



Singular optimal dividend control for the regime-switching Cramér–Lundberg model with credit and debit interest



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HIGHLIGHTS

- We study a regime-switching compound Poisson model with credit and debit interest.
- The surplus process is controlled by subtracting the cumulative dividends.
- Our objective is to find an optimal dividend strategy.
- We show that a regime-switching band strategy is optimal.

ARTICLE INFO

Article history:

Received 31 January 2013

Received in revised form 15 August 2013

Keywords:

Band strategy

Compound Poisson process

Cramér–Lundberg model

Dividend optimization

Regime switching

Singular control

ABSTRACT

We investigate the dividend optimization problem for a company whose surplus process is modeled by a regime-switching compound Poisson model with credit and debit interest. The surplus process is controlled by subtracting the cumulative dividends. The performance of a dividend distribution strategy which determines the timing and amount of dividend payments, is measured by the expectation of the total discounted dividends until ruin. The objective is to identify an optimal dividend strategy which attains the maximum performance. We show that a regime-switching band strategy is optimal.

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1. Introduction

The problem of finding the optimal strategy which determines when to pay out dividends and how much to pay out is an important topic in finance and actuarial science [1]. Most works on the dividend optimization problem are based on either the diffusion setting or the compound Poisson (Cramér–Lundberg) setting. The former setting is an approximation of the latter and has better mathematical tractability. There is a wealth of work studying the dividend optimization problem and its extensions under the diffusion setting (see [2–7] and the references therein). The classical compound Poisson model is more directly appealing for insurance modeling. Under such a setting, the dividend optimization along with optimal reinsurance problem was solved in [1]. Albrecher and Thonhauser [8] studied the optimization problem for the compound Poisson model by adding constant force of interest. Kulenko and Schmidli [9] considered the optimization problem for the compound Poisson model with capital injections. Most recently, [10] investigated the problem by allowing the insurance company to continue its business when the surplus is negative and above a critical level through refinancing and [11] solved the optimal dividend problem for the case of bounded dividend rates.

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The majority of the papers in the literature of dividend optimization consider only one source of uncertainty in modeling the evolution of the cash surplus, which is a Brownian motion in the diffusion setting and a compound Poisson arrival process in the Cramér–Lundberg setting. However, empirical studies showed that a company's earnings are also affected by the external environment regime, e.g. macroeconomic conditions (see [12] and the references therein for details). Hence, models with regime switching are more realistic. A substantial literature in econometrics supports that a finite-state Markov process is appropriate to model the external environment regime (see [13] and references therein). Markov regime-switching models have been used in different contexts including consumption–investment (see, for example, [14]), portfolio management (see, for example, [15]), option pricing (see, for example, [16,17]), risk theory (see, for example, [18]) etc. Recently, research devoted to studying the optimal dividend problem for models with regime switching has appeared. Sotomayor and Cadenillas [12] studied a diffusion process with two regimes and solved the optimal dividend problem for both the case of bounded dividend payment rates and unbounded dividend rates. Jiang and Pistorius [13] solved the dividend optimization problem for a regime-switching diffusion process with multiple regimes. Wei et al. [19] studied the compound Poisson model with regime switching and showed that the value function for the impulse control problem is the unique viscosity solution of the corresponding quasi-variational inequality. Yin et al. [20] studied a limit system of the compound Poisson model with regime switching and showed that the optimal dividend strategy for the limit system is asymptotically optimal for the original model. However, for the regime-switching compound Poisson model, no optimal strategies have been identified for either the singular or the classical control problem.

In this paper, we consider the singular dividend optimization problem for the compound Poisson model with Markov regime switching. Our goal is to find an optimal dividend strategy among a set of admissible strategies with unrestricted dividend rates so that the total expected discounted dividends are maximized. The inclusion of regime switching makes it hard to directly apply the viscosity approach used for the model with no regimes. Instead, we solve the problem by first studying an auxiliary optimization problem with a different optimization criterion where only the dividends up to the first regime switch plus a terminal random value are included in the performance functional. By building a connection between the original and the auxiliary optimization problem, we use the optimization results obtained for the latter to derive results for the original problem. We find that a strategy of a band type is optimal. Our results show the optimal strategy is stationary and depends on the level of surplus and the environment regime at the time.

This paper is organized as follows. In Section 2 we present the problem. In Section 3 we study an auxiliary optimization problem with a different performance functional and present the optimal solution for this new problem. In Section 4, we construct an optimal dividend strategy with the assistance of the optimization results for the auxiliary problem. A conclusion is provided in Section 5.

2. Problem formulation

Consider a company operating under an environment described by the finite state stochastic process $\{J_t; t \geq 0\}$. When the environment status (regime) is i , claims arrive according to a Poisson process with intensity rate λ_i and premiums are collected continuously at rate p_i . Let S_k denote the arrival time of the k th claim and U_k the size of this claim. Claim sizes, conditioning on the regimes of the arrival times of these claims, are independent and independent of the claim arrival process. And given $J_{S_k} = i$, the random variable U_k follows the distribution $F_i(\cdot)$. Let $N(t)$ denote the number of claims up to time t . Then, $N(t) = \#\{k : S_k \leq t\}$. Assume that the insurance company, when in regime i , earns interest under a constant force $r_i (> 0)$ for its positive surplus (reserve), and, if the surplus drops below zero, could borrow money with the amount equal to the deficit. The company will repay the debt and the interest charged at a force α_i continuously at the same rate as the incoming premium rate.

For any x , define $(x)^+ = \max\{x, 0\}$ and $(x)^- = -\min\{x, 0\}$. Let R_t denote the surplus at time t . Then the surplus process $\{R_t; t \geq 0\}$ follows the following dynamics:

$$dR_t = (p_{J_t} + r_{J_t}(R_t)^+ - \alpha_{J_t}(R_t)^-)dt - d\left(\sum_{k=1}^{N(t)} U_k\right). \quad (2.1)$$

Assume that all the above random quantities are defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. The environment process $\{J_t; t \geq 0\}$ is a Markov chain with the state space $\mathbb{E} = \{1, 2, \dots, \kappa\}$ and the transition intensity matrix $Q = (q_{ij})_{\kappa \times \kappa}$. We define $q_i = -q_{ii} = \sum_{j \neq i} q_{ij}$ for $i \in \mathbb{E}$.

We introduce the notations $P_{(x,i)}(\cdot) = P(\cdot | R_0 = x, J_0 = i)$ and $E_{(x,i)}[\cdot] = E[\cdot | R_0 = x, J_0 = i]$.

Suppose the company will pay out dividends to the shareholders from surplus. We use L_t to denote the cumulative dividends paid up to time t and call the stochastic process $L = \{L_t; t \geq 0\}$ a dividend strategy. Then, it is fair to set $L_0 = 0$ and let L be non-decreasing in t . The dividend payment decision at any time will be made based on past information only and not on any future information, and therefore the amount of dividend to be paid immediately after t depends on the information up to time t . As a result, it is reasonable to assume that the stochastic process L is predictable. It is natural to assume that L is also left continuous with limits from the right side (càglàd). Since the process L is nondecreasing and left continuous, it has the following decomposition: $L_t = L_t^c + \sum_{0 \leq s < t} (L_{s+} - L_s)$, where $\{L_t^c; t \geq 0\}$ is the continuous part of the process $\{L_t; t \geq 0\}$. Furthermore, $L_{t+} = L_t^c + \sum_{0 \leq s \leq t} (L_{s+} - L_s)$.

Let R_t^L denote the L -controlled surplus process, the surplus process when the dividend strategy L is applied. Then we have the following dynamics,

$$R_t^L = R_0 + \int_0^t (p_{j_{s-}} + r_{j_{s-}}(R_{s-}^L)^+ - \alpha_{j_{s-}}(R_{s-}^L)^-) ds - \sum_{k=1}^{N(t)} U_k - L_t. \quad (2.2)$$

For convenience, we use R, R^L, J and (R^L, J) to denote the stochastic processes $\{R_t; t \geq 0\}$, $\{R_t^L; t \geq 0\}$, $\{J_t; t \geq 0\}$ and $\{(R_t, J_t); t \geq 0\}$, respectively.

As commented in [10,21], for any regime i , if the company is in this regime when the surplus hits $-\frac{p_i}{\alpha_i}$ or any level below that and the company will stay in this regime forever, the surplus will never be able to return to a non-negative level again. However, if $\frac{p_i}{\alpha_i} < \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$, the company in regime i with the level of surplus falling in the interval $(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, -\frac{p_i}{\alpha_i}]$ may still be able to return to a positive level after switching to a different regime, say k with $-\frac{p_k}{\alpha_k}$ less than the current level of surplus, as the drift may be positive after switching to the regime k . If the surplus hits a level at or below $-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$, the surplus will never be able to return to a non-negative level again, no matter which regime it will switch to in the future. Hence, the time to (absolute) ruin for the L -controlled surplus process can be defined by:

$$T^L = \inf \left\{ t \geq 0 : R_t^L \leq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j} \right\}. \quad (2.3)$$

A dividend strategy L is said to be *admissible* if it is càglàd and adapted, and, under L , no dividends are allowed to be paid out after ruin and the amount of dividends paid instantly at any time is not bigger than the current level of surplus. Define Π to be the set of all admissible strategies. Then, for any $L \in \Pi$, $L_t \equiv L_{T^L}$ for $t \geq T^L$ and $0 \leq L_{t+} - L_t \leq X_t^L + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$ for $t \geq 0$.

A dividend strategy L is a *Markov strategy* if, given the history of the process (R^L, J) up to and including time t , the conditional probability distribution of L at any future time depends only on the current time t and the current value (R_t^L, J_t) . It is not hard to see that the set of admissible strategies Π consists of both Markov and non-Markov strategies.

Let $\delta(t)$ denote the risk discount rate at time t . Assume that $\delta(t) = \delta_i$ ($\delta_i > r_i$) when the regime is i . The discount factor at time t is $\Lambda_t = \int_0^t \delta(s) ds$.

Define the performance functional $\mathcal{J}$ by

$$\mathcal{J}(L)(x, i) = E_{(x, i)} \left[\int_0^{T^L} e^{-\Lambda_s} dL_s^c + \sum_{0 \leq s < T^L} e^{-\Lambda_s} (L_{s+} - L_s) \right]. \quad (2.4)$$

Here $\mathcal{J}(L)(x, i)$ is the expectation of the total discounted dividends until ruin given that the process starts from the initial capital x and the initial environment regime i .

Define

$$V(x, i) = \sup_{L \in \Pi} \mathcal{J}(L)(x, i). \quad (2.5)$$

We call $\mathcal{J}(L)(x, i)$ the return function associated with the strategy L and $V(x, i)$ the optimal return function.

Our ultimate goal is to investigate the existence of a dividend strategy $L^* \in \Pi$ such that $V(x, i) = V_{L^*}(x, i)$ for all x and i , and to identify such a strategy, if any. Such an L^* is called the *optimal dividend strategy*.

We start with looking at a useful property of V first.

Lemma 2.1. *There exists a constant $K > 0$ such that $V(x, i) \leq x + K$ for $x \in \mathbb{R}$ and $i \in \mathbb{E}$.*

Proof. Write $\tilde{p} = \max_{j \in \mathbb{E}} p_j$, $\tilde{\alpha} = \min_{j \in \mathbb{E}} \alpha_j$ and $\tilde{r} = \max_{j \in \mathbb{E}} r_j$. Consider the stochastic process $\{Y_t; t \geq 0\}$ defined by

$$Y_t = R_0 + \int_0^t (\tilde{p} + \tilde{r} \cdot (Y_{s-})^+ - \tilde{\alpha} \cdot (Y_{s-})^-) ds - \sum_{k=1}^{\tilde{N}(t)} \tilde{U}_k, \quad (2.6)$$

where $\sum_{k=1}^{\tilde{N}(t)} \tilde{U}_k$ is a compound Poisson random quantity with $\{\tilde{N}(t); t \geq 0\}$ being a Poisson process with intensity $\tilde{\lambda} = \min_{j \in \mathbb{E}} \lambda_j$ and the distribution function of $\tilde{U}_1$ being $\tilde{F}(x) = 1 - \prod_{j=1}^k (1 - F_j(x))$.

Let $\bar{U}_n$ denote the time when the n th jump of the Markov chain J occurs. Consider a fixed t . Write $t_0 = 0$. Notice that given $0 < \tau_1 = t_1 < \tau_2 = t_2 < \dots < \tau_n = t_n \leq t < \tau_{n+1}$ and $J_0 = i_0, J_{\tau_1} = i_1, J_{\tau_2} = i_2, \dots, J_{\tau_n} = i_n$ with $i_1, \dots, i_n \in \mathbb{E}$,

- $\{U_k; k = 1, 2, \dots, n\}$ is a sequence of independent random variables and is independent of $\{N(t); t \geq 0\}$;
- $N(t_1) - N(t_0), N(t_2) - N(t_1), \dots, N(t_n) - N(t_{n-1})$, and $N(t) - N(t_n)$ are a sequence of independent Poisson random variables with mean $\lambda_{i_0}(t_1 - t_0), \lambda_{i_1}(t_2 - t_1), \dots, \lambda_{i_{n-1}}(t_n - t_{n-1})$ and $\lambda_{i_n}(t - t_n)$, respectively;

- $\sum_{k=1}^{N(t_1)} U_k, \sum_{k=1}^{N(t_1)+1} U_k, \dots, \sum_{k=1}^{N(t_n)} U_k$ and $\sum_{k=1}^{N(t_n)+1} U_k$ are a sequence of independent compound Poisson quantities; and
- the last sequence of random sums have, respectively, the same distribution as the sequence of the following compound Poisson quantities $\sum_{k=1}^{N_{0,i_0}(t_1-t_0)} U_{0,i_0,k}, \sum_{k=1}^{N_{1,i_1}(t_2-t_1)} U_{1,i_1,k}, \dots, \sum_{k=1}^{N_{n-1,i_{n-1}}(t_n-t_{n-1})} U_{n-1,i_{n-1},k}$, and $\sum_{k=1}^{N_{n,i_n}(t-t_n)} U_{n,i_n,k}$, where $\{N_{0,i_0}(t); t \geq 0\}, \{N_{1,i_1}(t); t \geq 0\}, \dots$ are independent Poisson processes with intensity $\lambda_{i_0}, \lambda_{i_1}, \dots$, respectively, and $\{U_{0,i_0,k}; k \in \mathbb{N}\}, \{U_{1,i_1,k}; k \in \mathbb{N}\}, \dots$ are independent sequences of independently and identically distributed random variables with $U_{j,i_j,1}$ following distribution $F_{i_j}(\cdot)$.

Note that for each j and i_j , the Poisson process $\{\tilde{N}(t); t \geq 0\}$ has the same distribution as the thinning process of $\{N_{j,i_j}(t); t \geq 0\}$ with thinning probability $\frac{\tilde{\lambda}}{\lambda_{i_j}}$. Also note that $P(\tilde{U}_1 > x) = 1 - \tilde{F}(x) \leq 1 - F_{i_j}(x) = P(U_{j,i_j,1} > x)$ for $x \geq 0, j \in \mathbb{N}$ and $i_j \in \mathbb{E}$. For any two random variables, we say X is stochastically larger than Y if $P(X \geq z) \geq P(Y \geq z)$ for any $z \in \mathbb{R}$. For convenience, write $t_{n+1} = t$. Hence, for any $j = 0, 1, 2, \dots, n$, the compound Poisson random quantity $\sum_{k=1}^{N_{j,i_j}(t_{j+1}-t_j)} U_{j,i_j,k}$ is stochastically larger than the compound Poisson quantity $\sum_{k=1}^{\tilde{N}(t_{j+1}-t_j)} \tilde{U}_k$, which has the same distribution as $\sum_{k=\tilde{N}(t_j)+1}^{\tilde{N}(t_{j+1})} \tilde{U}_k$. Hence, the sum $\sum_{k=1}^{N_{0,i_0}(t_1)} U_{0,i_0,k} + \sum_{k=1}^{N_{1,i_1}(t_2-t_1)} U_{1,i_1,k} + \dots + \sum_{k=1}^{N_{n-1,i_{n-1}}(t_n-t_{n-1})} U_{n-1,i_{n-1},k} + \sum_{k=1}^{N_{n,i_n}(t-t_{n-1})} U_{n,i_n,k}$ is stochastically larger than $\sum_{k=\tilde{N}(t_0)+1}^{\tilde{N}(t_1)} \tilde{U}_k + \dots + \sum_{k=\tilde{N}(t_{n-1})+1}^{\tilde{N}(t_n)} \tilde{U}_k + \sum_{k=\tilde{N}(t_n)+1}^{\tilde{N}(t)} \tilde{U}_k = \sum_{k=1}^{\tilde{N}(t)} \tilde{U}_k$. Consequently, the compound sum $\sum_{k=1}^{N(t)} U_k$ is stochastically larger than $\sum_{k=1}^{\tilde{N}(t)} \tilde{U}_k$. Moreover, note $\tilde{p} + \tilde{r} \cdot (x)^+ - \tilde{\alpha} \cdot (x)^- \geq p_j + r_j \cdot (x)^+ - \alpha_j \cdot (x)^-$ for $x \in \mathbb{R}$ and $j \in \mathbb{E}$. We can conclude that $\{Y_t; t \geq 0\}$ is stochastically larger than $\{\tilde{R}_t; t \geq 0\}$. Let $\{Y_t^L; t \geq 0\}$ denote the corresponding controlled process under the dividend strategy L :

$$Y_t^L = R_0 + \int_0^t (\tilde{p} + \tilde{r} \cdot (Y_{s-}^L)^+ - \tilde{\alpha} \cdot (Y_{s-}^L)^-) ds - \sum_{k=1}^{\tilde{N}(t)} \tilde{U}_k. \quad (2.7)$$

Let T_Y^L denote the time of ruin: $T_Y^L = \inf\{Y_t^L \leq -\frac{\tilde{p}}{\tilde{\alpha}}\}$. We define admissible strategies for the stochastic process $\{Y_t; t \geq 0\}$ similarly to stochastic process R . We use $\Pi(Y)$ to denote the set of admissible dividend strategies for $\{Y_t; t \geq 0\}$. Define

$$\tilde{V}(x) = \sup_{L \in \Pi(Y)} E \left[\int_0^{T_Y^L} e^{-\tilde{\delta}t} dL_t^c + \sum_{0 \leq t < T_Y^L} e^{-\tilde{\delta}t} (L_{t+} - L_t) \middle| Y_0 = x \right], \quad (2.8)$$

where $\tilde{\delta} = \min_{j \in \mathbb{E}} \delta_j$. The function $\tilde{V}(x)$ can be interpreted as the value function of the dividend optimization problem of the controlled process (2.7) and the discount rate $\tilde{\delta}$.

