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Erratum

Erratum to: ''On the HSS iteration methods for positive definite Toeplitz linear systems'' [J. Comput. Appl. Math. 224 (2009) 709–718]

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ABSTRACT

In [1], Gu and Tian [Chuanqing Gu, Zhaolu Tian, On the HSS iteration methods for positive definite Toeplitz linear systems, J. Comput. Appl. Math. 224 (2009) 709–718] proposed the special HSS iteration methods for positive definite linear systems $Ax = b$ with $A \in \mathbb{C}^{n\times n}$ a complex Toeplitz matrix. But we find that the special HSS iteration methods are incorrect. Some examples are given in our paper.

1. Introduction

Recently, Gu and Tian [1] proposed the special HSS iteration methods for positive definite linear systems

$$Ax = b \tag{1}$$

with $A \in \mathbb{C}^{n\times n}$ a complex Toeplitz matrix and $x, b \in \mathbb{C}^{n\times n}$. Such systems arise in a variety of applications in mathematics and engineering; see Refs. [2,3] for details. Their aim was to apply the special HSS iteration methods to solve the large sparse non-Hermitian positive definite Toeplitz systems, which is a special version of the HSS iteration method in [4] and the splitting is

$$A = H + S, \tag{2}$$

where the symmetric part $H = \frac{1}{2}(A + A^T)$ is a centrosymmetric matrix, the skew-symmetric part $S = \frac{1}{2}(A - A^T)$ is a skew-centrosymmetric matrix and T denotes the transpose of the matrix. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be centrosymmetric if $JAJ = A$ and skew-centrosymmetric if $JAJ = -A$, where J is the permutation matrix with ones on the cross diagonal (bottom left to top right) and zero elsewhere. Then they proposed the modified Hermitian and skew-Hermitian (HSS) splitting iterative method for solving $Ax = b$ with $A \in \mathbb{C}^{n\times n}$: An initial vector $x^{(0)}$ is given. For $k = 0, 1, \ldots$ until $x(k)$ converges, compute

$$\begin{cases} (\beta I + H)x^{(k+\frac{1}{2})} = (\beta I - S)x^{(k)} + b, \\ (\beta I + S)x^{(k+1)} = (\beta I - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{3}$$

where β is a given positive constant, $H = \frac{1}{2}(A + A^T)$ and $S = \frac{1}{2}(A - A^T)$.

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2. Main results

We begin this section by recalling some definitions that will be used in this paper.

Definition 1. A complex matrix $A \in \mathbb{C}^{n\times n}$ is called positive definite if $\mathrm{R}(x^H Ax) > 0$ for all nonzero complex vectors x, where x^H denotes the conjugate transpose of the vector x [5, page 399].

A necessary and sufficient condition for a complex matrix A to be positive definite is that the Hermitian part $\frac{1}{2}(A^H + A)$ is positive definite.

Definition 2. For any complex matrix A, the 2-norm or spectral norm of A is defined as $\|A\|_2 = \sqrt{\rho(A^H A)}$, where $\rho(\cdot)$ represents the spectral radius of a matrix [5, page 295].

Definition 3. If a complex matrix A satisfies $A^T A = I$, then A is said to be a complex orthogonal matrix [6, page 63].

It should be noted that a complex orthogonal matrix A is not orthogonal, i.e., $A^H A \neq I$. Therefore, $\|A\|_2$ may not be equal to one for a complex orthogonal matrix A.

For the special HHS iteration methods, Gu and Tian [1] gave the following theorem:

Theorem 3 ([1]). *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $H = \frac{1}{2}(A + A^T)$ and $S = \frac{1}{2}(A - A^T)$ be its centrosymmetric and skew-centrosymmetric parts, respectively. Let*

$$M(\beta) = (\beta I + S)^{-1}(\beta I - H)(\beta I + H)^{-1}(\beta I - S), \tag{4}$$

be the iteration matrix of the HSS iteration (3) and $V(\beta) = (\beta I - H)(\beta I + H)^{-1}$. Then the spectral radius $\rho(M(\beta))$ is bounded by $\|V(\beta)\|_2$ and has the following relation:

$$\rho(M(\beta)) \leq \|V(\beta)\|_2 < 1 \quad \textit{for } \forall \beta > 0;$$

i.e., the HSS iteration (3) converges to the exact solution $x^* \in \mathbb{C}^n$ of the system of Toeplitz linear equations (1).

