2}, x_{j+1/2}]$; then our linearized scheme Eq. (11) takes the form

$$\alpha U_{j-3/2}^{n+1} + \beta U_{j-1/2}^{n+1} + \beta U_{j+1/2}^{n+1} + \alpha U_{j+3/2}^{n+1} = \alpha U_{j-3/2}^{n-1} + \beta U_{j-1/2}^{n-1} + \beta U_{j+1/2}^{n-1} + \alpha U_{j+3/2}^{n-1} + \frac{4\varepsilon \Delta t \tilde{u}^p}{9h} [U_{j-3/2}^n + 3U_{j-1/2}^n - 3U_{j+1/2}^n - U_{j+3/2}^n]. \quad (12)$$

Substituting the Fourier mode

$$U_j^n = \xi^n e^{ij\theta}, \quad \theta = \frac{hk}{2} \text{ and } i = \sqrt{-1} \quad (13)$$

where k is the mode number and h the space step size into the system of Eq. (12), we obtain

$$\xi^{n+1} = g \xi^n \quad (14)$$

where the growth factor, g , is determined by

$$g^2 + 2i \left(\frac{2\varepsilon \Delta t \tilde{u}^p}{9h} \frac{\sin 3\theta + \sin \theta}{\alpha \cos 3\theta + \beta \cos \theta} \right) g - 1 = 0 \quad (15)$$

where $\alpha = \frac{1}{18} \left(1 - \frac{12\delta}{h^2} \right)$ and $\beta = \frac{1}{18} \left(11 + \frac{12\delta}{h^2} \right)$. Let the roots of Eq. (15) be g_1 and g_2 so that $g_1 g_2 = -1$, and

$$g_1 + g_2 = -\frac{4\varepsilon \Delta t \tilde{u}^p i}{9h} \frac{\sin 3\theta + \sin \theta}{\alpha \cos 3\theta + \beta \cos \theta}.$$

If we prove that $\left| \frac{2\varepsilon \Delta t \tilde{u}^p}{9h} \frac{\sin 3\theta + \sin \theta}{\alpha \cos 3\theta + \beta \cos \theta} \right| \leq 1$, then Eq. (15) can be written as $g^2 + 2 \sin \eta g i - 1 = 0$, from which the roots must be of the form $g_1 = e^{\eta i}$, $g_2 = -e^{-\eta i}$; therefore $|g_1| = |g_2| = 1$ and our scheme is conditionally stable. Now we prove the following lemma.

Lemma. For h , Δt , $\tilde{u}$, and θ as defined above, then

$$\left| \frac{2\varepsilon \Delta t \tilde{u}^p}{9h} \frac{\sin 3\theta + \sin \theta}{\alpha \cos 3\theta + \beta \cos \theta} \right| \leq 1, \quad \text{for all } \theta$$

where $\alpha = \frac{1}{18} \left(1 - \frac{12\delta}{h^2} \right)$ and $\beta = \frac{1}{18} \left(11 + \frac{12\delta}{h^2} \right)$.

Proof. $\left| \frac{2\varepsilon \Delta t \tilde{u}^p}{9h} \frac{\sin 3\theta + \sin \theta}{\alpha \cos 3\theta + \beta \cos \theta} \right| = \frac{4\varepsilon \Delta t \tilde{u}^p}{h} \left| \frac{\sin \theta \cos \theta}{\left(1 - \frac{12\delta}{h^2}\right) \cos^2 \theta + 2\left(1 + \frac{6\delta}{h^2}\right)} \right|.$

Now, suppose $x = 1 - \cos \theta$; then

$$\sup_{\forall \theta} \left[\frac{4\varepsilon \Delta t \tilde{u}^p}{h} \frac{\sin \theta \cos \theta}{\left(1 - \frac{12\delta}{h^2}\right) \cos^2 \theta + 2\left(1 + \frac{6\delta}{h^2}\right)} \right]^2 = \left(\frac{4\varepsilon \Delta t \tilde{u}^p}{h} \right)^2 \sup_{0 \leq x \leq 2} \frac{(2x - x^2)(1 - x)^2}{\left[\left(1 - \frac{12\delta}{h^2}\right)(1 - x)^2 + 2\left(1 + \frac{6\delta}{h^2}\right)\right]^2}.$$

Let $f(x) = \frac{(2x - x^2)(1 - x)^2}{\left[\left(1 - \frac{12\delta}{h^2}\right)(1 - x)^2 + 2\left(1 + \frac{6\delta}{h^2}\right)\right]^2}$ and suppose $\left(1 - \frac{12\delta}{h^2}\right)(1 - x)^2 + 2\left(1 + \frac{6\delta}{h^2}\right) = 0$; then $(1 - x)^2 = \frac{2\left(1 + \frac{6\delta}{h^2}\right)}{\frac{12\delta}{h^2} - 1}$ and so $(1 - x)^2 \leq 1 \Rightarrow 3 \leq 0$, so we have a contradiction.

Table 1Invariants and error norms for the single solitary wave: $p = 2$, amplitude = 0.25, $\Delta t = 0.2$, $h = 0.1$, $\varepsilon = 3$, $\delta = 1$, $0 \leq x \leq 80$.

t	I_1	I_2	I_3	$L_2 \times 10^5$	$L_\infty \times 10^5$
0	0.785398	0.166662	0.00520829	0.0	0.0
5	0.785398	0.166664	0.00520827	0.0627308	0.0647797
10	0.785398	0.166665	0.00520827	0.128612	0.136823
15	0.785398	0.166667	0.00520829	0.190869	0.20854
20	0.785398	0.166669	0.00520829	0.250172	0.275164

Table 2Invariants and error norms for the single solitary wave: $p = 2$, amplitude = 1, $\Delta t = 0.2$, $h = 0.1$, $\varepsilon = 3$, $\delta = 1$, $0 \leq x \leq 80$.