The stochastic process $\{Y_t^L; t \geq 0\}$ is a controlled compound Poisson model with investment incomes and debit interest with absolute ruin and is same as the one studied in [10]. Hence, the function $\tilde{V}(x)$ is same as the value function studied in [10] with the premium rate, force of interest, debit rate and discount rate being $\tilde{p}, \tilde{r}, \tilde{\alpha}$ and $\tilde{\delta}$, respectively. Although it was assumed there that the debit rate is greater than the interest rate, the violation of such an assumption would not affect the major results there (all the results up to at least Theorem 4.2 there still hold). From Lemma 4.1(ii) we can see that the value function $\tilde{V}(x)$ is a linear function with slope 1 when x is large. Hence, we can conclude that there exists a $K > 0$ such that $V(x, i) \leq \tilde{V}(x) \leq x + K$ for $x \in \mathbb{R}$ and $i \in \mathbb{E}$. $\square$

3. An auxiliary optimization problem

In this section, we introduce an auxiliary performance functional and study the optimization problem under a new optimization criterion associated with this new performance functional. The results we obtain here will play an essential role in deriving results for the optimization problem under the original performance functional.

Let σ_1 denote the time when the first transition of the Markov chain J occurs,

$$\sigma_1 = \inf\{t > 0 : J_t \neq J_0\}. \quad (3.9)$$

Then given $J_0 = i$, σ_1 is an exponential random variable with mean $\frac{1}{q_i}$.

For any admissible strategy L and any $f : \mathbb{R} \times \mathbb{E} \rightarrow \mathbb{R}^+$, define a new performance functional $\mathcal{J}_f$ such that

$$\mathcal{J}_f(L)(x, i) = E_{(x,i)} \left[\int_0^{T^L \wedge \sigma_1} e^{-\Lambda t} dL_t^c + \sum_{0 \leq t < T^L \wedge \sigma_1} e^{-\Lambda t} (L_{t+} - L_t) + I\{\sigma_1 < T^L\} e^{-\Lambda \sigma_1} f(R_{\sigma_1}^L, J_{\sigma_1}) \right]. \quad (3.10)$$

Define

$$W_f(x, i) = \sup_{L \in \Pi} \mathcal{J}_f(L)(x, i). \quad (3.11)$$

Then $W_f(x, i)$ is the optimal return function under the new performance functional $\mathcal{J}_f$.

For convenience, we use $D_{(R,J)}^L(t_1, t_2)$ to denote the present value at time 0 of all the dividends payable between time t_1 and time t_2 under the strategy L . Then

$$D_{(R,J)}^L(t_1, t_2) = \int_{t_1}^{t_2} e^{-\Lambda_s} dL_s^c + \sum_{t_1 \leq s < t_2} e^{-\Lambda_s} (L_{s+} - L_s). \quad (3.12)$$

Hence,

$$\mathcal{J}_f(L)(x, i) = E_{(x,i)} [D_{(R,J)}^L(0, T^L \wedge \sigma_1) + I\{\sigma_1 < T^L\} e^{-\Lambda_{\sigma_1}} f(R_{\sigma_1}^L, J_{\sigma_1})]. \quad (3.13)$$

The remainder of this section is devoted to studying the optimization problem under the auxiliary performance functional $\mathcal{J}_f$. As the performance functional includes only the dividends paid out before time σ_1 (the time of the first regime switch) and the return function is not affected by the dividends paid out afterwards, this auxiliary optimization problem can be considered as a problem to optimize expected dividends until a random time (σ_1) plus a terminal value for the compound Poisson model with no regime switching.

Let $\mathcal{D}$ denote the set of functions $f: \mathbb{R} \times \mathbb{E} \rightarrow \mathbb{R}^+$, such that for any fixed $i \in \mathbb{E}$, $f(x, i) \equiv 0$ for $x \leq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$, $f(\cdot, i)$ is non-decreasing, and $f(x, i) \leq C_1 x + C_2$ for all x and for some constant $C_1, C_2 > 0$.

Lemma 3.1. For any $f \in \mathcal{D}$,

- (i) $y - x \leq W_f(y, i) - W_f(x, i) \leq W_f(x, i) \left(\left(\frac{p_j + r_j \cdot (y)^+}{p_i + r_i \cdot (x)^+} \right)^{\frac{\lambda_j + q_j + \delta_j}{r_i}} \left(\frac{p_i - \alpha_i \cdot (y)^-}{p_i - \alpha_i \cdot (x)^-} \right)^{\frac{\lambda_i + q_i + \delta_i}{\alpha_i}} - 1 \right)$ for $y > x > -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$ and $i \in \mathbb{E}$;
- (ii) $W_f \in \mathcal{D}$.

Proof. (i) For any $\epsilon > 0$, $x \geq 0$ and $i \in \mathbb{E}$, let $L_{x,i}^\epsilon$ be an ϵ -optimal strategy for the process (R, J) with initial value $(R_0, J_0) = (x, i)$, that is, $\mathcal{J}_f(L_{x,i}^\epsilon)(x, i) \geq W_f(x, i) - \epsilon$. For the process (R, J) starting from the initial value (y, i) ($y > x > -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$), construct a dividend payout strategy $\hat{L}$ as follows: pay a lump sum $y - x$ at the beginning and then apply the strategy $L_{x,i}^\epsilon$ immediately. As a result, we have

$$W_f(y, i) \geq \mathcal{J}_f(\hat{L})(y, i) = y - x + \mathcal{J}_f(L_{x,i}^\epsilon)(x, i) \geq y - x + W_f(x, i) - \epsilon, \quad (3.14)$$

which implies $W_f(y, i) - W_f(x, i) \geq y - x$ for all $y > x > -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$.

Define $\tau(y) = \inf\{t \geq 0 : R_t = y\}$ and recall $\sigma_1 = \inf\{t \geq 0 : J_t \neq J_0\}$. Construct the strategy $\bar{L}_{x,i}$ for the process (R, J) starting from the initial value (x, i) that pays no dividends until the level of surplus reaches y ($y > x$) and then treat the process from this moment onwards as a new process starting with the initial value $(R_{\tau(y)}^{\bar{L}_{x,i}}, J_{\tau(y)})$ and apply the strategy $L_{y, J_{\tau(y)}}^\epsilon$. We can see that the controlled process $(R^{\bar{L}_{x,i}}, J)$ has Markov property on the time interval $[0, \sigma_1]$ with respect to the probability measure $P_{(x,i)}$. Note that ruin occurs immediately when the surplus is $-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$ or smaller.

Hence, $\mathcal{J}_f(L)(y, i) = 0$ for any $y \leq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$, $i \in \mathbb{E}$ and $L \in \Pi$. Then it follows by noting $R_{T^{\bar{L}_{x,i}}}^{\bar{L}_{x,i}} \leq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$ and hence $\mathcal{J}_f(\bar{L}_{x,i})(R_{T^{\bar{L}_{x,i}}}^{\bar{L}_{x,i}}, i) = 0$ that

$$\begin{aligned} \mathcal{J}_f(\bar{L}_{x,i})(x, i) &= E_{(x,i)} [e^{-\Lambda_{\tau(y)}} \mathcal{J}_f(L_{y, J_{\tau(y)}}^\epsilon)(y, J_{\tau(y)}); \tau(y) < T^{\bar{L}_{x,i}} \wedge \sigma_1] + E_{(x,i)} [e^{-\Lambda_{\sigma_1}} f(R_{\sigma_1}^{\bar{L}_{x,i}}, J_{\sigma_1}); \sigma_1 < T^{\bar{L}_{x,i}} \wedge \tau(y)] \\ &\geq \mathcal{J}_f(L_{y,i}^\epsilon)(y, i) E_{(x,i)} [e^{-\delta_i \tau(y)}; \tau(y) < \sigma_1, \tau(y) < S_1], \end{aligned} \quad (3.15)$$

where the last inequality follows by the positivity of the function f and the fact that under strategy $\bar{L}_{x,i}$, $\tau(y) \wedge T^{\bar{L}_{x,i}} \wedge \sigma_1 \geq \tau(y) \wedge S_1 \wedge \sigma_1$.

Note that given there are no transitions of the state of the Markov chain J and no claims arrive before the reserve reaches y , the time it takes for the process (R, J) starting from the initial value (x, i) to reach the surplus level y for the first time is

$$t_i(x, y) = \frac{1}{\alpha_i} \log \left(\frac{\alpha_i \min\{y, 0\} + p_i}{\alpha_i \min\{x, 0\} + p_i} \right) + \frac{1}{r_i} \log \left(\frac{r_i \max\{y, 0\} + p_i}{r_i \max\{x, 0\} + p_i} \right). \quad (3.16)$$

Then, given $(R_0, J_0) = (x, i)$,

$$\tau(y) = t_i(x, y) \quad \text{on } \{\tau(y) < \sigma_1, \tau(y) < S_1\}. \quad (3.17)$$

As a result,

$$\begin{aligned} E_{(x,i)} [e^{-\delta_i \tau(y)} I\{\tau(y) < \sigma_1, \tau(y) < S_1\}] &= E_{(x,i)} [e^{-\delta_i t_i(x,y)} I\{\sigma_1 > t_i(x, y), S_1 > t_i(x, y)\}] \\ &= e^{-(\lambda_i + q_i + \delta_i) t_i(x,y)}, \end{aligned} \quad (3.18)$$

where the last equality follows by the conditional independence of S_1 and σ_1 given $(R_0, J_0) = (x, i)$.

Therefore, using (3.17) and (3.18) and the fact that $\mathcal{J}_f(L_{y,i}^\epsilon)(y, i) \geq W_f(y, i) - \epsilon$, from (3.15) we obtain $W_f(x, i) \geq \mathcal{J}_f(\bar{L}_{x,i})(x, i) \geq e^{-(\lambda_i + q_i + \delta_i)t_i(x, y)} \mathcal{J}_f(L_{y,i}^\epsilon)(y, i) \geq e^{-(\lambda_i + q_i + \delta_i)t_i(x, y)} (W_f(y, i) - \epsilon)$, which implies $W_f(y, i) - W_f(x, i) \leq W_f(x, i)(e^{(\lambda_i + q_i + \delta_i)t_i(x, y)} - 1)$. Combining this and (3.16) completes the proof.

(ii) Note that if $R_0 = x \leq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$, ruin occurs immediately. It follows from (3.10) and (3.11) that $W_f(x, i) = 0$ for $x \leq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$. The non-decreasing property of $W_f(\cdot, i)$ follows immediately by the result in (i).

It remains to show that the function $W_f(\cdot, i)$ is bounded by a linear function. Consider a stochastic process $R_t^{(i)}$ with the following dynamics:

$$R_0^{(i)} = R_0, \quad \text{and} \quad dR_t^{(i)} = (p_i + r_i R_t^{(i)}) dt. \quad (3.19)$$

We can see that given $J_0 = i$, for any $t \leq \sigma_i$, $R_t^{(i)} + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$ is the amount of the surplus at time t if no claims have arrived and no dividends have been paid out before time t . Hence, for any admissible strategy L , given $J_0 = i$, we have

$$L_{t+} \leq R_t^{(i)} + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j} \quad \text{for } t \leq T^L \wedge \sigma_1. \quad (3.20)$$

It follows by (3.19) that given $R_0 = x$,

$$R_t^{(i)} = \left(x + \frac{p_i}{r_i}\right) e^{r_i t} - \frac{p_i}{r_i} < \left(x + \frac{p_i}{r_i}\right) e^{r_i t} \quad \text{for } t \geq 0. \quad (3.21)$$

As L is left continuous with right limits, the process $\{L_{t+}; t \geq 0\}$ is right continuous with left limits and the left limit at time t is $L_{t-} = L_t$. Then $D_{(R,J)}^L(0, T^L \wedge \sigma_1) = \int_0^{T^L \wedge \sigma_1} e^{-\delta_i s} dL_{s+}$. Hence, it follows by integration by parts that

$$\begin{aligned} E_{(x,i)} [D_{(R,J)}^L(0, T^L \wedge \sigma_1)] &= E_{(x,i)} \left[L_{(T^L \wedge \sigma_1)+} e^{-\delta_i (T^L \wedge \sigma_1)} - L_{0+} + \int_0^{T^L \wedge \sigma_1} \delta_i L_{s+} e^{-\delta_i s} ds \right] \\ &\leq E_{(x,i)} \left[\left(\left(x + \frac{p_i}{r_i}\right) e^{r_i (T^L \wedge \sigma_1)} + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j} \right) e^{-\delta_i (T^L \wedge \sigma_1)} \right. \\ &\quad \left. + \delta_i \int_0^\infty \left(\left(x + \frac{p_i}{r_i}\right) e^{r_i s} + \frac{p_i}{r_i} \right) e^{-\delta_i s} ds \right] \\ &\leq C_3 x + C_4, \end{aligned} \quad (3.22)$$

where C_3 and C_4 are two positive constants, the second last inequality follows by (3.20) and (3.21), and the last inequality follows by noting $r_i < \delta_i$.

As $f \in \mathcal{D}$, we have

$$\begin{aligned} E_{(x,i)} [I\{\sigma_1 < T^L\} e^{-\Lambda \sigma_1} f(R_{\sigma_1}^L, J_{\sigma_1})] &\leq E_{(x,i)} [e^{-\delta_i \sigma_1} f(R_{\sigma_1}^L, J_{\sigma_1})] \\ &\leq E_{(x,i)} [e^{-\delta_i \sigma_1} (C_5 R_{\sigma_1}^L + C_6)] \\ &\leq E_{(x,i)} [e^{-\delta_i \sigma_1} (C_5 R_{\sigma_1}^{(i)} + C_6)] \\ &= C_7, \end{aligned} \quad (3.23)$$

where C_7 is a positive constant and the last equality follows by using (3.21) and $r_i < \delta_i$.

It follows by (3.11), (3.13), (3.22) and (3.23) that for some positive constants C_8 and C_9 , $W_f(x, i) \leq C_8 x + C_9$. Consequently, $W_f \in \mathcal{D}$. $\square$

From the above lemma, we can see that for any fixed $i \in \mathbb{E}$ and $f \in \mathcal{D}$, the function $W_f(x, i)$ is nonnegative, nondecreasing, continuous and locally Lipschitz continuous on $(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty)$. Hence, $W_f(x, i)$ is differentiable with respect to x almost everywhere on $[-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty)$ and the derivative at the point where it exists is greater than 1.

We can consider the optimization problem for any fixed $i \in \mathbb{E}$ and $f \in \mathcal{D}$ with respect to the auxiliary performance functional as a problem for a compound Poisson model to maximize the expected dividend payments up to the time of ruin or an exponential random time σ_1 , whichever is earlier, plus a terminal value. Applying the standard arguments in stochastic control theory (e.g. [22]) or a similar method in [1] we can see that for any fixed $i \in \mathbb{E}$ and $f \in \mathcal{D}$, the function $W_f(\cdot, i)$ satisfies the following dynamic programming principle: for any stopping time τ ,

$$\begin{aligned} W_f(x, i) &= \sup_{L \in \Pi} E_{(x,i)} [D_{(R,J)}^L(0, T^L \wedge \sigma_1 \wedge \tau) + e^{-\Lambda \sigma_1} f(R_{\sigma_1}^L, J_{\sigma_1}) I\{\sigma_1 \leq T^L \wedge \tau\} \\ &\quad + e^{-\Lambda T^L \wedge \tau} W_f(R_{T^L \wedge \tau}^L, J_{T^L \wedge \tau}) I\{\sigma_1 > T^L \wedge \tau\}] \end{aligned}$$

$$\begin{aligned}
&= \sup_{L \in \mathcal{I}} E_{(x,i)} [D_{(R,J)}^L(0, T^L \wedge \sigma_1 \wedge \tau) + e^{-\delta_i \sigma_1} f(R_{\sigma_1}^L, J_{\sigma_1}) I\{\sigma_1 \leq T^L \wedge \tau\} \\
&\quad + e^{-\delta_i(T^L \wedge \tau)} W_f(R_{T^L \wedge \tau}^L, i) I\{\sigma_1 > T^L \wedge \tau\}],
\end{aligned} \tag{3.24}$$

and that the associated Hamilton–Jacobi–Bellman (HJB) equation is

$$\max\{1 - \phi'(x), \mathcal{L}_i^f(\phi)(x)\} = 0 \quad \text{for } x \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \tag{3.25}$$

where $\mathcal{L}_i^f$ is an operator defined by

$$\begin{aligned}
\mathcal{L}_i^f(\phi)(x) &= (p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^-) \phi'(x) - (\lambda_i + q_i + \delta_i) \phi(x) \\
&\quad + \lambda_i \int_0^{x + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} \phi(x - u) dF_i(u) + \sum_{j \neq i} q_{ij} f(x, j).
\end{aligned} \tag{3.26}$$

As so far we only know that for any fixed $i \in \mathbb{E}$ and $f \in \mathcal{D}$ the function $W_f(\cdot, i)$ is differentiable with respect to x almost everywhere, so it may not be the classical solution to the Eq. (3.25) and therefore we need to study a weaker version of solution, the viscosity solution, which is defined as follows.