They have proved the theorem. Unfortunately, the proof of Theorem 3 in Ref. [1] is incorrect.

By similarity transformation, they first note that

$$\begin{aligned}
\rho(M(\beta)) &= \rho((\beta I + S)^{-1}(\beta I - H)(\beta I + H)^{-1}(\beta I - S)) \\
&= \rho((\beta I - S)(\beta I + S)^{-1}(\beta I - H)(\beta I + H)^{-1}) \\
&\leq \|(\beta I - S)(\beta I + S)^{-1}(\beta I - H)(\beta I + H)^{-1}\|_2 \\
&\leq \|(\beta I - S)(\beta I + S)^{-1}\|_2 \|(\beta I - H)(\beta I + H)^{-1}\|_2.
\end{aligned}$$

If $P \in \mathbb{C}^{n\times n}$ is a positive definite matrix, then it holds that $\|(\beta I - P)(\beta I + P)^{-1}\|_2 < 1$; see [7]. Since A is a positive definite matrix, $H = \frac{1}{2}(A + A^T)$ is also positive definite matrix. It then follows that $\|V(\beta)\|_2 < 1$. Letting $Q(\beta) = (\beta I - S)(\beta I + S)^{-1}$, we see that

$$\begin{aligned}
Q(\beta)^T Q(\beta) &= (\beta I - S)^{-1}(\beta I + S)(\beta I - S)(\beta I + S)^{-1} \\
&= (\beta I - S)^{-1}(\beta I - S)(\beta I + S)(\beta I + S)^{-1} \\
&= I.
\end{aligned}$$

That is to say, $Q(\beta)$ is a complex orthogonal matrix for $\forall \beta > 0$. So, they think that $\|Q(\beta)\|_2 = 1$. It then follows that

$$\rho(M(\beta)) \leq \|(\beta I - H)(\beta I + H)^{-1}\|_2 < 1.$$

In fact, $Q(\beta) = (\beta I - S)(\beta I + S)^{-1}$ is not an orthogonal matrix. So $\|Q(\beta)\|_2 = 1$ is incorrect. For example, $A = \begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$, thus $Q(\beta) = \begin{pmatrix} \beta & \mathrm{i} \\ -\mathrm{i} & \beta \end{pmatrix} \begin{pmatrix} \beta & -\mathrm{i} \\ \mathrm{i} & \beta \end{pmatrix}^{-1}$. $\|Q(\beta)\|_2 = 3 \neq 1$ for $\beta = 2$. It is because the definition of 2-norm or spectral norm of $Q(\beta)$ is $\|Q(\beta)\|_2 = \sqrt{\rho(Q(\beta)^H Q(\beta))}$. Therefore, even if $Q(\beta)$ is an complex orthogonal matrix, $\|Q(\beta)\|_2$ may not be equal to one.

Table 1
The spectral radius $\rho(M(\beta))$ of the iteration matrices (4).

$n = 64$					
β	0.5	2.5	4.5	6.5	8.5
$\rho(M(\beta))$	15.487	12.764	3.367	1.857	0.628
$n = 128$					
β	0.5	2.5	4.5	6.5	8.5
$\rho(M(\beta))$	15.360	13.176	3.343	1.888	0.627

3. Examples

Example 1. Now let us give a counterexample of Theorem 3 below. Consider the non-Hermitian positive definite Toeplitz linear systems $Ax = b$ with

$$A = \begin{pmatrix} 10 & 2\mathrm{i} & 3\mathrm{i} & & & & \\ -1-2\mathrm{i} & 10 & 2\mathrm{i} & 3\mathrm{i} & & & \\ -1-3\mathrm{i} & -1-2\mathrm{i} & 10 & 2\mathrm{i} & 3\mathrm{i} & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -1-3\mathrm{i} & -1-2\mathrm{i} & 10 & 2\mathrm{i} & 3\mathrm{i} \\ & & & -1-3\mathrm{i} & -1-2\mathrm{i} & 10 & 2\mathrm{i} \\ & & & & -1-3\mathrm{i} & -1-2\mathrm{i} & 10 \end{pmatrix}.$$

The spectral radius $\rho(M(\beta))$ of the iteration matrices (4) are shown in Table 1. From Table 1, we see that not all positive constants β satisfy $\rho(M(\beta)) < 1$ for a definite positive Toeplitz matrix A. But a good choice of β can make the modified HSS method converge.

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