t	I_1	I_2	I_3	L_2	L_∞
0	3.14159	2.66659	1.33332	0.0	0.0
5	3.14159	2.66666	1.33338	0.0057735	0.0047474
10	3.14159	2.66671	1.33340	0.0084179	0.0060423
15	3.14159	2.66673	1.33341	0.0103745	0.0073247
20	3.14159	2.66673	1.33341	0.0123326	0.0086082

Hence, $\left(1 - \frac{12\delta}{h^2}\right)(1-x)^2 + 2\left(1 + \frac{6\delta}{h^2}\right) \neq 0$. Therefore $f(x)$ is continuous on $[0, 2]$. Now differentiating f with respect to x , to find the maximum, $f'(x) = 0$ then gives us

$$x_1 = 1, \quad x_2 = \frac{-5h^2 + 12\delta + \sqrt{2}\sqrt{5h^4 + 42h^2\delta + 72\delta^2}}{5h^2 + 12\delta} \quad \text{and} \quad x_3 = \frac{-5h^2 + 12\delta - \sqrt{2}\sqrt{5h^4 + 42h^2\delta + 72\delta^2}}{5h^2 + 12\delta}.$$

Now

$$f''(x_1) = \frac{1}{2\left(1 + \frac{2\delta}{h^2}\right)} > 0 \quad \text{and} \quad f''(x_2) = f''(x_3) = -\frac{(5h^2 + 12\delta)^3}{216(h^3 + 6h\delta)^2} < 0.$$

$$\therefore f_{\max} = f(x_2) = f(x_3) = \frac{h^2}{24(h^2 + 6\delta)}.$$

Thus,

$$\sup_{\forall \theta} \left[\frac{4\varepsilon \Delta t \tilde{u}^p}{h} \frac{\sin \theta \cos \theta}{\left(1 - \frac{12\delta}{h^2}\right) \cos^2 \theta + 2\left(1 + \frac{6\delta}{h^2}\right)} \right]^2 = \left[\frac{4\varepsilon \Delta t \tilde{u}^p}{h} \right]^2 \times \frac{h^2}{24(h^2 + 6\delta)} = \frac{2\varepsilon^2 \Delta t^2 \tilde{u}^{2p}}{3(h^2 + 6\delta)} = R \text{ (say).}$$

The space step h is usually small and the time step Δt is also small. Also $\tilde{u}$ represents a single speed and is usually small. So, for any problem of practical significance, $R \leq 1$. Hence the lemma. $\square$

4. Numerical applications

The analytical solution of the GEW equation (2) can be obtained by using the transformation $u(x, t) = f(x - ct)$, where c represents the constant velocity of the wave traveling in the positive direction of the x -axis and the analytic solution can be expressed in the form [10,11]

$$u(x, t) = \sqrt[p]{\frac{c(p+1)(p+2)}{2\varepsilon} \sec h^2 \left[\frac{p}{2\sqrt{\delta}}(x - ct - x_0) \right]} \quad (16)$$

where x_0 is an arbitrary constant. Also the GEW equation (2) has the following three invariants of motion given in [10,11]:

$$I_1 = \int_a^b u dx, \quad I_2 = \int_a^b (u^2 + \delta u_x^2) dx, \quad I_3 = \int_a^b u^{p+2} dx \quad (17)$$

corresponding to conservation of mass, momentum and energy. These integrals are evaluated to measure the conservation properties of our scheme. The L_2 and L_∞ error norms are used to measure the accuracy of the present scheme and to compare our result with both exact values, Eq. (16), as well as other results in the literature whenever available.

Table 3Invariants and error norms for the single solitary wave: $p = 3$, amplitude = 0.15, $\Delta t = 0.2$, $h = 0.1$, $\varepsilon = 3$, $\delta = 1$, $0 \leq x \leq 80$.

t	I_1	I_2	$I_3 \times 10^4$	$L_2 \times 10^7$	$L_\infty \times 10^7$
0	0.418916	0.0549782	0.733047	0.0	0.0
5	0.418916	0.0549782	0.733047	0.153824	0.193359
10	0.418916	0.0549782	0.733047	0.320447	0.405343
15	0.418916	0.0549783	0.733047	0.474243	0.604016
20	0.418916	0.0549783	0.733047	0.640778	0.820602

Table 4Invariants and error norms for the single solitary wave: $p = 3$, amplitude = 1, $\Delta t = 0.2$, $h = 0.1$, $\varepsilon = 3$, $\delta = 1$, $0 \leq x \leq 80$.

t	I_1	I_2	I_3	L_2	L_∞
0	2.80436	2.4638	0.985537	0.0	0.0
5	2.80436	2.46386	0.985532	0.0020047	0.00214292
10	2.80437	2.46395	0.985564	0.00340545	0.00274122
15	2.80436	2.46395	0.985547	0.00416244	0.00322676
20	2.80436	2.46389	0.98556	0.00484271	0.00370926

Table 5Invariants and error norms for the single solitary wave: $p = 4$, amplitude = 1, $\Delta t = 0.2$, $h = 0.1$, $\varepsilon = 3$, $\delta = 1$, $0 \leq x \leq 80$.

t	I_1	I_2	I_3	L_2	L_∞
0	2.62206	2.35604	0.785343	0.0	0.0
5	2.62206	2.35617	0.785361	0.00099742	0.00115339
10	2.62206	2.35613	0.785347	0.00162186	0.00146642
15	2.62206	2.35613	0.785346	0.00199995	0.00167297
20	2.62206	2.35615	0.785344	0.00230499	0.00188285

Table 6Comparisons of results for the single solitary wave of the GEW equation with $h = 0.1$, $\Delta t = 0.2$, $\varepsilon = 3$, $\delta = 1$, $0 \leq x \leq 80$.