Definition 3.1. For any fixed $x_{\min}, x_{\max} \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty]$ with $x_{\min} < x_{\max}$, any $i \in \mathbb{E}$ and $f \in \mathcal{D}$,

- (i) a continuous function $\underline{u} : (x_{\min}, x_{\max}) \rightarrow \mathbb{R}$ is said to be a viscosity sub-solution (super-solution) of (3.25) on $(x_{\min}, x_{\max})$ if for any $x \in (x_{\min}, x_{\max})$, each continuously differentiable function $\phi : (x_{\min}, x_{\max}) \rightarrow \mathbb{R}$ with $\phi(x) = \underline{u}(x)$ such that $\underline{u} - \phi$ reaches the maximum (minimum) at x satisfies

$$\max\{1 - \phi'(x), \mathcal{L}_i^f(\phi)(x)\} \geq (\leq) 0; \tag{3.27}$$

- (ii) a continuous function $u : (x_{\min}, x_{\max}) \rightarrow \mathbb{R}$ is a viscosity solution of (3.25) on $(x_{\min}, x_{\max})$ if it is both a viscosity sub-solution and a viscosity super-solution of (3.25).

Remark 3.1. If we further define

$$\begin{aligned}
\mathcal{L}_i^f(\phi, u)(x) &= (p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^-) \phi'(x) - (\lambda_i + q_i + \delta_i) u(x) \\
&\quad + \lambda_i \int_0^{x + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} u(x - y) dF_i(y) + \sum_{j \neq i} q_{ij} f(x, j),
\end{aligned} \tag{3.28}$$

we have an equivalent version of definition for viscosity sub- and super-solutions [23,24] by replacing (3.27) by $\max\{1 - \phi'(x), \mathcal{L}_i^f(\phi, \bar{u})(x)\} \geq (\leq) 0$.

The function ϕ in the above definition is called a test function. The following remark states a useful property of test functions [25].

Remark 3.2. Let O be any open interval. For any viscosity super(sub)-solution u on O , there exists a continuously differentiable function $\phi : O \rightarrow \mathbb{R}$ such that $u - \phi$ reaches a minimum (maximum) at $x \in O$ with $\phi'(x) = q$ if and only if $\liminf_{y \downarrow x} \frac{u(y) - u(x)}{y - x} \geq q \geq \limsup_{y \uparrow x} \frac{u(y) - u(x)}{y - x}$ ($\liminf_{y \uparrow x} \frac{u(y) - u(x)}{y - x} \geq q \geq \limsup_{y \downarrow x} \frac{u(y) - u(x)}{y - x}$).

Remark 3.3. (i) We can replace “maximum” and “minimum” in Definition 3.1, Remarks 3.1 and 3.2 by “local maximum” and “local minimum”, respectively.

- (ii) Suppose u is a viscosity solution of (3.25) on $(x_{\min}, x_{\max})$. If for an $x \in (x_{\min}, x_{\max})$, $u(\cdot)$ is differentiable at x , then

$$\max\{1 - u'(x), \mathcal{L}_i^f(u)(x)\} = 0.$$

A function $g(x)$ is said to satisfy the linear growth condition if there exist some positive constants C_1 and C_2 such that $g(x) \leq C_1 x + C_2$ for all x .

We show in the following lemmas that for any fixed $i \in \mathbb{E}$ and $f \in \mathcal{D}$, the function $W_f(x, i)$ is indeed a viscosity solution of (3.25) and it is the unique solution that satisfies certain conditions.

Lemma 3.2. For any fixed $i \in \mathbb{E}$ and $f \in \mathcal{D}$, $W_f(\cdot, i)$ is a viscosity solution to (3.25) on $(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty)$.

Proof. We fix $i \in \mathbb{E}$ and $f \in \mathcal{D}$ throughout the proof. We first show that the function $W_f(\cdot, i)$ is a viscosity super-solution.

Define $a_{i,x,l}(h) = xe^{r_i h} + (p_i - l) \int_0^h e^{r_i(h-s)} ds$ if $x > 0$ and $l \geq 0$, or $x = 0$ and $0 \leq l \leq p_i$, and $a_{i,x,l}(h) = xe^{\alpha_i h} + (p_i - l) \int_0^h e^{\alpha_i(h-s)} ds$ if $-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j} < x < 0$ and $l \geq 0$, or $x = 0$ and $l > p_i$. For any fixed $x > -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$ and $l \geq 0$,

we can always choose a small enough $h > 0$ such that either $a_{i,x,l}(s) \geq 0$ for all $s \in (0, h]$ or $-\max_{j \in E} \frac{p_j}{\alpha_j} < a_{i,x,l}(s) < 0$ for all $s \in (0, h]$. Consider a dividend payout strategy $\bar{L}$ that pays out dividends continuously at a constant rate l until time $\sigma_1 \wedge S_1 \wedge h$. Then under the dividend strategy $\bar{L}$, we can see that given $(R_0, J_0) = (x, i)$,

$$R_t^{\bar{L}} = a_{i,x,l}(t) \quad \text{for } t < \sigma_1 \wedge S_1 \wedge h, \quad \text{and} \quad R_{S_1}^{\bar{L}} = a_{i,x,l}(S_1) - U_1 \text{ on } \{S_1 < \sigma_1\}, \quad (3.29)$$

where U_1 represents the amount of the first claim. From (3.29) it is clear that under the dividend strategy $\bar{L}$, ruin will not occur before time $\sigma_1 \wedge S_1 \wedge h$, and hence $T^{\bar{L}} \wedge \sigma_1 \wedge S_1 \wedge h = \sigma_1 \wedge S_1 \wedge h$. Then by the dynamic programme principle (3.24), we have

$$\begin{aligned} W_f(x, i) &= \sup_{L \in \Pi} E_{(x,i)}[D_{R,J}^L(0, T^L \wedge \sigma_1 \wedge S_1 \wedge h) + e^{-\delta_i \sigma_1} f(R_{\sigma_1}^L, J_{\sigma_1}) I\{\sigma_1 \leq T^L \wedge S_1 \wedge h\} \\ &\quad + e^{-\delta_i(T^L \wedge S_1 \wedge h)} W_f(R_{T^L \wedge S_1 \wedge h}^L, i) I\{\sigma_1 > T^L \wedge S_1 \wedge h\}] \\ &\geq E_{(x,i)}[D_{R,J}^{\bar{L}}(0, T^{\bar{L}} \wedge \sigma_1 \wedge S_1 \wedge h) + e^{-\delta_i \sigma_1} f(R_{\sigma_1}^{\bar{L}}, J_{\sigma_1}) I\{\sigma_1 \leq T^{\bar{L}} \wedge S_1 \wedge h\} \\ &\quad + e^{-\delta_i(T^{\bar{L}} \wedge S_1 \wedge h)} W_f(R_{T^{\bar{L}} \wedge S_1 \wedge h}^{\bar{L}}, i) I\{\sigma_1 > T^{\bar{L}} \wedge S_1 \wedge h\}] \\ &= E_{(x,i)}[D_{R,J}^{\bar{L}}(0, \sigma_1 \wedge S_1 \wedge h) + e^{-\delta_i \sigma_1} f(R_{\sigma_1}^{\bar{L}}, J_{\sigma_1}) I\{\sigma_1 \leq S_1 \wedge h\} \\ &\quad + e^{-\delta_i(S_1 \wedge h)} W_f(R_{S_1 \wedge h}^{\bar{L}}, i) I\{\sigma_1 > S_1 \wedge h\}]. \end{aligned} \quad (3.30)$$

Write $A_1 = \{\sigma_1 \wedge S_1 \wedge h = h\}$, $A_2 = \{\sigma_1 \wedge S_1 \wedge h = S_1\}$ and $A_3 = \{\sigma_1 \wedge S_1 \wedge h = \sigma_1\}$, and define

$$\begin{aligned} I_k &= E_{(x,i)} \left[D_{R,J}^{\bar{L}}(0, \sigma_1 \wedge S_1 \wedge h) + e^{-\delta_i(S_1 \wedge h)} W_f(R_{S_1 \wedge h}^{\bar{L}}, i) I\{\sigma_1 > S_1 \wedge h\}; A_k \right], \quad k = 1, 2, \\ I_3 &= E_{(x,i)} \left[D_{R,J}^{\bar{L}}(0, \sigma_1 \wedge S_1 \wedge h) + e^{-\delta_i \sigma_1} f(R_{\sigma_1}^{\bar{L}}, J_{\sigma_1}) I\{\sigma_1 \leq S_1 \wedge h\}; A_3 \right]. \end{aligned} \quad (3.31)$$

Then it follows by (3.30) that

$$W_f(x, i) \geq I_1 + I_2 + I_3. \quad (3.32)$$

Notice that given $J_0 = i$, σ_1 follows the exponential distribution with mean $\frac{1}{\lambda_i}$ and $\sigma_1 \wedge S_1$ follows the exponential distribution with mean $\frac{1}{\lambda_i + q_i}$, respectively. Then using (3.29) we get

$$\begin{aligned} I_1 &= E_{(x,i)} \left[\int_0^h e^{-\delta_i s} l ds + e^{-\delta_i h} W_f(R_h^{\bar{L}}, i); A_1 \right] \\ &= \frac{l}{\delta_i} e^{-(\lambda_i + q_i)h} (1 - e^{-\delta_i h}) + e^{-(\lambda_i + q_i + \delta_i)h} W_f(a_{i,x,l}(h), i). \end{aligned} \quad (3.33)$$

Note $W_f(y, i) = 0$ for $y < -\max_{j \in E} \frac{p_j}{\alpha_j}$. By first conditioning on σ_1 and then S_1 , using (3.29) we obtain

$$I_2 = \int_0^\infty q_i e^{-q_i t} dt \int_0^{t \wedge h} \lambda_i e^{-\lambda_i s} ds \left(\int_0^s e^{-\delta_i u} l du + \int_0^{a_{i,x,l}(s) + \max_{j \in E} \frac{p_j}{\alpha_j}} W_f(a_{i,x,l}(s) - u, i) dF_i(u) \right). \quad (3.34)$$

By conditioning on σ_1 and using (3.29) we can obtain

$$I_3 = \int_0^h q_i e^{-q_i t} e^{-\lambda_i t} \left(\int_0^t e^{-\delta_i s} l ds + e^{-\delta_i t} \sum_{j \neq i} \frac{q_{ij}}{q_i} f(a_{i,x,l}(t), j) \right) dt. \quad (3.35)$$

Let ϕ be any continuously differentiable function with $\phi(x) = W_f(x, i)$ such that $W_f(\cdot, i) - \phi(\cdot)$ reaches the minimum at x . Hence, $W_f(y, i) \geq \phi(y)$ for all $y \geq -\max_{j \in E} \frac{p_j}{\alpha_j}$. Then from (3.33) we have

$$\begin{aligned} \lim_{h \downarrow 0} \frac{I_1 - W_f(x, i)}{h} &= \lim_{h \downarrow 0} \frac{I_1 - \phi(x)}{h} \geq \lim_{h \downarrow 0} \left(\frac{l}{\delta_i} e^{-(\lambda_i + q_i)h} \frac{1 - e^{-\delta_i h}}{h} + \frac{e^{-(\lambda_i + q_i + \delta_i)h} \phi(a_{i,x,l}(h)) - \phi(x)}{h} \right) \\ &= l + (p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^- - l) \phi'(x) - (\lambda_i + q_i + \delta_i) \phi(x). \end{aligned} \quad (3.36)$$

From (3.34) and (3.35) it follows that

$$\lim_{h \downarrow 0} \frac{I_2}{h} = \lambda_i \int_0^{x + \max_{j \in E} \frac{p_j}{\alpha_j}} W_f(x - u, i) dF_i(u), \quad \lim_{h \downarrow 0} \frac{I_3}{h} = \sum_{j \neq i} q_{ij} f(x, j). \quad (3.37)$$

Combining (3.32), (3.36) and (3.37) gives

$$0 \geq l(1 - \phi'(x)) + (p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^-)\phi'(x) - (\lambda_i + q_i + \delta_i)W_f(x, i) \\ + \lambda_i \int_0^{x + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} W_f(x - u, i) dF_i(u) + \sum_{j \neq i} q_{ij} f(x, j). \quad (3.38)$$

By setting $l = 0$ in (3.38) we get

$$(p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^-)\phi'(x) - (\lambda_i + q_i + \delta_i)W_f(x, i) \\ + \lambda_i \int_0^{x + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} W_f(x - u, i) dF_i(u) + \sum_{j \neq i} q_{ij} f(x, j) \leq 0. \quad (3.39)$$

By choosing l large enough, the inequality in (3.38) implies $\phi'(x) \geq 1$. By using this and (3.39) we can conclude that $W_f(x, i)$ is a viscosity super-solution.

The proof for $W_f(\cdot, i)$ being a viscosity sub-solution follows closely with the proof of the viscosity sub-solution property of the optimal return function V for the compound Poisson model, subject to minor adaptation. We do not repeat the proof here. Readers are directed to Proposition 5.1 in [1] or Section 2.3 of [8] for details. $\square$

Lemma 3.3. For any fixed $i \in \mathbb{E}$, $f \in \mathcal{D}$ and $x_{\max} \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty]$, if $\bar{u}(x)$ is a locally Lipschitz continuous nondecreasing (i) viscosity super-solution ((ii) viscosity solution) to the Eq. (3.25) on $(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max})$ that satisfies $\bar{u}(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}) = 0$ and in the case $x_{\max} = +\infty$, the linear growth condition, then $\bar{u}(x) \geq W_f(x, i)$ ((ii) $\bar{u}(x) = W_f(x, i)$) for all $x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max})$.

Proof. (i) Suppose $\bar{u}(x)$ is a viscosity super-solution. It is sufficient to show that for any small $h > 0$, $\bar{u}(x) \geq W_f(x, i)$ for $x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}$. We use a proof by contradiction. Suppose for some $x_0 \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}$,

$$W_f(x_0, i) > \bar{u}(x_0). \quad (3.40)$$

For any constant $\gamma > 0$, define functions for $x \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$,

$$\bar{u}_\gamma(x) = e^{-\gamma x} \bar{u}(x) \quad \text{and} \quad W_{f,i,\gamma}(x) = e^{-\gamma x} W_f(x, i). \quad (3.41)$$

As the functions $\bar{u}(\cdot)$ and $W_f(\cdot, i)$ are locally Lipschitz continuous, and in the case $x_{\max} = +\infty$, bounded by a linear function, it follows immediately that $\bar{u}_\gamma(x)$ and $W_{f,i,\gamma}(x)$ are both Lipschitz continuous on $(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}$ and, in the case $x_{\max} = +\infty$,

$$\lim_{x \rightarrow \infty} W_{f,i,\gamma}(x) = 0, \quad \lim_{x \rightarrow \infty} \bar{u}_\gamma(x) = 0. \quad (3.42)$$

As a result, we can find some constant $K > 0$ such that for $x, y \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}$,

$$|\bar{u}_\gamma(y) - \bar{u}_\gamma(x)| \leq K|y - x|, \quad |W_{f,i,\gamma}(y) - W_{f,i,\gamma}(x)| \leq K|y - x|. \quad (3.43)$$

We use a similar method used in [1] to construct viscosity sub- and super-solutions. For any $t > 0$, define m_t by

$$m_t(x, y) = W_{f,i,\gamma}(x) - \bar{u}_\gamma(y) - \frac{t}{2}(x - y)^2 - \frac{2K}{t^2(y - x)^2 + t}. \quad (3.44)$$

Note that by (3.40) we have $W_{f,i,\gamma}(x_0) - \bar{u}_\gamma(x_0) > 0$ for small $\gamma > 0$, and that

$$\sup_{x, y \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} m_t(x, y) \geq \sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} m_t(x, x) \\ \geq W_{f,i,\gamma}(x_0) - \bar{u}_\gamma(x_0) - \frac{2K}{t}. \quad (3.45)$$

These together with (3.42) imply that $\liminf_{t \uparrow \infty} \sup_{x, y \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} m_t(x, y) > 0$, and that we can find $\sigma > 0$, large enough t and $x_t, y_t \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}$ such that

$$m_t(x_t, y_t) = \sup_{x, y \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} m_t(x, y) > \sigma. \quad (3.46)$$

Let t_n be a sequence tending to ∞ as n goes to ∞ , such that the limits of x_{t_n} and y_{t_n} exist. Define $\bar{x} = \lim_{n \uparrow \infty} x_{t_n}$ and $\bar{y} = \lim_{n \uparrow \infty} y_{t_n}$. We can show that

$$\bar{x} = \bar{y}. \quad (3.47)$$

If this is not true, then $\lim_{n \rightarrow \infty} (x_{t_n} - y_{t_n})^2 > 0$, and hence

$$\begin{aligned} \lim_{n \rightarrow \infty} m_{t_n}(x_{t_n}, y_{t_n}) &= \lim_{n \rightarrow \infty} \left(W_{f,i,\gamma}(x_{t_n}) - \bar{u}_\gamma(y_{t_n}) - \frac{t_n}{2}(x_{t_n} - y_{t_n})^2 - \frac{2K}{t_n^2(y_{t_n} - x_{t_n})^2 + t_n} \right) \\ &= W_{f,i,\gamma}(\bar{x}) - \bar{u}_\gamma(\bar{y}) - \lim_{n \rightarrow \infty} \frac{t_n}{2}(x_{t_n} - y_{t_n})^2 = -\infty, \end{aligned} \quad (3.48)$$

which is a contradiction to (3.46).

Notice that

$$\begin{aligned} \limsup_{n \rightarrow \infty} m_{t_n}(x_{t_n}, y_{t_n}) &= \lim_{n \rightarrow \infty} \left(W_{f,i,\gamma}(x_{t_n}) - \bar{u}_\gamma(y_{t_n}) - \frac{t_n}{2}(x_{t_n} - y_{t_n})^2 - \frac{2K}{t_n^2(y_{t_n} - x_{t_n})^2 + t_n} \right) \\ &\leq W_{f,i,\gamma}(\bar{x}) - \bar{u}_\gamma(\bar{x}), \end{aligned} \quad (3.49)$$

which together with (3.46) implies

$$W_{f,i,\gamma}(\bar{x}) - \bar{u}_\gamma(\bar{x}) > 0. \quad (3.50)$$

It follows by Lemma 3.1(ii) that $W_{f,i,\gamma}(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}) - \bar{u}_\gamma(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}) = 0$. Then $\bar{x} > -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$. Otherwise we will obtain a contradiction to (3.50). In the case $x_{\max} = +\infty$, we have $W_{f,i,\gamma}(+\infty) - \bar{u}_\gamma(+\infty) = 0$ by (3.42). Combining this and (3.50) we can see $\bar{x} < \infty$, in the case $x_{\max} = +\infty$. In conclusion, $\bar{x} = \bar{y} \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}$. Therefore, we can find a sequence $\{t_n\}$ tending to $+\infty$ such that both $\{x_{t_n}\}$ and $\{y_{t_n}\}$ are convergent and

$$x_{t_n}, y_{t_n} \in \left(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h \right] - \{\infty\}. \quad (3.51)$$

We can show that

$$W_{f,i,\gamma}(\bar{x}) - \bar{u}_\gamma(\bar{x}) = \sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x)). \quad (3.52)$$

Suppose the contrary, that is

$$W_{f,i,\gamma}(\bar{x}) - \bar{u}_\gamma(\bar{x}) < \sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x)). \quad (3.53)$$

By noting that $W_{f,i,\gamma}(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}) - \bar{u}_\gamma(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}) = 0$, that $W_{f,i,\gamma}(x) - \bar{u}_\gamma(x)$ is bounded and that $\lim_{x \rightarrow \infty} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x)) = 0$ (see (3.42)), we can find an $x_1 \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}$ and $x_1 \neq \bar{x}$ such that

$$W_{f,i,\gamma}(x_1) - \bar{u}_\gamma(x_1) = \sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x)). \quad (3.54)$$

Let $\{x'_{t_n}\}$ be a sequence converging to x_1 and define $y'_{t_n} = x'_{t_n} + y_{t_n} - x_{t_n}$. Then by (3.47) we have $\lim_{n \rightarrow \infty} y'_{t_n} = x_1$. Hence, it follows by (3.53) and (3.54) that for large enough n , $W_{f,i,\gamma}(x'_{t_n}) - \bar{u}_\gamma(y'_{t_n}) > W_{f,i,\gamma}(x_{t_n}) - \bar{u}_\gamma(y_{t_n})$. As a result, for large enough n ,

$$\begin{aligned} m_{t_n}(x'_{t_n}, y'_{t_n}) &= W_{f,i,\gamma}(x'_{t_n}) - \bar{u}_\gamma(y'_{t_n}) - \frac{t_n}{2}(x'_{t_n} - y'_{t_n})^2 - \frac{2K}{t_n^2(x'_{t_n} - y'_{t_n})^2 + t_n} \\ &> W_{f,i,\gamma}(x_{t_n}) - \bar{u}_\gamma(y_{t_n}) - \frac{t_n}{2}(x_{t_n} - y_{t_n})^2 - \frac{2K}{t_n^2(x_{t_n} - y_{t_n})^2 + t_n} \\ &= m_{t_n}(x_{t_n}, y_{t_n}), \end{aligned} \quad (3.55)$$

which contradicts (3.46).

Now consider two functions $e^{\gamma x} \bar{\phi}_{t,\gamma}(x)$ and $e^{\gamma x} \phi_{t,\gamma}(x)$, where

$$\begin{aligned}\phi_{t,\gamma}(x) &= \bar{u}_\gamma(y_t) + \frac{t}{2}(x - y_t)^2 + \frac{2K}{t^2(y_t - x)^2 + t} + m_t(x_t, y_t), \\ \bar{\phi}_{t,\gamma}(y) &= W_{f,i,\gamma}(x_t) - \frac{t}{2}(x_t - y)^2 - \frac{2K}{t^2(x_t - y)^2 + t} - m_t(x_t, y_t).\end{aligned}$$

We can see that $\phi_{t_n,\gamma}$ and $\bar{\phi}_{t_n,\gamma}$ are both continuously differentiable, that $W_f(x, i) - e^{\gamma x} \phi_{t_n,\gamma}(x) = e^{\gamma x} (m_{t_n}(x, y_{t_n}) - m_{t_n}(x_{t_n}, y_{t_n}))$ attains its local maximum 0 at $x = x_{t_n}$, and that $\bar{u}(y) - e^{\gamma y} \bar{\phi}_{t_n,\gamma}(y) = e^{\gamma y} (-m_{t_n}(x_{t_n}, y) + m_{t_n}(x_{t_n}, y_{t_n}))$ attains its local minimum 0 at $y = y_{t_n}$. By the fact that the functions $W_f(\cdot, i)$ and $\bar{u}(\cdot)$ are respectively the viscosity solution and super-solution of (3.25) on $(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max})$ and by the definition for viscosity solutions and (3.51) we have for small $\gamma > 0$ and large n ,

$$\max\{\xi_1(f, i, \gamma, n), \zeta_1(f, i, \gamma, n)\} \geq 0 \geq \max\{\xi_2(f, i, \gamma, n), \zeta_2(f, i, \gamma, n)\}, \quad (3.56)$$

where

$$\begin{aligned}\xi_1(f, i, \gamma, n) &= 1 - e^{\gamma x_{t_n}} \left(\gamma \phi_{t_n,\gamma}(x_{t_n}) + \phi'_{t_n,\gamma}(x_{t_n}) \right), \\ \zeta_1(f, i, \gamma, n) &= (p_i + r_i \cdot (x_{t_n})^+ - \alpha_i \cdot (x_{t_n})^-) \left(\gamma \phi_{t_n,\gamma}(x_{t_n}) + \phi'_{t_n,\gamma}(x_{t_n}) \right) e^{\gamma x_{t_n}} - (\lambda_i + q_i + \delta_i) W_f(x_{t_n}, i) \\ &\quad + \lambda_i \int_0^{x_{t_n} + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} W_f(x_{t_n} - y, i) dF_i(y) + \sum_{j \neq i} q_{ij} f(x, j), \\ \xi_2(f, i, \gamma, n) &= 1 - e^{\gamma y_{t_n}} \left(\gamma \bar{\phi}_{t_n,\gamma}(y_{t_n}) + \bar{\phi}'_{t_n,\gamma}(y_{t_n}) \right),\end{aligned}$$

and

$$\begin{aligned}\zeta_2(f, i, \gamma, n) &= (p_i + r_i \cdot (y_{t_n})^+ - \alpha_i \cdot (y_{t_n})^-) \left(\gamma \bar{u}_\gamma(y_{t_n}) + \bar{\phi}'_{t_n,\gamma}(y_{t_n}) \right) e^{\gamma y_{t_n}} \\ &\quad - (\lambda_i + q_i + \delta_i) \bar{u}(y_{t_n}) + \lambda_i \int_0^{y_{t_n} + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} \bar{u}(y_{t_n} - y) dF_i(y) + \sum_{j \neq i} q_{ij} f(x, j).\end{aligned}$$

Notice that

$$\phi'_{t_n,\gamma}(x_{t_n}) = \bar{\phi}'_{t_n,\gamma}(y_{t_n}) = t_n(x_{t_n} - y_{t_n}) + \frac{4K(y_{t_n} - x_{t_n})}{(t_n(y_{t_n} - x_{t_n})^2 + 1)^2}. \quad (3.57)$$

Recall $\lim_{n \rightarrow \infty} x_{t_n} = \bar{x} = \bar{y} = \lim_{n \rightarrow \infty} y_{t_n}$ and $\bar{x} \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}$. Therefore, using (3.41) we have

$$\begin{aligned}\lim_{n \rightarrow \infty} (e^{-\gamma y_{t_n}} \zeta_2(f, i, \gamma, n) - e^{-\gamma x_{t_n}} \zeta_1(f, i, \gamma, n)) &= (p_i + r_i \cdot (\bar{x})^+ - \alpha_i \cdot (\bar{x})^-) (\gamma \bar{u}_\gamma(\bar{x}) - \gamma W_{f,i,\gamma}(\bar{x})) + (\lambda_i + q_i + \delta_i) (W_{f,i,\gamma}(\bar{x}) - \bar{u}_\gamma(\bar{x})) \\ &\quad + \lambda_i \int_0^{\bar{x} + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} (\bar{u}_\gamma(\bar{x} - y) - W_{f,i,\gamma}(\bar{x} - y)) e^{-\gamma y} dF_i(y) \\ &\geq [\lambda_i + q_i + \delta_i - \gamma (p_i + r_i \cdot (\bar{x})^+ - \alpha_i \cdot (\bar{x})^-)] (W_{f,i,\gamma}(\bar{x}) - \bar{u}_\gamma(\bar{x})) \\ &\quad - \lambda_i \int_0^{\bar{x} + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} \sup_{x \in [-\frac{p_i}{\alpha_i}, x_{\max} - h] - \{\infty\}} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x)) e^{-\gamma y} dF_i(y).\end{aligned} \quad (3.58)$$

It follows by (3.46) that for any $x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}$,

$$m_{t_n}(x_{t_n}, y_{t_n}) = \max_{x, y \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} m_{t_n}(x, y) \geq m_{t_n}(x, x) = W_{f,i,\gamma}(x) - \bar{u}_\gamma(x) - \frac{2K}{t_n}, \quad (3.59)$$

which implies

$$m_{t_n}(x_{t_n}, y_{t_n}) \geq \sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x)) - \frac{2K}{t_n}. \quad (3.60)$$

Therefore,

$$\lim_{n \rightarrow \infty} m_{t_n}(x_{t_n}, y_{t_n}) \geq \sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x)). \quad (3.61)$$

By choosing small enough $\gamma \in (0, \frac{\delta_i}{2|p_i + r_i \cdot (\bar{x})^+ - \alpha_i \cdot (\bar{x})^-|})$, it follows immediately from (3.58) and (3.52) that

$$\begin{aligned} \lim_{n \rightarrow \infty} (\zeta_2(f, i, \gamma, n) - \zeta_1(f, i, \gamma, n)) &\geq [\lambda_i + q_i + \delta_i - \gamma(p_i + r_i \cdot (\bar{x})^+ - \alpha_i \cdot (\bar{x})^-) \vee 0 - \lambda_i] \\ &\quad \times \sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x)) \\ &> \left(q_i + \frac{\delta_i}{2}\right) \sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x)) > 0, \end{aligned} \quad (3.62)$$

where the last inequality follows by

$$\sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x)) > 0 \quad (3.63)$$

due to (3.40). Hence, $\zeta_2(f, i, \gamma, n) > \zeta_1(f, i, \gamma, n)$ for large n and small $\gamma > 0$. This combined with (3.56) implies $\xi_1(f, i, \gamma, n) \geq \xi_2(f, i, \gamma, n)$. Then by noticing $W_{f,i,\gamma}(x_{t_n}) = \underline{\phi}_{t_n,\gamma}(x_{t_n})$ and $\bar{u}_\gamma(y_{t_n}) = \bar{\phi}_{t_n,\gamma}(y_{t_n})$, we can obtain

$$e^{\gamma x_{t_n}}(\gamma W_{f,i,\gamma}(x_{t_n}) + \underline{\phi}'_{t_n,\gamma}(x_{t_n})) \leq e^{\gamma y_{t_n}}(\gamma \bar{u}_\gamma(y_{t_n}) + \bar{\phi}'_{t_n,\gamma}(y_{t_n})). \quad (3.64)$$

It follows immediately from (3.57) and (3.64) that

$$e^{\gamma x_{t_n}} W_{f,i,\gamma}(x_{t_n}) - e^{\gamma y_{t_n}} \bar{u}_\gamma(y_{t_n}) \leq \frac{\frac{4K}{(t_n(y_{t_n} - x_{t_n})^2 + 1)^2} - t_n}{\gamma} (y_{t_n} - x_{t_n})(e^{\gamma y_{t_n}} - e^{\gamma x_{t_n}}). \quad (3.65)$$

By noticing (3.63) and $\lim_{n \rightarrow \infty} t_n = +\infty$, we can find a positive integer N_1 such that for all $n \geq N_1$,

$$t_n > \max \left(4K, \frac{6K}{\sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x))} \right). \quad (3.66)$$

Since $(y_{t_n} - x_{t_n})(e^{\gamma y_{t_n}} - e^{\gamma x_{t_n}})$ is always nonnegative, then it follows by (3.65) that

$$e^{\gamma x_{t_n}} W_{f,i,\gamma}(x_{t_n}) - e^{\gamma y_{t_n}} \bar{u}_\gamma(y_{t_n}) \leq 0 \quad \text{for } n \geq N_1. \quad (3.67)$$

For any $\epsilon > 0$, let $N_2(\epsilon)$ be an integer such that for all $n \geq N_2(\epsilon)$,

$$|e^{\gamma x_{t_n}} - e^{\gamma \bar{x}}| < \epsilon, \quad |e^{\gamma y_{t_n}} - e^{\gamma \bar{x}}| < \epsilon \quad \text{and} \quad |\bar{u}_\gamma(x_{t_n}) - \bar{u}_\gamma(y_{t_n})| < \epsilon.$$

Then for $n \geq N_2(\epsilon)$, we have

$$\begin{aligned} &W_{f,i,\gamma}(x_{t_n})(1 - e^{\gamma x_{t_n}}) - \bar{u}_\gamma(y_{t_n})(1 - e^{\gamma y_{t_n}}) \\ &= W_{f,i,\gamma}(x_{t_n})(1 - e^{\gamma x_{t_n}}) - \bar{u}_\gamma(x_{t_n})(1 - e^{\gamma y_{t_n}}) + (\bar{u}_\gamma(x_{t_n}) - \bar{u}_\gamma(y_{t_n}))(1 - e^{\gamma y_{t_n}}) \\ &< (W_{f,i,\gamma}(x_{t_n}) - \bar{u}_\gamma(x_{t_n}))(1 - e^{\gamma \bar{x}}) + (W_{f,i,\gamma}(x_{t_n}) + \bar{u}_\gamma(x_{t_n}) + 1)\epsilon. \end{aligned} \quad (3.68)$$

Note that $1 - e^{\gamma \bar{x}} < \frac{1}{3}$ for $0 < \gamma < \frac{\ln(3/2)}{(\bar{x})^-}$. Hence, for $0 < \gamma < \frac{\ln(3/2)}{(\bar{x})^-}$,

$$(W_{f,i,\gamma}(x_{t_n}) - \bar{u}_\gamma(x_{t_n}))(1 - e^{\gamma \bar{x}}) < \frac{1}{3} \sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x)). \quad (3.69)$$

As $W_{f,i,\gamma}$ and $\bar{u}_\gamma$ are both bounded, by (3.63) we have

$$\frac{\sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x))}{\sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} |W_{f,i,\gamma}(x) + \bar{u}_\gamma(x) + 1|} > 0. \quad (3.70)$$

It follows by (3.44) and (3.66)–(3.70) that for

$$\epsilon < \frac{\sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x))}{3 \sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} |W_{f,i,\gamma}(x) + \bar{u}_\gamma(x) + 1|},$$

for any large $n \geq \max\{N_1, N_2(\epsilon)\}$ and for any small $\gamma \in (0, \frac{\ln(3/2)}{(\bar{x})} \wedge 1)$,

$$\begin{aligned} m_{t_n}(x_{t_n}, y_{t_n}) &\leq W_{f,i,\gamma}(x_{t_n}) - \bar{u}_\gamma(y_{t_n}) \\ &= e^{\gamma x_{t_n}} W_{f,i,\gamma}(x_{t_n}) - e^{\gamma y_{t_n}} \bar{u}_\gamma(y_{t_n}) + W_{f,i,\gamma}(x_{t_n})(1 - e^{\gamma x_{t_n}}) - \bar{u}_\gamma(y_{t_n})(1 - e^{\gamma y_{t_n}}) - \frac{2K}{t_n} + \frac{2K}{t_n} \\ &< \sup_{x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max} - h] - \{\infty\}} (W_{f,i,\gamma}(x) - \bar{u}_\gamma(x)) - \frac{2K}{t_n}, \end{aligned} \quad (3.71)$$

which is a contradiction to (3.60).

(ii) If $\bar{u}(x)$ is a viscosity solution, it is also a viscosity super-solution and therefore by (i) we know $\bar{u}(x) \geq W_f(x, i)$ for all $x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max}]$. By interchanging the positions of $\bar{u}(\cdot)$ and $W_f(\cdot, i)$ in the proof of (i) we can show $W_f(x, i) \geq \bar{u}(x)$ for $x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_{\max}]$. This completes the proof. $\square$

Remark 3.4. It follows by Lemma 3.3(ii) that for any fixed $i \in \mathbb{E}$ and $f \in \mathcal{D}$, $W_f(\cdot, i)$ is the unique viscosity solution to (3.25) on $(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty)$ that is locally Lipschitz continuous, takes value 0 at $x = -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$, and satisfies the linear growth condition.