Ref.	Method	p -value	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
Ours	Petrov–Galerkin	2	0.785398	0.166662	0.0052083	0.0025017	0.00275164
[10]	Collocation (quadratic)		0.7852864	0.1665818	0.00520602	0.1569539	0.2021476
[11]	Collocation (cubic)		0.7846676	0.1664340	0.0051938	0.19588780	0.17443300
Ours	Petrov–Galerkin	3	0.418916	0.0549783	0.0000733047	0.000064078	0.00008206
[11]	Collocation (cubic)		0.6590833	0.5938137	0.000068718	0.5149677	0.32060590

Table 7Invariants for interaction of two solitary waves with $p = 2$, $c_1 = 0.5$, $c_2 = 0.125$, $x_1 = 15$, $x_2 = 30$, $h = 0.1$, $\Delta t = 0.025$, $0 \leq x \leq 80$.

t	I_1	I_2	I_3
0	4.71239	3.33324	1.141666
10	4.71239	3.33324	1.141666
20	4.71239	3.33324	1.141666
30	4.71239	3.33324	1.141664
40	4.71239	3.33333	1.141665
50	4.71239	3.33338	1.141666
60	4.71239	3.33333	1.141666

Table 8Invariants for interaction of two solitary waves with $p = 3$, $c_1 = 0.3$, $c_2 = 0.0375$, $x_1 = 15$, $x_2 = 30$, $h = 0.1$, $\Delta t = 0.025$, $0 \leq x \leq 80$.

t	I_1	I_2	I_3
0	4.20655	3.97977	1.01634
10	4.20655	2.07986	1.01634
20	4.20655	3.07982	1.01634
30	4.20655	3.07980	1.01634
40	4.20655	3.07986	1.01634
50	4.20655	3.07981	1.01633
60	4.20655	3.07987	1.01633
70	4.20655	3.07976	1.01634
80	4.20655	3.07991	1.01633
90	4.20655	3.07974	1.01633
100	4.20655	3.07972	1.01634

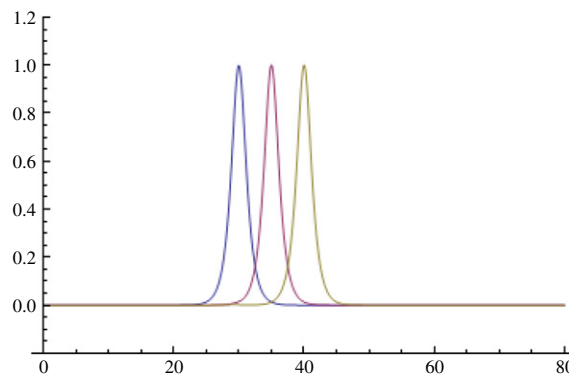


Fig. 1. Single solitary wave with $p = 2$, $c = 1/2$, $x_0 = 30$, $0 \leq x \leq 80$ at level time $t = 0, 10, 20$.

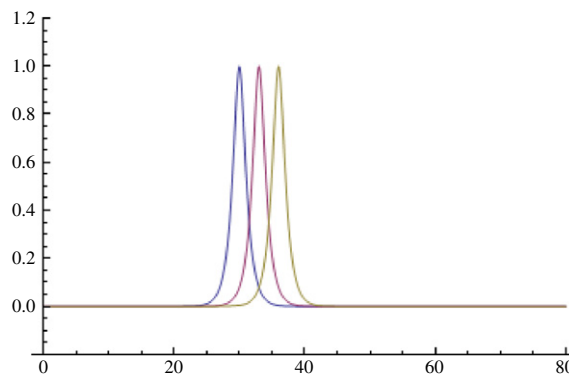


Fig. 2. Single solitary wave with $p = 3$, $c = 0.3$, $x_0 = 30$, $0 \leq x \leq 80$ at level time $t = 0, 10, 20$.

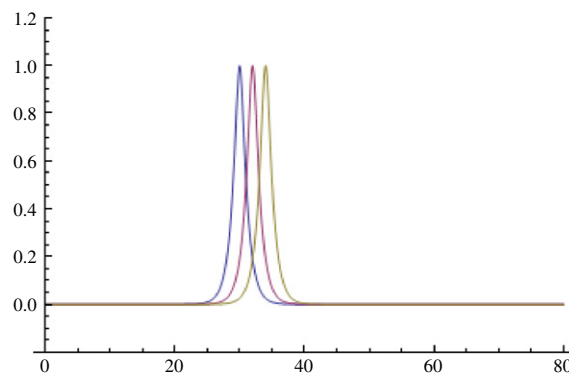


Fig. 3. Single solitary wave with $p = 4$, $c = 0.2$, $x_0 = 30$, $0 \leq x \leq 80$ at level time $t = 0, 10, 20$.

4.1. The single solitary wave

First we take $p = 2$. In this case two sets of parameters have been considered for our computational work. The first set is chosen to be $c = (1/32)$, $h = 0.1$, $\Delta t = 0.2$, $\varepsilon = 3$, $\delta = 1$ and $x_0 = 30$ with range $[0, 80]$ to coincide with the collocation methods of [10,11]. Thus the solitary wave has amplitude 0.25 and the simulations are done up to $t = 20$. Values of the three invariants as well as L_2 and L_∞ error norms from our method have been computed and reported in Table 1. The changes of the invariants $I_1 \times 10^5$, $I_2 \times 10^5$ and $I_3 \times 10^5$ from their initial values are less than 0.001, 0.8 and 0.003, respectively. Error deviations are changed in the range of $-1.86541 \times 10^{-6} < \text{error} < 2.75154 \times 10^{-6}$. The second set is chosen to be $c = 1/2$, $h = 0.1$, $\Delta t = 0.2$, $\varepsilon = 3$, $\delta = 1$ and $x_0 = 30$ with range $[0, 80]$; then the solitary wave has amplitude 1.0. The simulations are done up to $t = 20$. The motion of a single solitary wave using the above scheme corresponding to the second set of parameters has been computed and plotted in Fig. 1. Table 2 represents values of the three invariants and the error norms from the given method. The changes of the invariants $I_1 \times 10^3$, $I_2 \times 10^3$ and $I_3 \times 10^3$ from their initial values are less

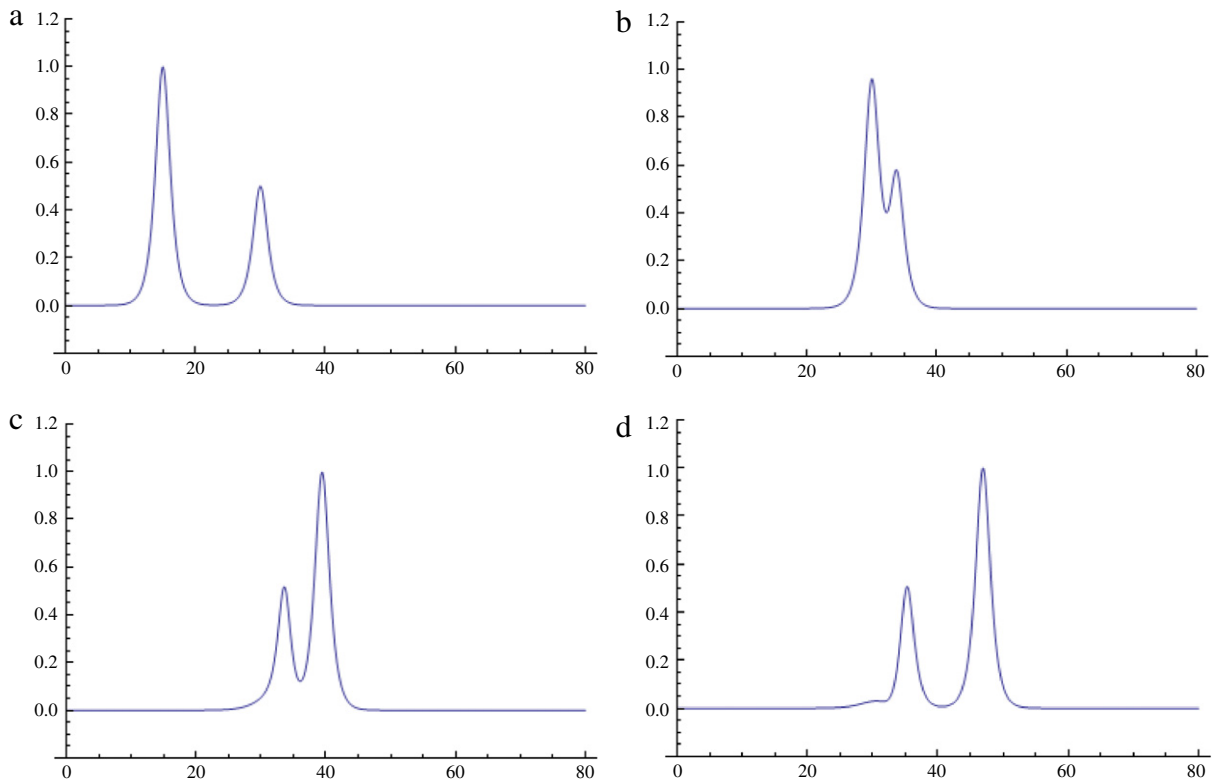


Fig. 4. $p = 2$: interaction of two solitary waves at (a) $t = 0$, (b) $t = 30$, (c) $t = 45$, (d) $t = 60$.

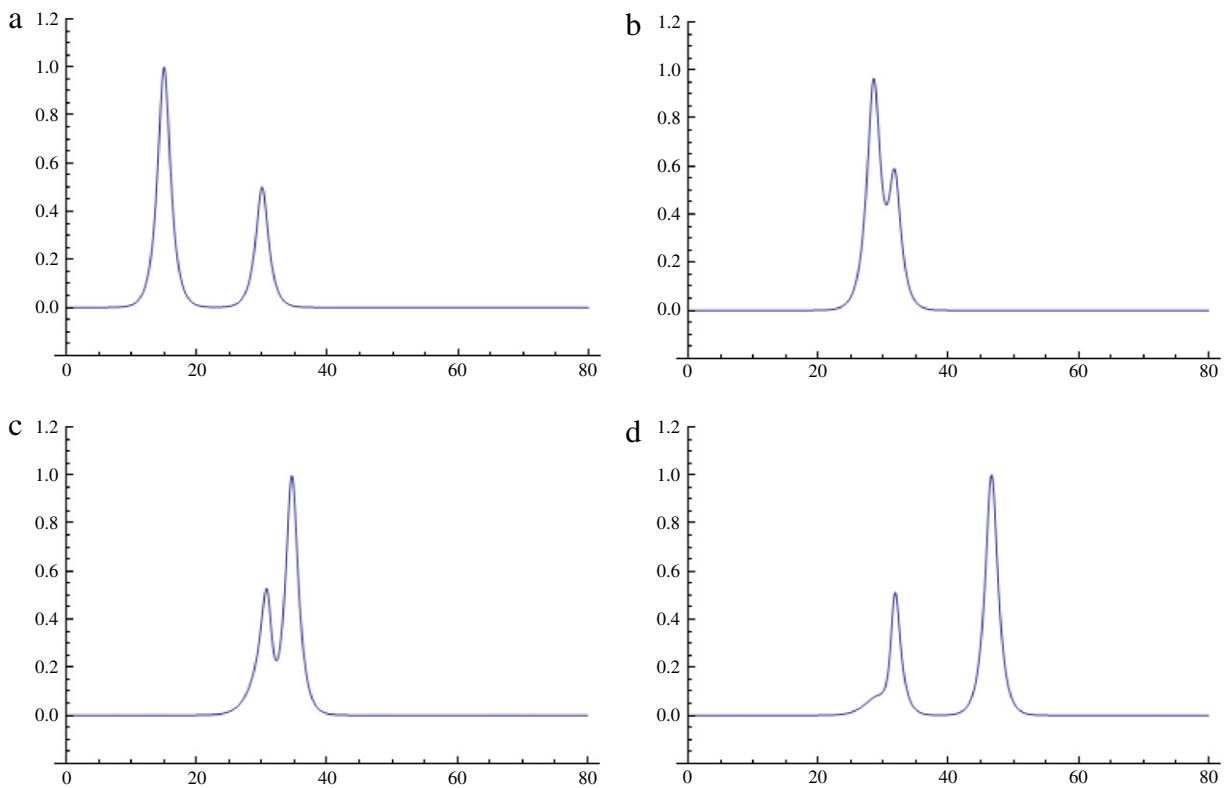


Fig. 5. $p = 3$: interaction of two solitary waves at (a) $t = 0$, (b) $t = 45$, (c) $t = 60$, (d) $t = 100$.