For any fixed i and f , define an operator $\mathcal{G}_i^f$ by

$$\mathcal{G}_i^f(\phi)(x) = p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^- - (\lambda_i + q_i + \delta_i)\phi(x) + \lambda_i \int_0^{x + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} \phi(x - y) dF_i(y) + \sum_{j \neq i} q_{ij} f(x, j). \quad (3.72)$$

We can define the crucial sets similar to the compound Poisson case that are essential for the construction of the candidate for the optimal strategy.

Definition 3.2. Define for $f \in \mathcal{D}$ and $i \in \mathbb{E}$,

$$\begin{aligned} \mathcal{A}_i^f &= \{x \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty) : \mathcal{G}_i^f(W_f(\cdot, i))(x) = 0\}, \\ \mathcal{B}_i^f &= \{x \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty) : W_f'(x, i) = 1 \text{ and } \mathcal{G}_i^f(W_f(\cdot, i))(x) < 0\}, \text{ and} \\ \mathcal{C}_i^f &= (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty) - \mathcal{A}_i^f \cup \mathcal{B}_i^f. \end{aligned}$$

Recall that the set $\mathcal{D}$ is defined right above Lemma 3.1. Define a subset $\tilde{\mathcal{D}} = \{f \in \mathcal{D} : \text{for any fixed } i \in \mathbb{E}, f(x, i) \leq x + C \text{ for all } x \text{ and some constant } C > 0\}$.

Lemma 3.4. For fixed $f \in \mathcal{D}$ and $i \in \mathbb{E}$,

- (a) the set $\mathcal{B}_i^f$ is left-open;
- (b) if $f \in \tilde{\mathcal{D}}$, the set $\mathcal{B}_i^f$ is not empty and there exists a y such that $(y, \infty) \subset \mathcal{B}_i^f$;
- (c) the set $\mathcal{A}_i^f$ is closed;
- (d) if $(x_0, x_1] \subset \mathcal{B}_i^f$ and $x_0 \notin \mathcal{B}_i^f \cup \{-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}\}$, then $x_0 \in \mathcal{A}_i^f$;
- (e) the set $\mathcal{A}_i^f$ is not empty;
- (f) the set $\mathcal{C}_i^f$ is right-open.

Proof. (a) It is sufficient to show that for any $x \in \mathcal{B}_i^f$, $(x - h, x) \subset \mathcal{B}_i^f$ for small enough $h > 0$. Define for any $z \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$, a function

$$B_{f,i,z}(y) = \begin{cases} W_f(y, i) & -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j} \leq y < z, \\ W_f(z, i) + y - z & y \geq z. \end{cases} \quad (3.73)$$

For any $y \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x - h)$, we use $\phi_y(z)$ to denote any continuously differentiable function such that $\phi_y(y) = B_{f,i,x-h}(y)$ and $\phi_y(z) \leq B_{f,i,x-h}(z)$ for all $z \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$. As $W_f(\cdot, i)$ is a viscosity solution on $(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty)$, then

by Remark 3.1 and (3.73) we have

$$\max \left\{ 1 - \phi'_y(y), \mathcal{L}_i^f(\phi_y, B_{f,i,x-h})(y) \right\} \leq 0 \quad \text{for } y \in \left(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x - h \right). \quad (3.74)$$

For any y , if $B_{f,i,x-h}(y)$ is differentiable at y , it is not hard to see that, $\phi'_y(y) = B'_{f,i,x-h}(y)$. Otherwise, the function $\phi_y(\cdot)$ is larger than $B_{f,i,x-h}(\cdot)$ either in the left-hand side neighborhood or in the right-hand side neighborhood of the point y . Note that

$$B'_{f,i,x-h}(y) = 1 \quad \text{for } y \in (x - h, x]. \quad (3.75)$$

Hence,

$$\phi'_y(y) = B'_{f,i,x-h}(y) = 1 \quad \text{for } y \in (x - h, x]. \quad (3.76)$$

As $\phi_{x-h}(x - h) = B_{f,i,x-h}(x - h)$ and $\phi_{x-h}(\cdot) \leq B_{f,i,x-h}(\cdot)$, it follows by (3.73) and Lemma 3.1(i) that

$$\phi'_{x-h}(x - h) \geq \limsup_{z \uparrow (x-h)} \frac{B_{f,i,x-h}(x - h) - B_{f,i,x-h}(z)}{x - h - z} = \limsup_{z \uparrow (x-h)} \frac{W_f(x - h, i) - W_f(z, i)}{x - h - z} \geq 1, \quad (3.77)$$

and

$$\phi'_{x-h}(x - h) \leq \liminf_{z \downarrow (x-h)} \frac{B_{f,i,x-h}(x - h) - B_{f,i,x-h}(z)}{x - h - z} = 1.$$

Hence,

$$\phi'_{x-h}(x - h) = 1. \quad (3.78)$$

It follows by (3.28), (3.76) and (3.78) that

$$\begin{aligned} \mathcal{L}_i^f(\phi_y, B_{f,i,x-h})(y) &= (p_i + r_i \cdot (y)^+ - \alpha_i \cdot (y)^-) - (\lambda_i + q_i + \delta_i) B_{f,i,x-h}(y) \\ &\quad + \lambda_i \int_0^{y + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} B_{f,i,x-h}(y - u) dF_i(u) + \sum_{j \neq i} q_{ij} f(y, j) \quad \text{for } y \in [x - h, x]. \end{aligned} \quad (3.79)$$

As $W_f(\cdot, i)$ and $B_{f,i,x-h}(\cdot)$ are continuous, for any $\epsilon > 0$ we can find a small $h > 0$ such that

$$(\lambda_i + q_i + \delta_i) |B_{f,i,x-h}(y) - W_f(x, i)| < \frac{\epsilon}{3} \quad \text{for } y \in [x - h, x], \quad (3.80)$$

$$\sum_{j \neq i} q_{ij} |f(y, j) - f(x, j)| < \frac{\epsilon}{3} \quad \text{for } y \in [x - h, x]. \quad (3.81)$$

By (3.73) and Lemma 3.1(i) we can see that $B_{f,i,x-h}(\cdot)$ is nondecreasing and $B_{f,i,x-h}(\cdot) \leq W_f(\cdot, i)$ and therefore,

$$\begin{aligned} &\lambda_i \left(\int_0^{y + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} B_{f,i,x-h}(y - u) dF_i(u) - \int_0^{x + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} W_f(x - u, i) dF_i(u) \right) \\ &\leq \lambda_i \left(\int_0^{y + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} B_{f,i,x-h}(x - u) dF_i(u) - \int_0^{x + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} W_f(x - u, i) dF_i(u) \right) \\ &\leq 0 \quad \text{for } y \in [x - h, x]. \end{aligned} \quad (3.82)$$

Note that $p_i + r_i \cdot (y)^+ - \alpha_i \cdot (y)^- \leq p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^-$ for $y \in [x - h, x]$. This combined with (3.72) and (3.79)–(3.82) shows

$$\mathcal{L}_i^f(\phi_y, B_{f,i,x-h})(y) < \mathcal{G}_i^f(W_f(\cdot, i))(x) + \epsilon \quad \text{for } y \in [x - h, x] \text{ and small } h > 0. \quad (3.83)$$

Note that $\mathcal{G}_i^f(W_f(\cdot, i))(x) < 0$ due to $x \in \mathcal{B}_i^f$. Let $\epsilon < -\frac{\mathcal{G}_i^f(W_f(\cdot, i))(x)}{2}$. It follows by the arbitrariness of $\epsilon > 0$ and (3.83) that

$$\mathcal{L}_i^f(\phi_y, B_{f,i,x-h})(y) < 0 \quad \text{for } y \in [x - h, x] \text{ and small } h > 0. \quad (3.84)$$

Combining (3.76), (3.78) and (3.84) shows $\max\{1 - \phi'_y(y), \mathcal{L}_i^f(\phi_y, B_{f,i,x-h})(y)\} \leq 0$ for $y \in [x - h, x]$ and small $h > 0$, which together with (3.74) implies that $B_{f,i,x-h}(\cdot)$ is a viscosity super-solution on $(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x]$. As $W_f(\cdot, i)$ is locally Lipschitz

continuous, so is $B_{f,i,x-h}(\cdot)$. Then it follows by Lemma 3.3 that $B_{f,i,x-h}(y) \geq W_f(y, i)$ for $y \in [-\frac{p_i}{\alpha_i}, x]$ and small $h > 0$. By Lemma 3.1(i) and the definition of $B_{f,i,x-h}(\cdot)$, it is obvious that $B_{f,i,x-h}(y) \leq W_f(y, i)$ for $y \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$. As a result,

$$B_{f,i,x-h}(y) = W_f(y, i) \quad \text{for } y \in \left[-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x\right] \text{ and small } h > 0. \quad (3.85)$$

Hence, we can find $h > 0$ such that for any $y \in (x-h, x)$, $\mathcal{G}_i^f(W_f(\cdot, i))(y) = \mathcal{G}_i^f(B_{f,i,x-h})(y) = \mathcal{L}_i^f(\phi_y, B_{f,i,x-h})(y) < 0$, where the last equality is due to (3.79) and the last inequality due to (3.84). Note $B_{f,i,x-h}(y) = 1$ for $y \in (x-h, x)$. Consequently, $(x-h, x) \subset \mathcal{B}_i^f$.

(b) Note that for $x > y > 0$,

$$\begin{aligned} \mathcal{G}_i^f(B_{f,i,y})(x) &= p_i + r_i y - (\lambda_i + q_i + \delta_i) B_{f,i,y}(x) + \lambda_i \int_0^{x+\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} B_{f,i,y}(x-u) dF_i(u) + \sum_{j \neq i} q_{ij} f(x, j) \\ &\leq p_i + r_i x - (\lambda_i + q_i + \delta_i) B_{f,i,y}(x) + \lambda_i B_{f,i,y}(x) \int_0^{x+\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} dF_i(u) + \sum_{j \neq i} q_{ij} f(x, j) \\ &\leq p_i + r_i x - (q_i + \delta_i)(W_f(y, i) + x - y) + \sum_{j \neq i} q_{ij} f(x, j). \end{aligned} \quad (3.86)$$

Note that $W_f(y, i) \geq y + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$ due to Lemma 3.1(i), that $f(x, j) \leq x + K$ for some $K > 0$ and that $\sum_{j \neq i} q_{ij} = q_i$. Then it follows by (3.86) that for large y ,

$$\mathcal{G}_i^f(B_{f,i,y})(x) \leq p_i + (r_i - \delta_i)x - (q_i + \delta_i) \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j} + q_i K < 0, \quad \text{for } x > y > 0, \quad (3.87)$$

where the last inequality follows by noticing $\delta_i > r_i$. By applying the same lines (starting from the Eq. (3.73)) in the proof for (3.85) in (a) by replacing $[-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x]$ and $x-h$ there by $[-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty)$ and y (y large enough), respectively, we can show that for large y ,

$$B_{f,i,y}(x) = W_f(x, i) \quad \text{for } x \in \left[-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty\right). \quad (3.88)$$

Therefore, for large enough y , we have $W_f'(x, i) = B_{f,i,y}'(x) = 1$ for $x > y$ and $\mathcal{G}_i^f(W_f(\cdot, i))(x) = \mathcal{G}_i^f(B_{f,i,y})(x) < 0$ for all $x > y$ (due to (3.87)). Consequently, we can conclude by Definition 3.2 that for large enough y , $(y, \infty) \subset \mathcal{B}_i^f$. The existence of such a y also indicates that $\mathcal{B}_i^f$ is not empty.

(c) The continuity of the function $\mathcal{G}_i^f(W_f(\cdot, i))(x)$ implies that $\mathcal{A}_i^f$ is closed.

(d) Assume that $x_1 > x_0 > -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$, $(x_0, x_1] \subset \mathcal{B}_i^f$ and $x_0 \notin \mathcal{B}_i^f$. We need to show $x_0 \in \mathcal{A}_i^f$.

Note $W_f'(x, i) = 1$ for all $x \in (x_0, x_1]$. Therefore, for any $\epsilon > 0$, $W_f(x, i) = W_f(x_0 + \epsilon, i) + x - (x_0 + \epsilon)$ for $x \in [x_0 + \epsilon, x_1]$. Letting ϵ converge to 0, we get $W_f(x, i) = W_f(x_0, i) + x - x_0$ for $x \in [x_0, x_1]$. As a result,

$$\lim_{x \downarrow x_0} \frac{W_f(x, i) - W_f(x_0, i)}{x - x_0} = 1. \quad (3.89)$$

Write $a = \liminf_{x \uparrow x_0} \frac{W_f(x, i) - W_f(x_0, i)}{x - x_0}$. By Lemma 3.1(i) we can see $a \geq 1$.

Suppose $a > 1$. Then for any b with $1 < b \leq a$, we have

$$\limsup_{x \downarrow x_0} \frac{W_f(x, i) - W_f(x_0, i)}{x - x_0} = 1 < b \leq \liminf_{x \uparrow x_0} \frac{W_f(x, i) - W_f(x_0, i)}{x - x_0}. \quad (3.90)$$

Note that $W_f(\cdot, i)$ is a viscosity sub-solution, it follows by Remark 3.2 that there exists a continuously differentiable function $\phi : (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty) \rightarrow \mathbb{R}$ such that $W_f(x, i) - \phi(x, i)$ reaches a maximum at $x = x_0$ with $\phi'(x_0) = b$. Therefore, by

Remark 3.1 it follows that $(p_i + r_i \cdot (x_0)^+ - \alpha_i \cdot (x_0)^-)b - (\lambda_i + q_i + \delta_i)W_f(x_0, i) + \lambda_i \int_0^{x_0 + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} W_f(x_0 - y, i) dF_i(y) + \sum_{j \neq i} q_{ij} f(x, j) \geq 0$. By letting $b \rightarrow 1$ we get $\mathcal{G}_i^f(W_f(\cdot, i))(x_0) \geq 0$. Note that $\mathcal{G}_i^f(W_f(\cdot, i))(x) < 0$ for $x \in (x_0, x_1]$ because $(x_0, x_1] \subset \mathcal{B}_i^f$. By the continuity of $\mathcal{G}_i^f(W_f(\cdot, i))(x)$ we can see that $\mathcal{G}_i^f(W_f(\cdot, i))(x_0) = 0$. Hence, $x_0 \in \mathcal{A}_i^f$.

Suppose $a = 1$. We can find a positive sequence $\{h_n\}$ with $h_n \downarrow 0$ such that

$$\lim_{n \rightarrow \infty} \frac{W_f(x_0, i) - W_f(x_0 - h_n, i)}{h_n} = 1. \quad (3.91)$$

Define $d_n = \frac{W_f(x_0, i) - W_f(x_0 - h_n, i)}{h_n} - 1$. Then $\lim_{n \rightarrow \infty} d_n = 0$. By Lemma 3.1(i) we know $d_n \geq 0$. If there exists an n_0 such that $d_{n_0} = 0$, then $W_f(x_0, i) - W_f(x_0 - h_{n_0}, i) = h_{n_0}$. Note that by Lemma 3.1(i) again, $W_f(x, i) - W_f(x_0 - h_{n_0}, i) \geq x - (x_0 - h_{n_0})$ for $x \geq x_0 - h_{n_0}$. As a result,

$$\begin{aligned} W_f(x_0, i) - W_f(x, i) &= W_f(x_0, i) - W_f(x_0 - h_{n_0}, i) - (W_f(x, i) - W_f(x_0 - h_{n_0}, i)) \\ &\leq x_0 - x \quad \text{for } x \in [x_0 - h_{n_0}, x_0]. \end{aligned} \quad (3.92)$$

As by Lemma 3.1(i) we have $W_f(x_0, i) - W_f(x, i) \geq x_0 - x$ for $x \leq x_0$, we conclude

$$W_f(x_0, i) - W_f(x, i) = x_0 - x \quad \text{for } x \in [x_0 - h_{n_0}, x_0]. \quad (3.93)$$

It follows by (3.89) and (3.93) that $W'_f(x_0, i) = 1$. Therefore, by noticing $\mathcal{G}_i^f(W_f(\cdot, i))(x) < 0$ for $x \in (x_0, x_1]$ because $(x_0, x_1] \subset \mathcal{B}_i^f$, $x_0 \notin \mathcal{B}_i^f$ and the continuity of $\mathcal{G}_i^f(W_f(\cdot, i))(x)$ we can see that $\mathcal{G}_i^f(W_f(\cdot, i))(x_0) = 0$, implying $x_0 \in \mathcal{A}_i^f$.