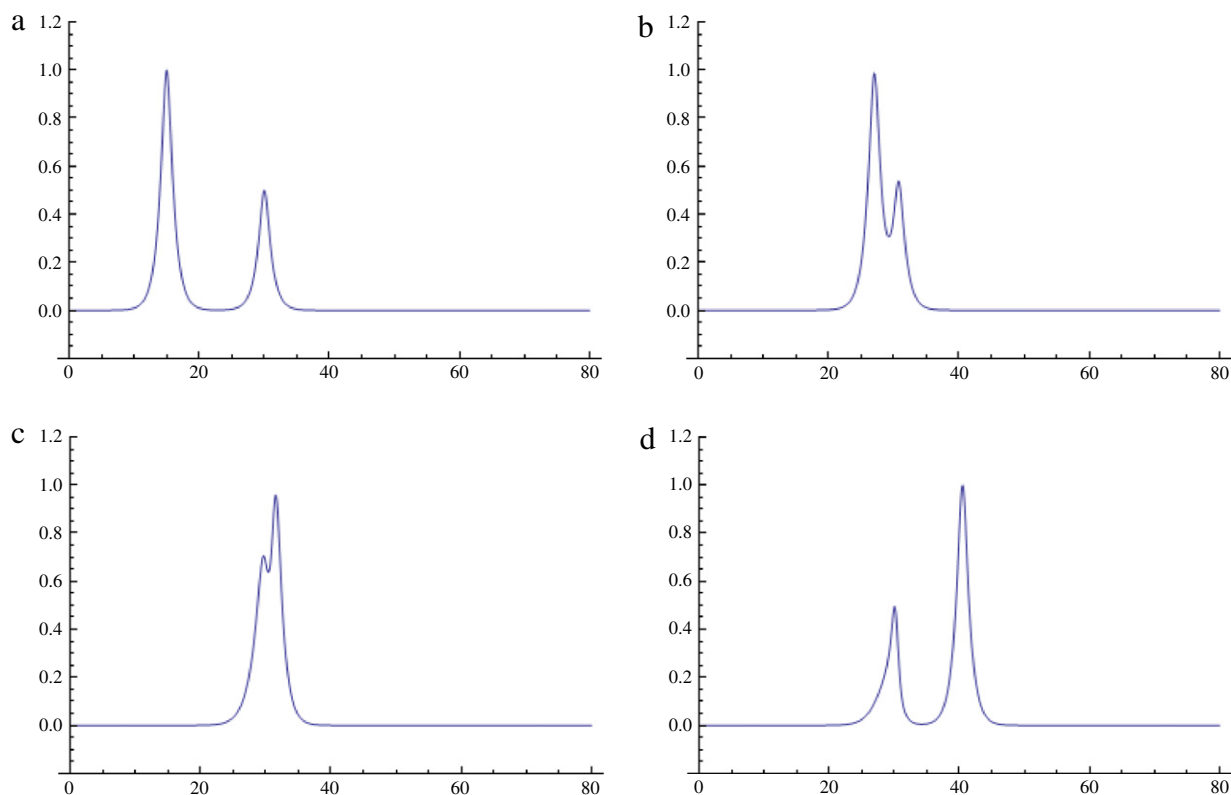


Fig. 6. $p = 4$: interaction of two solitary waves at (a) $t = 0$, (b) $t = 60$, (c) $t = 75$, (d) $t = 120$.

Table 9

Invariants for interaction of two solitary waves with $p = 4$, $c_1 = 0.2$, $c_2 = 1/80$, $x_1 = 15$, $x_2 = 30$, $h = 0.1$, $\Delta t = 0.025$, $0 \leq x \leq 80$.

t	I_1	I_2	I_3
0	3.93309	2.94512	0.797614
10	3.93309	2.94518	0.797612
20	3.93309	2.94517	0.797611
30	3.93309	2.94510	0.797612
40	3.93309	2.94515	0.797612
50	3.93309	2.94504	0.797616
60	3.93309	2.94505	0.797622
70	3.93307	2.94510	0.797636
80	3.93309	2.94506	0.797613
90	3.93309	2.94520	0.797612
100	3.93309	2.94508	0.797611
110	3.93309	2.94517	0.797610
120	3.93308	2.94511	0.797611

than 0.001, 0.15 and 0.1 respectively. The error deviations are changed in the ranges of $-0.0059062 < \text{error} < 0.0086082$ in this case.

Secondly we take $p = 3$. In this case also two sets of parameters have been considered. In the first case we take $c = 0.001$, $h = 0.1$, $\Delta t = 0.2$, $\varepsilon = 3$, $\delta = 1$ and $x_0 = 30$ with range $[0, 80]$ to coincide with the collocation method of [11]. Thus the solitary wave has amplitude 0.15 and the simulations are done up to $t = 20$. Table 3 represents values of the three invariants and error norms. The changes of the invariants $I_1 \times 10^6$, $I_2 \times 10^6$ and $I_3 \times 10^6$ from their initial values are less than 0.01, 0.2 and 0.1×10^{-3} respectively. Error deviations are changed in the range of $-7.99209 \times 10^{-8} < \text{error} < 8.20602 \times 10^{-8}$. For the second set we take $c = 0.3$ and the same values are taken for other parameters. The solitary wave has amplitude 1.0 and the simulations are done up to $t = 20$. The motion of a single solitary wave using the above scheme corresponding to the second set of parameters has been computed and plotted in Fig. 2. Table 4 presents values of the three invariants and error norms. The changes of the invariants $I_1 \times 10^3$, $I_2 \times 10^3$ and $I_3 \times 10^3$ from their initial values are less than 0.02, 0.16 and 0.028 respectively. Error deviations are changed in the range of $-0.00230152 < \text{error} < 0.00370926$.

Finally, we take $p = 4$, and the parameters $c = 0.2$, $h = 0.1$, $\Delta t = 0.2$, $\varepsilon = 3$, $\delta = 1$ and $x_0 = 30$ with range $[0, 80]$ are taken. Thus the solitary wave has amplitude 1.0 and the simulations are done up to $t = 20$. The motion of a single solitary

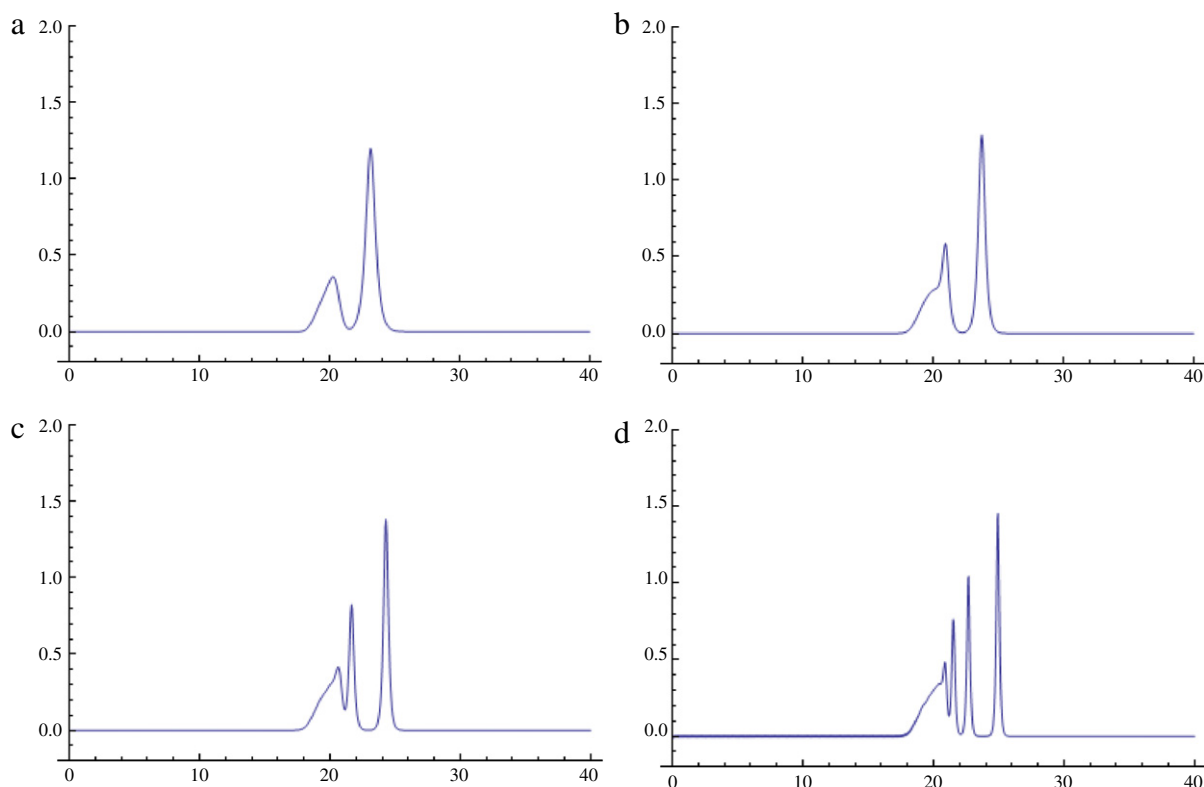


Fig. 7. Maxwellian initial condition $p = 2$: (a) at $t = 12$, $\delta = 0.1$, (b) at $t = 12$, $\delta = 0.05$, (c) at $t = 12$, $\delta = 0.025$, (d) at $t = 12$, $\delta = 0.01$.

wave using the above scheme corresponding to the above set of parameters has been computed and reported in Table 5. Fig. 3 represents the motion of a single solitary wave at different time levels. The changes of the invariants $I_1 \times 10^3$, $I_2 \times 10^3$ and $I_3 \times 10^3$ from their initial values are less than 0.001, 0.14 and 0.019, respectively. Error deviations are changed in the range of $-0.00105526 < \text{error} < 0.00186285$.

Comparisons of our results with exact solution as well as the recorded values in [10,11] have been made and tabulated in Table 6 at $t = 20$. The parameters used in the present paper are the same as those used in the cited papers. The superiority of the present method over the previous ones is clear.

4.2. The interaction of two GEW solitary waves

Here we study the interaction of two well separated solitary waves having different amplitudes and traveling in the same direction. The initial condition is given by

$$u(x, 0) = \sum_{i=1}^2 \sqrt{\frac{c_i(p+1)(p+2)}{2\varepsilon}} \sec h^2 \left[\frac{p}{2\sqrt{\delta}} (x - x_i) \right] \quad (18)$$

where c_i and x_i , $i = 1, 2$, are arbitrary constants. In this case also we take $p = 2, 3$ and 4. For all values of p , the following parameters: $h = 0.1$, $\Delta t = 0.025$, $\varepsilon = 3$, $\delta = 1$, $x_1 = 15$ and $x_2 = 30$ have been chosen with range $[0, 80]$. For $p = 2$, we take $c_1 = 0.5$ and $c_2 = 0.125$ so that the amplitudes of two well separated solitary waves are in the ratio 2:1. In this case the simulation is done up to $t = 60$. The computed values of the three invariants are recorded in Table 7. The changes of the invariants $I_1 \times 10^3$, $I_2 \times 10^3$ and $I_3 \times 10^3$ from their initial values are less than 0.001, 0.15 and 0.003 respectively. Fig. 4(a)–(d) show the computer plots of the interaction of two solitary waves at different time levels. For $p = 3$, we take $c_1 = 0.3$ and $c_2 = 0.0375$ so that the amplitudes of the two solitary waves are in the ratio 2:1. The simulation is done up to $t = 100$ and the computed values of the three invariants of motion are reported in Table 8. The changes of the invariants $I_1 \times 10^3$, $I_2 \times 10^3$ and $I_3 \times 10^3$ from their initial values are less than 0.001, 0.15 and 0.02 respectively. Fig. 5(a)–(d) show the computer plots of the interaction of two solitary waves at different time levels for this case. Finally for $p = 4$ we take $c_1 = 0.2$ and $c_2 = 1/80$ so that the amplitudes of the two solitary waves are in the ratio 2:1. In this case the simulation is done up to $t = 120$ and the computed values of the three invariants of motion are recorded in Table 9. The changes of the invariants $I_1 \times 10^4$, $I_2 \times 10^4$ and $I_3 \times 10^4$ from their initial values are less than 0.3, 0.9 and 0.23 respectively. In Fig. 6(a)–(d) we show the interaction of two solitary waves at different time levels.