If $d_n > 0$ for all n , instead. Define $A_n = \{x \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, h_n] : W'_f(x, i) \text{ exists and } W'_f(x, i) \geq 1 + 2d_n\}$. Note that $W_f(\cdot, i)$ is differentiable almost everywhere, and that $W'_f(\cdot, i)$, if exists, is greater than 1. Therefore, $d_n + 1 = (\int_{A_n} W'_f(x, i) dx + \int_{[0, h_n] \setminus A_n} W'_f(x, i) dx) / h_n \geq (|A_n|(1 + 2d_n) + (h_n - |A_n|)) / h_n$, where $|A_n|$ denotes the Lebesgue measure of the set A_n . Hence, $|A_n| \leq \frac{h_n}{2} \rightarrow 0$ as $n \rightarrow \infty$. Therefore we can find a sequence $y_n \uparrow x_0$ such that $W'_f(y_n, i)$ exists and $1 \leq W'_f(y_n, i) < 1 + 2d_n$. Consequently, $\lim_{n \rightarrow \infty} W'_f(y_n, i) = 1$. If there exists a subsequence $\{y_{n_k}\}$ with $y_{n_k} \uparrow x_0$ such that $W'_f(y_{n_k}, i) > 1$, then $\mathcal{L}_i^f(W_f(\cdot, i))(y_{n_k}) = 0$ due to (3.25). Note $\lim_{k \rightarrow \infty} W'_f(y_{n_k}, i) = 1$. Hence $\mathcal{G}_i^f(W_f(\cdot, i))(x_0) = \lim_{k \rightarrow \infty} \mathcal{G}_i^f(W_f(\cdot, i))(y_{n_k}) = \lim_{k \rightarrow \infty} \mathcal{L}_i^f(W_f(\cdot, i))(y_{n_k}) = 0$, which implies $x_0 \in \mathcal{A}_i^f$. If, on the other hand, there exists an integer $n_0 > 0$, such that for all $n \geq n_0$, $W'_f(y_n, i) = 1$, then $\mathcal{G}_i^f(W_f(\cdot, i))(x_0) \leq 0$ by (3.25). In this case, we can show $\mathcal{G}_i^f(W_f(\cdot, i))(x_0) = 0$ by using proof by contradiction. Suppose $\mathcal{G}_i^f(W_f(\cdot, i))(x_0) < 0$. Note $\lim_{n \rightarrow \infty} y_n = x_0$ and $y_n \leq x_0$. Further note that $B_{f, i, y_n}(x) = W_f(x, i)$ for $x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, y_n]$ and that $B_{f, i, y_n}(x)$ converges to $W_f(x, i)$ for $x \in [y_n, x_0]$ as $y_n \uparrow x_0$. Hence, $\lim_{n \rightarrow \infty} \mathcal{G}_i^f(B_{f, i, y_n})(x_0) = \mathcal{G}_i^f(W_f(\cdot, i))(x_0) < 0$. Therefore, we can find an n large enough such that

$$\mathcal{G}_i^f(B_{f, i, y_n})(x_0) < 0 \quad \text{and} \quad x_0 - y_n \leq -\frac{\mathcal{G}_i^f(B_{f, i, y_n})(x_0)}{2(\lambda_i + q_i + \delta_i)}. \quad (3.94)$$

Note that $W_f(x, i) \geq W_f(y_n, i) + x - y_n = B_{f, i, y_n}(x)$ for all $x \geq y_n$, where the last function is defined in (3.73). Then by the increasing property of $B_{f, i, y_n}(\cdot)$ and $f(\cdot, j)$ and non-negativity of $B_{f, i, y_n}(\cdot)$, we can obtain that for all $x \in [y_n, x_0]$,

$$\begin{aligned} \mathcal{G}_i^f(B_{f, i, y_n})(x) &= p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^- - (\lambda_i + q_i + \delta_i) B_{f, i, y_n}(x) \\ &\quad + \lambda_i \int_0^{x + \frac{p_i}{\alpha_i}} B_{f, i, y_n}(x - y) dF_i(y) + \sum_{j \neq i} q_{ij} f(x, j) \\ &\leq p_i + r_i \cdot (x_0)^+ - \alpha_i \cdot (x_0)^- - (\lambda_i + q_i + \delta_i) B_{f, i, y_n}(x) \\ &\quad + \lambda_i \int_0^{x_0 + \frac{p_i}{\alpha_i}} B_{f, i, y_n}(x_0 - y) dF_i(y) + \sum_{j \neq i} q_{ij} f(x_0, j) \\ &= \mathcal{G}_i^f(B_{f, i, y_n})(x_0) + (\lambda_i + q_i + \delta_i)(B_{f, i, y_n}(x_0) - B_{f, i, y_n}(x)) \\ &= \mathcal{G}_i^f(B_{f, i, y_n})(x_0) + (\lambda_i + q_i + \delta_i)(x_0 - x) \\ &< 0, \end{aligned} \quad (3.95)$$

where the last inequality follows by (3.94).

For any $x \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_0]$, use $\phi_x(\cdot)$ to denote any continuously differentiable function such that $B_{f, i, y_n}(\cdot) - \phi_x(\cdot)$ reaches a local minimum 0 at x . As $W_f(\cdot, i)$ is a viscosity solution on $(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty)$ and $B_{f, i, y_n}(x) = W_f(x, i)$ for $x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, y_n]$ by (3.73), it follows by Remark 3.1 that

$$\max \left\{ 1 - \phi'_x(x), \mathcal{L}_i^f(\phi_x, B_{f, i, y_n})(x) \right\} \leq 0 \quad \text{for } x \in \left(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, y_n \right). \quad (3.96)$$

Noting that $B'_{f,i,y_n}(x) = 1$ for $x > y_n$, and hence due to the same reason mentioned right above the Eq. (3.75), $\phi'_x(x) = B'_{f,i,y_n}(x) = 1$ for $x > y_n$. This together with (3.95) implies

$$\max \left\{ 1 - \phi'_x(x), \mathcal{L}_i^f(\phi_x, B_{f,i,y_n})(x) \right\} \leq 0 \quad \text{for } x \in (y_n, x_0]. \quad (3.97)$$

Note that it follows by Remark 3.2 that any test function ϕ_{y_n} , if it exists, fulfills

$$1 \leq \limsup_{z \uparrow y_n} \frac{W_f(y_n, i) - W_f(z, i)}{y_n - z} \leq \phi'_{y_n}(y_n) \leq \liminf_{z \downarrow y_n} \frac{B_{f,i,y_n}(y_n) - B_{f,i,y_n}(z)}{y_n - z} = 1,$$

which combined with (3.95) implies

$$\max \left\{ 1 - \phi'_{y_n}(y_n), \mathcal{L}_i^f(\phi_{y_n}, B_{f,i,y_n})(y_n) \right\} \leq 0. \quad (3.98)$$

Combining (3.96)–(3.98) shows that $B_{f,i,y_n}(x)$ is a viscosity super-solution on the interval $(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_0)$. Then it follows by Lemma 3.3 and the fact $B_{f,i,y_n}(x) \leq W_f(x, i)$ for all $x \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$ that $W_f(x, i) = B_{f,i,y_n}(x)$ for $x \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_0]$. As a result,

$$\lim_{x \uparrow x_0} \frac{W_f(x, i) - W_f(x_0, i)}{x - x_0} = \lim_{x \uparrow x_0} \frac{B_{f,i,y_n}(x, i) - B_{f,i,y_n}(x_0, i)}{x - x_0} = 1, \quad (3.99)$$

which together with (3.89) implies $W'_f(x_0, i) = 1$. This combined with (3.95) implies $x_0 \in \mathcal{B}_i^f$. This contradicts the fact that $x_0 \notin \mathcal{B}_i^f$. Consequently, $\mathcal{G}_i^f(W_f(\cdot, i))(x_0) = 0$, which implies $x_0 \in \mathcal{A}_i^f$.

(e) The non-empty property follows immediately by (b) and (d).

(f) Consider any fixed $x \in \mathcal{C}_i^f$. By definition we have $\mathcal{G}_i^f(W_f(\cdot, i))(x) < 0$. Since $\mathcal{G}_i^f(W_f(\cdot, i))(y)$ is continuous, there exists an ϵ small enough such that

$$\mathcal{G}_i^f(W_f(\cdot, i))(y) < 0 \quad \text{for all } y \in [x, x + \epsilon). \quad (3.100)$$

If $(x, x + \epsilon) \cap \mathcal{B}_i^f$ is empty, then $[x, x + \epsilon) \subset \mathcal{C}_i^f$. If, on the other hand, there exists an $x_1 \in (x, x + \epsilon)$ such that $x_1 \in \mathcal{B}_i^f$, then according to (c), there exists an $x_0 < x_1$ such that $x_0 \in \mathcal{A}_i^f$ and $(x_0, x_1] \subset \mathcal{B}_i^f$. As $x < x_1$ and $x \notin \mathcal{B}_i^f$, we conclude that $x_0 \in (x, x_1) \subset (x, x + \epsilon)$. Furthermore, we have $\mathcal{G}_i^f(W_f(\cdot, i))(x_0) = 0$ due to $x_0 \in \mathcal{A}_i^f$, which is a contradiction to (3.100). This completes the proof. $\square$

Remark 3.5. By the Eq. (3.88) in the proof for Lemma 3.4(b), we can see that, for any $f \in \tilde{D}$ and $i \in E$, there exists a large y , $W_f(x, i) = B_{f,i,y}(x) = x - y + W_f(y, i)$ for $x \geq y$. Hence, there exists some constant $C > 0$ such that $W_f(x, i) \leq x + C$ for all x and i .

Lemma 3.5. For any fixed $f \in \mathcal{D}$ and $i \in \mathbb{E}$, the function $W_f(x, i)$ is differentiable from the right hand side with respect to x if $x \in \mathcal{C}_i^f$ and differentiable if $x \in \text{int}(\mathcal{C}_i^f)$ (the interior of $\mathcal{C}_i^f$), and $\mathcal{L}_i^f(W_f(\cdot, i))(x) = 0$ for $x \in \text{int}(\mathcal{C}_i^f)$.

Proof. Since $W_f(\cdot, i)$ is a viscosity solution to the Eq. (3.25) on $(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty)$ and therefore a super-solution, it follows by Remark 3.1 that for any $y \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty)$ and any continuously differentiable function ϕ with $\phi(y) = W_f(y, i)$ and $\phi(\cdot) \leq W_f(\cdot, i)$,

$$\max\{1 - \phi'(y), \mathcal{L}_i^f(\phi, W_f(\cdot, i))(y)\} \leq 0, \quad (3.101)$$

which implies $\mathcal{L}_i^f(\phi, W_f(\cdot, i))(y) \leq 0$. It follows by Remark 3.2 that

$$\phi'(y) \geq \limsup_{h \downarrow 0} \frac{W_f(y, i) - W_f(y - h, i)}{h} \geq 1 \quad \text{for } y \in \left(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty\right), \quad (3.102)$$

where the last inequality follows by Lemma 3.1(i). Hence, by the definitions in (3.28) and (3.72) we have $\mathcal{G}_i^f(W_f(\cdot, i))(y) = (p_i + r_i \cdot (y)^+ - \alpha_i \cdot (y)^-)(1 - \phi'(y)) + \mathcal{L}_i^f(\phi, W_f(\cdot, i))(y) \leq 0$ for $y \in (-\frac{p_i}{\alpha_i}, \infty)$. This together with Definition 3.2 implies that

$$\mathcal{G}_i^f(W_f(\cdot, i))(x) < 0 \quad \text{for } x \in \mathcal{C}_i^f \cap \left[-\frac{p_i}{\alpha_i}, \infty\right). \quad (3.103)$$

Now we will show that

$$\mathcal{G}_i^f(W_f(\cdot, i))(x) > 0 \quad \text{for } x \in \mathcal{C}_i^f \cap \left[-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, -\frac{p_i}{\alpha_i}\right). \quad (3.104)$$

Suppose the contrary. That is, there exists an $x \in \mathcal{C}_i^f \cap [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, -\frac{p_i}{\alpha_i})$ such that

$$\mathcal{G}_i^f(W_f(\cdot, i))(x) \leq 0.$$

Then by the definition of $\mathcal{A}_i^f$ and $\mathcal{C}_i^f$, we have $\mathcal{G}_i^f(W_f(\cdot, i))(x) < 0$, because otherwise we will have $x \in \mathcal{A}_i^f$. Noting that $\mathcal{C}_i^f$ is right open, we can find $x_0 \in \mathcal{C}_i^f \cap [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, -\frac{p_i}{\alpha_i})$ with $x_0 > x$ such that

$$\mathcal{G}_i^f(W_f(\cdot, i))(y) < 0 \quad \text{for } y \in [x, x_0). \quad (3.105)$$

Note that the function $W_f(\cdot, i)$ is differentiable almost everywhere. So we can find at least one point $y_0 \in [x, x_0)$ such that $W'_f(y_0, i)$ exists. Therefore, it follows by the definitions in (3.26) and (3.72) that

$$\mathcal{L}_i^f(W_f(\cdot, i))(y_0) = \mathcal{G}_i^f(W_f(\cdot, i))(y_0) + (p_i + r_i \cdot (y_0)^+ - \alpha_i \cdot (y_0)^-)(W'_f(y_0, i) - 1) < 0, \quad (3.106)$$

where the last inequality follows by using (3.105) and noting $p_i + r_i \cdot (y_0)^+ - \alpha_i \cdot (y_0)^- = p_i - \alpha_i \cdot (y_0)^- < 0$ and $W'_f(y_0, i) \geq 1$ (by Lemma 3.1(i)). As $W_f(\cdot, i)$ is a viscosity solution to the HJB equation (3.25), then

$$\max \left\{ 1 - W'_f(y_0, i), \mathcal{L}_i^f(W_f(\cdot, i))(y_0) \right\} = 0, \quad (3.107)$$

which combined with (3.106) implies $W'_f(y_0, i) = 1$. Note it follows from (3.105) that

$$\mathcal{G}_i^f(W_f(\cdot, i))(y_0) < 0.$$

As a result, $y_0 \in \mathcal{B}_i^f$, which contradicts the fact that $y_0 \in [x, x_0) \subset \mathcal{C}_i^f$.

Now consider any fixed $x \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$ and $x \in \mathcal{C}_i^f$. Define $\bar{x} = \sup\{y > x : [x, y) \subset \mathcal{C}_i^f\}$. As $\mathcal{C}_i^f$ is right open, we can show $\bar{x} > x$. Consider the initial value integral differential equation

$$\begin{aligned} & (p_i + r_i \cdot (y)^+ - \alpha_i \cdot (y)^-)u'(y) - (\lambda_i + q_i + \delta_i)u(y) + \lambda_i \int_0^{y-x} u(y-z)dF_i(z) \\ & + \lambda_i \int_{y-x}^{y+\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} W_f(y-z, i)dF_i(z) + \sum_{j \neq i} q_{ij}f(y, j) = 0 \quad \text{for } y \in (x, \bar{x}), \end{aligned} \quad (3.108)$$

$$u(x) = W_f(x, i). \quad (3.109)$$

Define $\bar{u}_i(s) = \frac{(\lambda_i + q_i + \delta_i)u(s) - \lambda_i \int_0^{s-x} u(s-z)dF_i(z) - \lambda_i \int_{s-x}^{s+\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} W_f(s-z, i)dF_i(z) - \sum_{j \neq i} q_{ij}f(s, j)}{p_i + r_i \cdot (s)^+ - \alpha_i \cdot (s)^-}$. The existence and uniqueness of a differentiable and increasing positive solution u to the above equation can be proven using the same method in Lemma 3.1 of [8]: consider an operator $\mathcal{T}(u)(y) = \int_x^y \bar{u}_i(s)ds + W_f(x, i)$ on the space of continuous and increasing functions over a small interval $[x, x+h)$ (h is independent of x), then verify that it is a contraction on the above-mentioned space and therefore the existence of a fixed point. Now we know the existence and uniqueness of continuous and increasing positive function u such that $u(y) = \int_x^y \bar{u}_i(s)ds + W_f(x, i)$ on $[x, x+h)$. Such a function u is the desired solution. Applying the same argument to $[x+h, x+2h)$, ... completes the proof.

Let $u_{i,x}$ denote the unique solution to the above initial value equation on $(x, \bar{x})$. Define a function

$$U_{f,i,x}(y) = \begin{cases} W_f(y, i) & y \leq x \\ u_{i,x}(y) & y > x. \end{cases} \quad (3.110)$$

We can see

$$\mathcal{L}_i^f(U_{f,i,x})(y) = \mathcal{L}_i^f(u_{i,x})(y) = 0 \quad \text{for } y \in (x, \bar{x}). \quad (3.111)$$

By the continuity of $u_{i,x}(\cdot)$ and $W_f(\cdot, i)$, we know that $\mathcal{G}_i^f(U_{f,i,x})(\cdot)$ is also continuous. It follows by (3.103) and (3.104) that $\mathcal{G}_i^f(U_{f,i,x})(x) = \mathcal{G}_i^f(W_f(\cdot, i))(x) < 0$ if $x \geq -\frac{p_i}{\alpha_i}$ and $\mathcal{G}_i^f(U_{f,i,x})(x) = \mathcal{G}_i^f(W_f(\cdot, i))(x) > 0$ if $x < -\frac{p_i}{\alpha_i}$. Hence, we can find $y > x$

such that $\mathcal{G}_i^f(U_{f,i,x})(z) < 0$ for $z \in [x, y]$ if $x \geq -\frac{p_i}{\alpha_i}$ and $\mathcal{G}_i^f(U_{f,i,x})(z) > 0$ for $z \in [x, y]$ if $x < -\frac{p_i}{\alpha_i}$. Define

$$x_1 = \begin{cases} \inf \left\{ y > x : \mathcal{G}_i^f(U_{f,i,x})(z) = 0 \right\} & x \geq -\frac{p_i}{\alpha_i} \\ \inf \left\{ y > x : \mathcal{G}_i^f(U_{f,i,x})(z) = 0 \right\} \wedge \left(-\frac{p_i}{\alpha_i} \right) & x < -\frac{p_i}{\alpha_i}. \end{cases}$$

Obviously, $x_1 > x$,

$$(x, x_1] \subset \left(-\frac{p_i}{\alpha_i}, \infty \right) \quad \text{if } x \geq -\frac{p_i}{\alpha_i}, \quad \text{and} \quad (x, x_1] \subset \left(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, -\frac{p_i}{\alpha_i} \right] \quad \text{if } x < -\frac{p_i}{\alpha_i}, \quad (3.112)$$

and

$$\mathcal{G}_i^f(U_{f,i,x})(y) \geq 0 \quad \text{for } y \in [x, x_1], \text{ if } x < -\frac{p_i}{\alpha_i}, \quad (3.113)$$

$$\mathcal{G}_i^f(U_{f,i,x})(y) \leq 0 \quad \text{for } y \in [x, x_1], \text{ if } x \geq -\frac{p_i}{\alpha_i}. \quad (3.114)$$