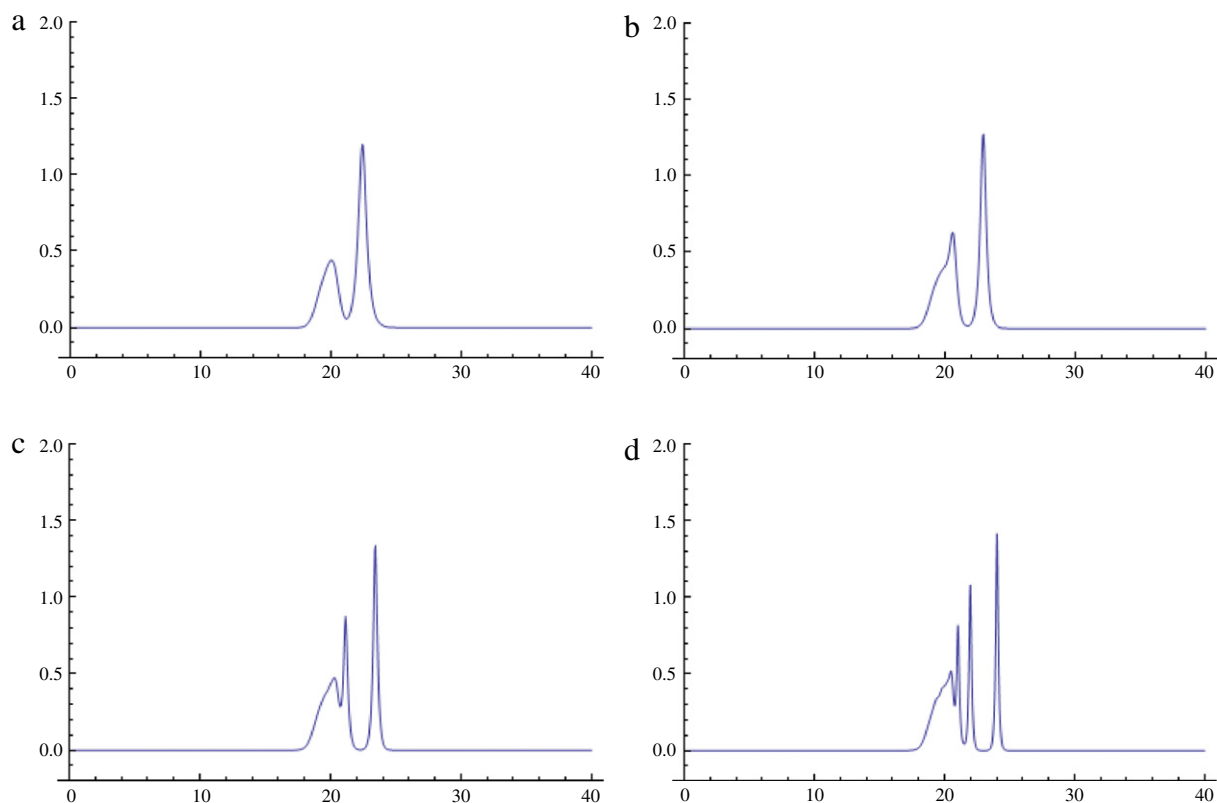


Fig. 8. Maxwellian initial condition $p = 3$: (a) at $t = 12$, $\delta = 0.1$, (b) at $t = 12$, $\delta = 0.05$, (c) at $t = 12$, $\delta = 0.025$, (d) at $t = 12$, $\delta = 0.01$.

Table 10

Invariants of the GEW equation using the Maxwellian initial condition.

δ	t	$p = 2$			$p = 3$			$p = 4$		
		I_1	I_2	I_3	I_1	I_2	I_3	I_1	I_2	I_3
0.1	0	1.7724	1.3786	0.8862	1.7724	1.3786	0.7927	1.7786	1.3786	0.7235
	4	1.7724	1.3790	0.8863	1.7724	1.3787	0.7928	1.7723	1.3793	0.7231
	8	1.7724	1.3787	0.8862	1.7724	1.3788	0.7926	1.7712	1.3863	0.7207
	12	1.7724	1.3785	0.8861	1.7724	1.3787	0.7926	1.7734	1.3836	0.7224
0.05	0	1.7724	1.3159	0.8862	1.7724	1.3159	0.7927	1.7724	1.3160	0.7236
	4	1.7724	1.3161	0.8862	1.7724	1.3161	0.7925	1.7725	1.3174	0.7236
	8	1.7724	1.3161	0.8861	1.7724	1.3156	0.7922	1.7725	1.3154	0.7240
	12	1.7724	1.3160	0.8861	1.7724	1.3156	0.7922	1.7724	1.3177	0.7245
0.025	0	1.7724	1.2846	0.8862	1.7724	1.2846	0.7927	1.7724	1.2846	0.7236
	4	1.7724	1.2846	0.8859	1.7725	1.2844	0.7914	1.7724	1.2855	0.7246
	8	1.7724	1.2849	0.8857	1.7725	1.2838	0.7911	1.7725	1.2855	0.7244
	12	1.7724	1.2835	0.8856	1.7723	1.2834	0.7910	1.7724	1.2849	0.7243
0.01	0	1.7724	1.2658	0.8862	1.7724	1.2658	0.7927	1.7724	1.2658	0.7236
	4	1.7724	1.267	0.8837	1.7724	1.2674	0.7950	1.7725	1.2669	0.7251
	8	1.7723	1.2646	0.8846	1.7724	1.2665	0.7947	1.7723	1.2683	0.7251
	12	1.7724	1.2658	0.8862	1.7724	1.2665	0.7947	1.7725	1.2669	0.7253