Hence,

$$p_i + r_i \cdot (y)^+ - \alpha_i \cdot (y)^- > 0 \quad \text{for } y \in (x, x_1], \text{ if } x \geq -\frac{p_i}{\alpha_i}, \quad (3.115)$$

$$p_i + r_i \cdot (y)^+ - \alpha_i \cdot (y)^- < 0 \quad \text{for } y \in (x, x_1), \text{ if } x < -\frac{p_i}{\alpha_i}. \quad (3.116)$$

By the definitions of $\mathcal{G}_i^f$, (3.114) and (3.113) we can see that if $x \geq -\frac{p_i}{\alpha_i}$, then for $y \in (x, x_1)$,

$$\begin{aligned} p_i + r_i \cdot (y)^+ - \alpha_i \cdot (y)^- &\leq (\lambda_i + q_i + \delta_i)U_{f,i,x}(y) - \lambda_i \int_0^{y+\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} U_{f,i,x}(y-z) dF_i(z) - \sum_{j \neq i} q_{ij}f(y, j) \\ &= (\lambda_i + q_i + \delta_i)u_{i,x}(y) - \lambda_i \int_0^{y-x} u_{i,x}(y-z) dF_i(z) \\ &\quad - \lambda_i \int_{y-x}^{y+\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} W_f(y-z, i) dF_i(z) - \sum_{j \neq i} q_{ij}f(y, j) \end{aligned} \quad (3.117)$$

and that if $x < -\frac{p_i}{\alpha_i}$, then for $y \in (x, x_1)$,

$$\begin{aligned} p_i + r_i \cdot (y)^+ - \alpha_i \cdot (y)^- &\geq (\lambda_i + q_i + \delta_i)u_{i,x}(y) - \lambda_i \int_0^{y-x} u_{i,x}(y-z) dF_i(z) \\ &\quad - \lambda_i \int_{y-x}^{y+\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} W_f(y-z, i) dF_i(z) - \sum_{j \neq i} q_{ij}f(y, j). \end{aligned} \quad (3.118)$$

Notice that for $y \in (x, x_1]$,

$$\begin{aligned} U'_{f,i,x}(y) &= u'_{i,x}(y) \\ &= \frac{(\lambda_i + q_i + \delta_i)u_{i,x}(y) - \lambda_i \int_0^{y-x} u_{i,x}(y-z) dF_i(z) - \lambda_i \int_{y-x}^{y+\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} W_f(y-z, i) dF_i(z)}{p_i + r_i \cdot (y)^+ - \alpha_i \cdot (y)^-} \\ &\quad - \frac{\sum_{j \neq i} q_{ij}f(y, j)}{p_i + r_i \cdot (y)^+ - \alpha_i \cdot (y)^-}. \end{aligned} \quad (3.119)$$

Combining (3.115)–(3.119) we can obtain that

$$U'_{f,i,x}(y) \geq 1 \quad \text{for } y \in (x, x_1). \quad (3.120)$$

Similarly,

$$U'^+_{f,i,x}(x) \geq 1. \quad (3.121)$$

Combining (3.120) with (3.111) we conclude that $U_{f,i,x}(y)$ is a classical solution to the HJB equation (3.25) on (x, x_1) . For any y , we use $\phi_y(\cdot)$ ($\underline{\phi}_y(\cdot)$) to denote any continuously differentiable equation such that $U_{f,i,x}(\cdot) - \phi_y(\cdot)$ ($U_{f,i,x}(\cdot) - \underline{\phi}_y(\cdot)$) obtains a local minimum (maximum) 0 at y . Then,

$$\max\{1 - \phi'_y(y), \mathcal{L}_i^f(\phi_y, U_{f,i,x})(y)\} = 0 = \max\{1 - \underline{\phi}'_y(y), \mathcal{L}_i^f(\underline{\phi}_y, U_{f,i,x})(y)\} \quad \text{for } y \in (x, x_1). \quad (3.122)$$

It follows by Remark 3.2 and (3.110) that

$$\phi'_x(x) \leq \liminf_{z \downarrow x} \frac{U_{f,i,x}(z) - U_{f,i,x}(x)}{z - x} = u_{i,x}^{'+}(x), \quad (3.123)$$

$$\phi'_x(x) \geq \limsup_{z \uparrow x} \frac{U_{f,i,x}(x) - U_{f,i,x}(z)}{x - z} = \limsup_{z \uparrow x} \frac{W_f(x, i) - W_f(z, i)}{x - z} \geq 1, \quad (3.124)$$

$$\underline{\phi}'_x(x) \geq \limsup_{z \downarrow x} \frac{U_{f,i,x}(z) - U_{f,i,x}(x)}{z - x} = u_{i,x}^{'+}(x), \quad (3.125)$$

$$\underline{\phi}'_x(x) \leq \liminf_{z \uparrow x} \frac{U_{f,i,x}(x) - U_{f,i,x}(z)}{x - z} = \liminf_{z \uparrow x} \frac{W_f(x, i) - W_f(z, i)}{x - z}. \quad (3.126)$$

It follows by (3.123) that if $x \geq -\frac{p_i}{\alpha_i}$,

$$\begin{aligned} \mathcal{L}_i^f(\phi_x, U_{f,i,x})(x) &\leq (p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^-) u_{i,x}^{'+}(x) - (\lambda_i + q_i + \delta_i) U_{f,i,x}(x) \\ &\quad + \lambda_i \int_0^{x + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} U_{f,i,x}(x - z) dF_i(z) + \sum_{j \neq i} q_{ij} f(x, j) = 0, \end{aligned} \quad (3.127)$$

and by (3.125) that if $x \geq -\frac{p_i}{\alpha_i}$,

$$\begin{aligned} \mathcal{L}_i^f(\underline{\phi}_x, U_{f,i,x})(x) &\geq (p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^-) u_{i,x}^{'+}(x) - (\lambda_i + q_i + \delta_i) U_{f,i,x}(x) \\ &\quad + \lambda_i \int_0^{x + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} U_{f,i,x}(x - z) dF_i(z) + \sum_{j \neq i} q_{ij} f(x, j) = 0, \end{aligned} \quad (3.128)$$

where the last equalities in (3.127) and (3.128) both follow by noting $U_{f,i,x}(x - z) = W_f(x - z, i)$ for $z \geq 0$ (see (3.110)), $U_{f,i,x}(x) = u_{i,x}(x)$ and the fact that $u_{i,x}$ solves the Eqs. (3.108) and (3.109).

We can find test functions $\psi_x(\cdot)$ ($\underline{\psi}_x(\cdot)$) for $W_f(\cdot, i)$ such that $W_f(\cdot, i) - \psi_x(\cdot)$ ($W_f(\cdot, i) - \underline{\psi}_x(\cdot)$) reaches local minimum (maximum) 0 at x , and $\psi'_x(x) = \limsup_{z \uparrow x} \frac{W_f(x, i) - W_f(z, i)}{x - z}$ and $\underline{\psi}'_x(x) = \liminf_{z \uparrow x} \frac{W_f(x, i) - W_f(z, i)}{x - z}$. It follows by (3.124) and (3.126) that $\psi'_x(x) \leq \phi'_x(x)$ and $\underline{\psi}'_x(x) \geq \underline{\phi}'_x(x)$. Therefore, if $x \in (-\max_{i \in \mathbb{E}} \frac{p_i}{\alpha_i}, -\frac{p_i}{\alpha_i})$,

$$\begin{aligned} \mathcal{L}_i^f(\phi_x, U_{f,i,x})(x) &\leq (p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^-) \psi'_x(x) - (\lambda_i + q_i + \delta_i) U_{f,i,x}(x) \\ &\quad + \lambda_i \int_0^{x + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} U_{f,i,x}(x - z) dF_i(z) + \sum_{j \neq i} q_{ij} f(x, j) \\ &= \mathcal{L}_i^f(\psi_x, W_f(\cdot, i))(x) \leq 0, \end{aligned} \quad (3.129)$$

where the last equality follows by noticing $U_{f,i,x}(x - z) = W_f(x - z, i)$ for $z \geq 0$ (see (3.110)), $U_{f,i,x}(x) = W_f(x, i)$ and the last inequality follows by the fact that $W_f(\cdot, i)$ is a viscosity super-solution. Similarly, if $x \in (-\max_{i \in \mathbb{E}} \frac{p_i}{\alpha_i}, -\frac{p_i}{\alpha_i})$,

$$\begin{aligned} \mathcal{L}_i^f(\underline{\phi}_x, U_{f,i,x})(x) &\geq (p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^-) \underline{\psi}'_x(x) - (\lambda_i + q_i + \delta_i) U_{f,i,x}(x) \\ &\quad + \lambda_i \int_0^{x + \max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}} U_{f,i,x}(x - z) dF_i(z) + \sum_{j \neq i} q_{ij} f(x, j) \\ &= \mathcal{L}_i^f(\underline{\psi}_x, W_f(\cdot, i))(x) \geq 0, \end{aligned} \quad (3.130)$$

where the last inequality follows by the fact that $W_f(\cdot, i)$ is a viscosity sub-solution.

As $U_{f,i,x}(y) = W_f(y, i)$ is a viscosity solution to the HJB equation (3.25) on $(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x)$, we have

$$\max\{1 - \phi'_x(y), \mathcal{L}_i^f(\phi_x, U_{f,i,x})(x)\} \leq 0 \quad \text{for } y \in \left(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x\right), \quad (3.131)$$

$$\max\{1 - \underline{\phi}'_x(y), \mathcal{L}_i^f(\underline{\phi}_x, U_{f,i,x})(x)\} \geq 0 \quad \text{for } y \in \left(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x\right). \quad (3.132)$$

Combining (3.131), (3.122), (3.120), (3.127), (3.124) and (3.129) yields

$$\max\{1 - \phi'_x(y), \mathcal{L}_i^f(\phi_x, U_{f,i,x})(x)\} \leq 0 \quad \text{for } y \in \left(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_1\right), \quad (3.133)$$

and combining (3.122), (3.128), (3.130) and (3.132) yields

$$\max\{1 - \underline{\phi}'_x(y), \mathcal{L}_i^f(\underline{\phi}_x, U_{f,i,x})(x)\} \geq 0 \quad \text{for } y \in \left(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_1\right). \quad (3.134)$$

Therefore, we can conclude that $U_{f,i,x}(\cdot)$ is a viscosity solution on $(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_1)$. Hence, it follows by Lemma 3.4 that $U_{f,i,x}(y) = W_f(y, i)$ for $y \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_1]$. Using the continuity of both $U_{f,i,x}(\cdot)$ and $W_f(\cdot)$, we can obtain

$$U_{f,i,x}(y) = W_f(y, i) \quad \text{for } y \in \left[-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, x_1\right]. \quad (3.135)$$

We can show $x_1 \geq \bar{x}$. Suppose the contrary, $x_1 < \bar{x}$. Note $\mathcal{G}_i^f(W_f(\cdot, i))(x_1) = \mathcal{G}_i^f(U_{f,i,x})(x_1) = 0$ and therefore, $x_1 \in \mathcal{A}_i^f$, which contradicts the fact $[x, \bar{x}] \subset \mathcal{C}_i^f$. As a result, $U_{f,i,x}(y) = W_f(y, i)$ for $y \in [-\frac{p_i}{\alpha_i}, \bar{x}]$ and therefore, $W_f(y, i) = u_{i,x}(y)$ for $y \in [x, \bar{x}]$. Hence, $W_f(\cdot, i)$ is a continuously differentiable function on $(x, \bar{x})$. We can further show $W'_f(y, i) > 1$ for $y \in (x, \bar{x})$. Otherwise, if there exists a point $y_0 \in (x, \bar{x})$ at which the inequality does not hold, then $W'_f(y_0, i) = 1$ and hence $\mathcal{L}_i^f(W_f(\cdot, i))(y_0, i) = \mathcal{G}_i^f(W_f(\cdot, i))(y_0, i) \leq 0$, implying $y_0 \in \mathcal{A}_i^f \cup \mathcal{B}_i^f$, which contradicts the assumption $[x, \bar{x}] \subset \mathcal{C}_i^f$. Therefore, $W_f(y, i) = U_{f,i,x}(y)$ and $\mathcal{L}_i^f(W_f(\cdot, i))(y) = 0$ for $y \in (x, \bar{x})$. Note that $U_{f,i,x}(\cdot)$ is differentiable from the right hand side at x . Hence, $W_f(\cdot, i)$ is differentiable from the right hand side at x . If $x \in \text{int}(\mathcal{C}_i^f)$, since $\mathcal{C}_i^f$ is right open, we can find $x_0 \in \mathcal{C}_i^f$ such that $x \in (x_0, \bar{x}_0)$ where $\bar{x}_0 = \sup\{y > x_0 : [x_0, y] \subset \mathcal{C}_i^f\}$. Repeating the same argument above by replacing x and $\bar{x}$ there by x_0 and $\bar{x}_0$, respectively, we can conclude $W_f(y, i) = U_{f,i,x_0}(y)$ and $\mathcal{L}_i^f(W_f(\cdot, i))(y) = 0$ for $y \in (x_0, \bar{x}_0)$. Therefore, $W_f(\cdot, i)$ is differentiable at x and $\mathcal{L}_i^f(W_f(\cdot, i))(x) = 0$. $\square$

Now we can construct a dividend strategy $L^{f,i}$ for the risk process starting from state $i \in \mathbb{E}$ as follows.

Lemma 3.6. For any fixed $f \in \tilde{\mathcal{D}}$ and $i \in \mathbb{E}$, let $L^{f,i}$ denote the dividend strategy under which, at any time t , the insurer

- (a) pays out dividends continuously at rate $p_i + r_i(R_t^{L^{f,i}})^+ - \alpha_i(R_t^{L^{f,i}})^-$ if $R_t^{L^{f,i}} \in \mathcal{A}_i^f$;
- (b) pays out a lump sum $R_t^{L^{f,i}} - x_0$ as dividends immediately after time t , where $x_0 \in \mathcal{A}_i^f \cup \{-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}\}$ is the point that satisfies $x_0 < R_t^{L^{f,i}}$ and $(x_0, R_t^{L^{f,i}}] \subset \mathcal{B}_i^f$, if $R_t^{L^{f,i}} \in \mathcal{B}_i^f$;
- (c) pays out no dividends at the moment if $R_t^{L^{f,i}} \in \mathcal{C}_i^f$.

Let L^f denote the strategy that at any time t , pays dividends according to $L^{f,i}$ if $J_t = i$. For any fixed $f \in \tilde{\mathcal{D}}$, the strategy L^f is optimal with respect to the auxiliary optimization criterion, i.e. $W_f(x, i) = \mathcal{J}_f(L^f)(x, i) = \mathcal{J}_f(L^{f,i})(x, i)$ for all $x \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$ and $i \in \mathbb{E}$.

Proof. Recall the function space

$$\tilde{\mathcal{D}} = \{g \in \mathcal{D} : \text{for any fixed } i \in \mathbb{E}, g(x, i) \leq x + C \text{ for all } x \geq 0 \text{ and some constant } C > 0\}.$$

Define the distance $d(g_1, g_2) = \sup_{x \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, i \in \mathbb{E}} |g_1(x, i) - g_2(x, i)|$ for any $g_1, g_2 \in \tilde{\mathcal{D}}$.

Fix $f \in \tilde{\mathcal{D}}$. Define the operator $\mathcal{K}_f$ by

$$\begin{aligned} \mathcal{K}_f(g)(x, i) &= E_{(x,i)}[D_{(R,J)}^{L^f}(0, S_1 \wedge \sigma_1 \wedge T^{L^f}) + e^{-\Lambda S_1} g(R_{S_1}^{L^f}, J_{S_1}^{L^f}) I\{S_1 < \sigma_1 \wedge T^{L^f}\} \\ &\quad + e^{-\Lambda \sigma_1} f(R_{\sigma_1}^L, J_{\sigma_1}) I\{\sigma_1 \leq S_1 \wedge T^L\}]. \end{aligned} \quad (3.136)$$

We will use the fixed point theory to prove the result. It is sufficient to show (i) the operator $\mathcal{K}_f$ is a contraction on the metric space $(\tilde{\mathcal{D}}, d)$; (ii) $\mathcal{J}_f(L^f) \in \tilde{\mathcal{D}}$ and $W_f \in \tilde{\mathcal{D}}$; and (iii) both $\mathcal{J}_f(L^f)$ and W_f are fixed point of the operator $\mathcal{K}_f$ on $\tilde{\mathcal{D}}$. Then the operator $\mathcal{K}_f$ has a unique fixed point on the space $\tilde{\mathcal{D}}$ and as a result, $\mathcal{J}_f(L^f) = W_f$.

We prove (ii) first. (ii) For any $f \in \tilde{\mathcal{D}}$, $W_f \in \tilde{\mathcal{D}}$ follows immediately by Lemma 3.1(ii) and Remark 3.5.