4.3. A Maxwellian initial condition

In this section, evolution of a train of solitary waves of the GEW equation has been studied using the Maxwellian initial condition

$$u(x, 0) = \text{Exp}(-(x - 20)^2) \quad (19)$$

for various values of δ . As it is known that there is a Maxwellian initial condition, the behaviour of the solution depends on the value of δ [10,11]. For $\delta \gg \delta_c$ where δ_c is some critical value, the Maxwellian initial condition does not break up into solitons but exhibits rapidly oscillating wave packets. When $\delta = \delta_c$ a mixed type of solution is found which consists of a leading soliton and an oscillating tail. For $\delta < \delta_c$, the Maxwellian breaks up into a number of solitons according to the values

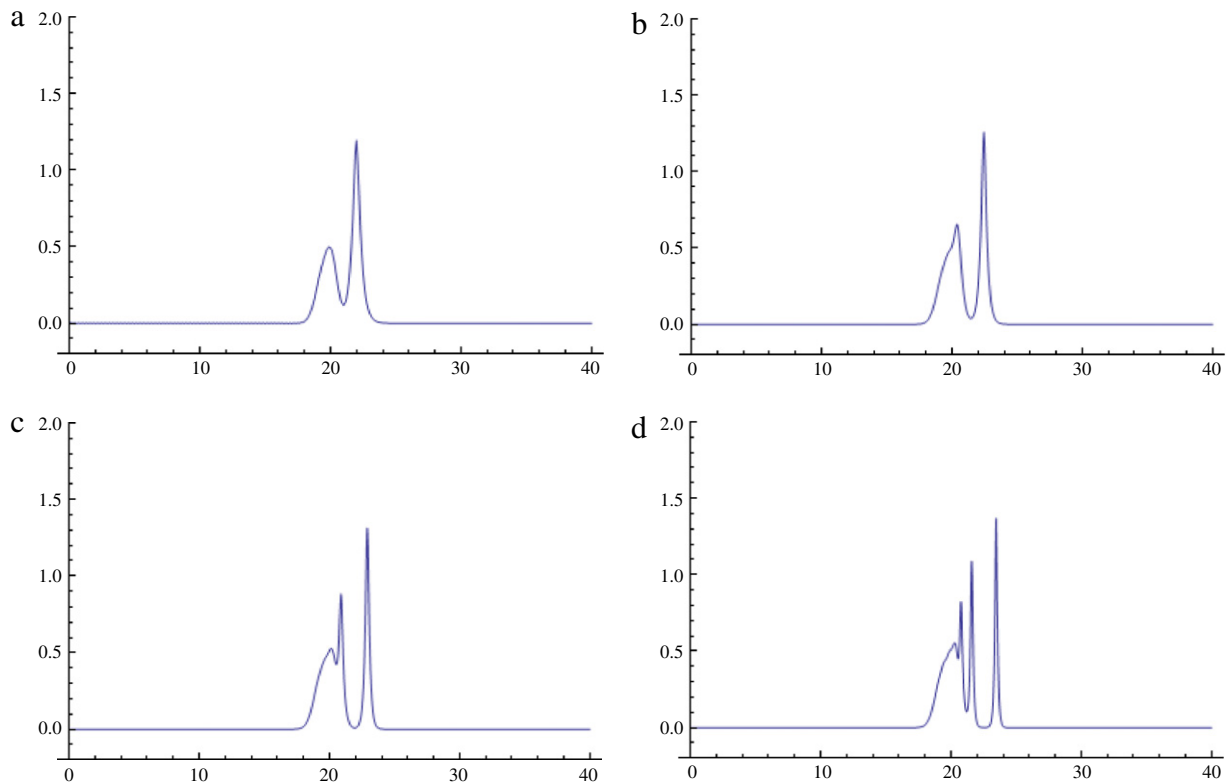


Fig. 9. Maxwellian initial condition $p = 4$: (a) at $t = 12$, $\delta = 0.1$, (b) at $t = 12$, $\delta = 0.05$, (c) at $t = 12$, $\delta = 0.025$, (d) at $t = 12$, $\delta = 0.01$.

of δ . We take $\delta = 0.1, 0.05, 0.025$ and 0.01 for all values of p . The computed values of the three invariants of motion for different values of δ are recorded in Table 10. For $p = 2$ the computer plot of the breaking of solitons is shown in Fig. 7(a)–(d). Similarly, for $p = 3$ the breaking of solitons from the Maxwellian are shown in Fig. 8(a)–(b). Finally, for $p = 4$ the breaking of solitons is shown in Fig. 9(a)–(d). All the simulations are done up to $t = 12$. For $p = 2$, the changes of the three invariants I_1 , I_2 and I_3 from their initial values are less than 0.0001, 0.0013 and 0.0026 respectively. Similarly for $p = 3$, the changes of the invariants I_1 , I_2 and I_3 from their initial values are less than 0.0001, 0.0017 and 0.0024 respectively. Finally for $p = 4$, the changes of the invariants from their initial values are less than 0.0075, 0.006 and 0.0029 respectively.

5. Conclusion

The Petrov–Galerkin method using the linear hat function and quadratic B -spline function as test and trial functions respectively has been successfully implemented to study the solitary waves of the GEW equation. It has been shown that our scheme is conditionally stable and it is more accurate than the other methods available in the literature. We tested our scheme using single solitary waves for which the analytic solution is known and then extended this to study the interaction of two solitons as well as Maxwellian initial conditions where no analytic solution is known. It is also observed that the conservation laws were reasonably well satisfied for the interaction of two solitons as well as Maxwellian initial conditions.

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