By noting that ruin occurs immediately if the initial value of the surplus is less than or equal to $-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$ we can obtain $\mathcal{J}_f(L^f)(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, i) = 0$. Note that under strategy L^f , for any fixed i and $y > x \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$, the value of the controlled process starting with initial surplus y and initial regime i will always be greater than that of the process starting with initial surplus x and initial regime i . The same relationship holds for the value of the corresponding dividend process $\{L_t^f; t \geq 0\}$. Hence, $\mathcal{J}_f(L^f)(y, i) \geq \mathcal{J}_f(L^f)(x, i)$ for $y > x \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$ and $i \in \mathbb{E}$, implying that for any fixed $i \in \mathbb{E}$ the function $\mathcal{J}_f(L^f)(\cdot, i)$ is non-decreasing. By Lemma 3.4(b) we know that there exists a $y_i \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$ such that $(y_i, \infty) \subset \mathcal{B}_i^f$ for $i \in \mathbb{E}$. Define $\tilde{x} = \inf\{y \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j} : (y, \infty) \subset \mathcal{B}_i^f\}$ for $i \in \mathbb{E}$. Obviously, $\tilde{x} \in [-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, y_i]$. As $\mathcal{B}_i^f$ is left open (by Lemma 3.4(a)), $\tilde{x} \notin \mathcal{B}_i^f$. Then it follows by Lemma 3.4(d) that $\tilde{x} \in \mathcal{A}_i^f$. According to the definition of the strategy L^f , we know that under L^f , a lump sum $x - \tilde{x}_i$ will be paid out as dividends immediately after time 0 given $R_0 = x > \tilde{x}$ and $J_0 = i$. As a result, $\mathcal{J}_f(L^f)(x, i) = x - \tilde{x} + \mathcal{J}_f(L^f)(\tilde{x}, i)$ for $x > \tilde{x}$. Therefore, we can conclude $\mathcal{J}_f(L^f) \in \tilde{\mathcal{D}}$.

(i) By using same reasoning as above we can show $\mathcal{K}(g) \in \tilde{\mathcal{D}}$ for $g \in \tilde{\mathcal{D}}$. Notice

$$\begin{aligned} d(\mathcal{K}_f(g_1) - \mathcal{K}_f(g_2)) &= \sup_{x \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, i \in \mathbb{E}} \left| E_{(x,i)} \left[e^{-\Lambda_{S_1}} (g_1(R_{S_1}^{L^f}, J_{S_1}^{L^f}) - g_2(R_{S_1}^{L^f}, J_{S_1}^{L^f})) I\{S_1 < \sigma_1 \wedge T^L\} \right] \right| \\ &\leq d(g_1, g_2) \sup_{x \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, i \in \mathbb{E}} \left| E_{(x,i)} \left[e^{-\Lambda_{S_1 \wedge \sigma_1}} \right] \right|. \end{aligned} \quad (3.137)$$

Note that given $J_0 = i$, $S_1 \wedge \sigma_1$ has the same distribution as the minimum of two independent exponential random variables with mean $\frac{1}{\lambda_i}$ and $\frac{1}{q_i}$ by the model assumption. Hence, $\sup_{x \geq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, i \in \mathbb{E}} |E_{(x,i)} [e^{-\Lambda_{S_1 \wedge \sigma_1}}]| = \max_{i \in \mathbb{E}} \frac{\lambda_i + q_i}{\lambda_i + \delta_i + q_i} < 1$. Therefore,

$\mathcal{K}_f$ is a contraction on $(\tilde{\mathcal{D}}, d)$.

(iii) It is not hard to see that the dividend strategy L^f and the controlled surplus process R^{L^f} have Markov property with respect to $\{\mathcal{F}_t\}$. Therefore, it follows by the definition of $\mathcal{J}_f$ in (2.4) and the Eq. (3.10) that

$$\mathcal{J}_f(L^f)(x, i) = \mathcal{K}_f(\mathcal{J}_f(L^f))(x, i). \quad (3.138)$$

Hence, $\mathcal{J}_f(L^f)$ is fixed point of $\mathcal{K}_f$.

Now we proceed to show that W_f is also a fixed point of $\mathcal{K}_f$.

Given $R_0 = x \in \mathcal{A}_i^f$ and $J_0 = i$, as all the incoming premiums are paid out and there are no claims before time $\sigma_1 \wedge S_1$, the controlled surplus under L^f remains at the level x until time $\sigma_1 \wedge S_1$ and then the company pays out dividends continuously at rate $p_i \cdot (x)^+ + r_i \cdot (x)^+ - \alpha_i \cdot (x)^-$. Further note $T^{L^f} \geq S_1$. Hence, given $R_0 = x \in \mathcal{A}_i^f$, $R_{S_1}^{L^f} = x - U_1$ on $\{S_1 < \sigma_1\}$, $R_{\sigma_1}^L = x$ on $\{\sigma_1 \leq S_1\}$, and

$$\begin{aligned} E_{(x,i)}[D_{(R,J)}^{L^f}(0, S_1 \wedge \sigma_1 \wedge T^{L^f})] &= E_{(x,i)} \left[\int_0^{S_1 \wedge \sigma_1} e^{-\delta_i t} (p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^-) dt \right] \\ &= \int_0^\infty e^{-(\lambda_i + \delta_i + q_i)t} (p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^-) dt. \end{aligned}$$

As a result, for $x \in \mathcal{A}_i^f$,

$$\begin{aligned} \mathcal{K}_f(W_f)(x, i) &= \int_0^\infty e^{-(\lambda_i + \delta_i + q_i)t} (p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^-) dt \\ &\quad + E_{(x,i)} [e^{-\delta_i S_1} W_f(x - U_1, i) I\{S_1 < \sigma_1\} + e^{-\delta_i \sigma_1} f(x, J_{\sigma_1}) I\{\sigma_1 \leq S_1\}] \\ &= \int_0^\infty e^{-(\lambda_i + \delta_i + q_i)t} (p_i + r_i \cdot (x)^+ - \alpha_i \cdot (x)^-) dt + \int_0^\infty q_i e^{-q_i t} \\ &\quad \times \left(\int_0^t \lambda_i e^{-(\delta_i + \lambda_i)s} \int_0^\infty W_f(x - y, i) dF_i(y) ds \right) dt + \sum_{j \neq i} \frac{q_{ij}}{q_i} f(x, j) \int_0^\infty q_i e^{-(\lambda_i + q_i + \delta_i)t} dt \\ &= \int_0^\infty e^{-(\lambda_i + \delta_i + q_i)t} \left(\mathcal{G}_i^f(W_f(\cdot, i))(x) + (\lambda_i + \delta_i + q_i) W_f(x, i) \right) dt \\ &= W_f(x, i), \end{aligned} \quad (3.139)$$

where the second last equality follows by (3.72) and the last equality by Definition 3.2.

Note

$$\mathcal{K}_f(W_f) \left(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, i \right) = 0 = W_f \left(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, i \right). \quad (3.140)$$

For $x \in \mathcal{B}_i^f$, there exists an $x_0 < x$ such that $x_0 \in \mathcal{A}_i^f \cup \{-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}\}$ and $(x_0, x] \subset \mathcal{B}_i^f$. By the definition for $\mathcal{B}_i^f$, we have $W_f(x, i) = x - x_0 + W_f(x_0, i)$. Hence, by the structure of L^f , we have

$$\mathcal{K}_f(W_f)(x, i) = x - x_0 + \mathcal{K}_f(W_f)(x_0, i) = x - x_0 + W_f(x_0, i) = W_f(x, i) \quad \text{for } x \in \mathcal{B}_i^f, \quad (3.141)$$

where the second last equality follows by (3.139) in the case $x_0 \in \mathcal{A}_i^f$ and by (3.140) in the case $x_0 = -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}$.

Now we consider $x \in \mathcal{C}_i^f$. As $\mathcal{C}_i^f$ is right open, we can find an $x_0 > x$ such that $x_0 \in \mathcal{A}_i^f \cup \mathcal{B}_i^f$ and $[x, x_0) \subset \mathcal{C}_i^f$. Let τ_{x_0} denote the first time that the controlled process R^{Lf} hits x_0 . Note that no dividends will be paid out when the surplus is in $\mathcal{C}_i^f$. Define $a_{x,i}(s)$ to be the solution to the ordinary differential equation $g'(s) = (p_i + r_i \cdot (g(s))^+ - \alpha_i \cdot (g(s))^-)ds$ with $g(0) = x$. Let t_0 be the quantity satisfying $a_{x,i}(t_0) = x_0$. Note that $R_s^{Lf} = a_{x,i}(s)$ for $x < S_1 \wedge \sigma_1$ and $R_{\sigma_1}^{Lf} = a_{x,i}(\sigma_1)$ if $\sigma_1 < S_1$. By the Markov property,

$$\begin{aligned} \mathcal{K}_f(W_f)(x, i) &= E_{(x,i)} \left[e^{-\delta_i S_1} W_f(a_{x,i}(S_1) - U_1, i) I\{S_1 < \sigma_1 \wedge \tau_{x_0}\} + e^{-\delta_i \tau_{x_0}} \mathcal{K}_f(W_f)(x_0, i) I\{\tau_{x_0} < S_1 \wedge \sigma_1\} \right] \\ &\quad + e^{-\delta_i \sigma_1} f(a_{x,i}(\sigma_1), J_{\sigma_1}) I\{\sigma_1 \leq S_1 \wedge \tau_{x_0}\} \\ &= \int_0^\infty q_i e^{-q_i t} \left(\int_0^{t \wedge t_0} \lambda_i e^{-\lambda_i s} e^{-\delta_i s} \int_0^\infty W_f(a_{x,i}(s) - y, i) dF_i(y) ds \right) dt \\ &\quad + e^{-(\lambda_i + q_i + \delta_i)t_0} \mathcal{K}_f(W_f)(x_0, i) + \sum_{j \neq i} \frac{q_{ij}}{q_i} \int_0^{t_0} q_i e^{-(\lambda_i + q_i + \delta_i)t} f(a_{x,i}(t), j) dt \\ &= \int_0^{t_0} e^{-(\lambda_i + q_i + \delta_i)s} \left(\lambda_i \int_0^\infty W_f(a_{x,i}(s) - y, i) dF_i(y) + \sum_{j \neq i} q_{ij} f(a_{x,i}(s), j) \right) ds \\ &\quad + e^{-(\lambda_i + q_i + \delta_i)t_0} \mathcal{K}_f(W_f)(x_0, i). \end{aligned} \quad (3.142)$$

Noticing by Lemma 3.5 we have $\mathcal{L}_i^f(W_f(\cdot, i))(y, i) = 0$ for $y \in (x, x_0)$. Hence,

$$\begin{aligned} &\int_0^{t_0} e^{-(\lambda_i + q_i + \delta_i)s} \left(\lambda_i \int_0^\infty W_f(a_{x,i}(s) - y, i) dF_i(y) + \sum_{j \neq i} q_{ij} f(a_{x,i}(s), j) \right) ds \\ &= \int_0^{t_0} e^{-(\lambda_i + q_i + \delta_i)s} \left((\lambda_i + q_i + \delta_i) W_f(a_{x,i}(s), i) - (p_i + r_i \cdot (a_{x,i}(s))^+ - \alpha_i \cdot (a_{x,i}(s))^-) W_f'(a_{x,i}(s), i) \right) ds \\ &= - \int_0^{t_0} d \left(e^{-(\lambda_i + q_i + \delta_i)s} W_f(a_{x,i}(s), i) \right) \\ &= W_f(a_{x,i}(0), i) - e^{-(\lambda_i + q_i + \delta_i)t_0} W_f(a_{x,i}(t_0), i) \\ &= W_f(x, i) - e^{-(\lambda_i + q_i + \delta_i)t_0} W_f(x_0, i) \end{aligned} \quad (3.143)$$

where the second equality follows by noticing $a'_{x,i}(s) = p_i + r_i \cdot (a_{x,i}(s))^+ - \alpha_i \cdot (a_{x,i}(s))^-$ and the last equality by $a_{x,i}(0) = x$ and $a_{x,i}(t_0) = x_0$.

Since, $x_0 \in \mathcal{A}_i^f \cup \mathcal{B}_i^f$, it follows by (3.139) and (3.141) that $\mathcal{K}_f(W_f)(x_0, i) = W_f(x_0, i)$, which together with (3.142) and (3.143) shows

$$\mathcal{K}_f(W_f)(x, i) = W_f(x, i) \quad \text{for } x \in \mathcal{C}_i^f. \quad (3.144)$$

Combining (3.139)–(3.141) and (3.144) implies that $W_f(\cdot, i)$ is a fixed point of $\mathcal{K}_f$. $\square$

4. The optimal dividend strategy

In this section we will construct a candidate for the optimal dividend distribution strategy and show that the candidate strategy is optimal with respect to the original optimization criterion.

Since the original optimal return function V belongs to $\tilde{\mathcal{D}}$, all the results in Section 3 hold when we set f there to be V . Hence, we know that given the initial regime i , it is optimal, with respect to the performance functional $\mathcal{J}_V$, to follow strategy

$L^{V,i}$ before time σ_1 , the time that the first regime switch occurs. It is natural to suspect that during the period between the consecutive switches of the regime, a similar strategy dependent on the regime may be optimal.

Applying standard arguments in stochastic control for Markov process, we can see that the optimal return function V satisfies the dynamic programming principle, that is, for any stopping time τ :

$$V(x, i) = \sup_{L \in \Pi} E_{(x,i)} \left[\int_0^{T^L \wedge \tau} e^{-\Lambda_t} dL_t^c + \sum_{0 \leq t < T^L \wedge \tau} e^{-\Lambda_t} (L_{t+} - L_t) + e^{-\Lambda_{T^L \wedge \tau}} V(R_{T^L \wedge \tau}^L, J_{T^L \wedge \tau}) \right]. \quad (4.145)$$

By setting $\tau = \sigma_1$, we have

$$\begin{aligned} V(x, i) &= \sup_{L \in \Pi} E_{(x,i)} \left[\int_0^{T^L \wedge \sigma_1} e^{-\Lambda_t} dL_t^c + \sum_{0 \leq t < T^L \wedge \sigma_1} e^{-\Lambda_t} (L_{t+} - L_t) + e^{-\Lambda_{T^L \wedge \sigma_1}} V(R_{T^L \wedge \sigma_1}^L, J_{T^L \wedge \sigma_1}) \right] \\ &= \sup_{L \in \Pi} E_{(x,i)} \left[\int_0^{T^L \wedge \sigma_1} e^{-\Lambda_t} dL_t^c + \sum_{0 \leq t < T^L \wedge \sigma_1} e^{-\Lambda_t} (L_{t+} - L_t) + I\{\sigma_1 < T^L\} e^{-\Lambda_{\sigma_1}} V(R_{\sigma_1}^L, J_{\sigma_1}) \right], \end{aligned} \quad (4.146)$$

where the last equality follows by noticing $0 \leq V(R_{T^L}^L, J_{T^L}) \leq V(-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, J_{T^L}) = 0$.

It follows by (3.11) and (4.146) that

$$V(x, i) = W_V(x, i) \quad \text{for } x \in \mathbb{R} \text{ and } i \in \mathbb{E}. \quad (4.147)$$

Define $\mathcal{A}_i = \mathcal{A}_i^V$, $\mathcal{B}_i = \mathcal{B}_i^V$, and $\mathcal{C}_i = \mathcal{C}_i^V$.

Remark 4.1. By noticing (4.147), we have the following equivalent version of definition: $\mathcal{A}_i = \{x \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty) : \mathcal{G}_i^V(V)(x) = 0\}$, $\mathcal{B}_i = \{x \in (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty) : V'(x, i) = 1 \text{ and } \mathcal{G}_i^V(V)(x) < 0\}$, and $\mathcal{C}_i = (-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}, \infty) - \mathcal{A}_i \cup \mathcal{B}_i$. Furthermore, it follows by Lemma 3.4 that

- (a) $\mathcal{A}_i$ is nonempty and closed;
- (b) $\mathcal{B}_i$ is nonempty and left-open. And there exists a y such that $(y, \infty) \subset \mathcal{B}_i$;
- (c) if $(x_0, x_1] \subset \mathcal{B}_i$ and $x_0 \notin \mathcal{B}_i$, then $x_0 \in \mathcal{A}_i \cup \{-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}\}$;
- (d) $\mathcal{C}_i$ is right-open.

Now we can construct a dividend strategy with similar structure to the optimal strategy for the auxiliary optimization problem when the process is in a certain regime.

Definition 4.1. Define the strategy L^* such that at any time $t \geq 0$,

- (a) if $R_t^{L^*} \in \mathcal{A}_{J_t}$, the insurer pays out dividends continuously at rate $p_{J_t} + r_{J_t} \cdot (R_t^{L^*})^+ - \alpha_{J_t} \cdot (R_t^{L^*})^-$;
- (b) if $R_t^{L^*} \in \mathcal{B}_{J_t}$, the insurer pays out a lump sum $R_t^{L^*} - x_0$ as dividends, where $x_0 \in \mathcal{A}_{J_t} \cup \{-\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j}\}$ is the point that satisfy $x_0 < R_{J_t}^{L^*}$ and $(x_0, R_{J_t}^{L^*}] \subset \mathcal{B}_{J_t}$;
- (c) if $R_t^{L^*} \in \mathcal{C}_{J_t}$, the insurer pays out no dividends at the moment.

It is not hard to see that the strategy L^* is a Markov strategy. Moreover, it is a stationary strategy. Note the process (R, J) without dividend payments is itself a Markov process. Therefore, the controlled process (R^{L^*}, J) is also a Markov process.

We proceed to show that the strategy L^* is optimal with respect to the original optimization criterion (defined in Section 2). Let $(R^{L^*}, J) : (\mathbb{R} \times \mathbb{E})^{\mathbb{R}_+} \rightarrow (\mathbb{R} \times \mathbb{E})^{\mathbb{R}_+}$ be the canonical process and let $\mathcal{F}$ denote the right-continuous canonical filtration induced by (R^{L^*}, J) . Define the shift operators $\theta_t : (\mathbb{R} \times \mathbb{E})^{\mathbb{R}_+} \rightarrow (\mathbb{R} \times \mathbb{E})^{\mathbb{R}_+}$ for $t \geq 0$ by $(\theta_t \omega)_s = w_{s+t}$, $s, t \in \mathbb{R}_+$, $w \in (\mathbb{R} \times \mathbb{E})^{\mathbb{R}_+}$. For any two random variables X and Y , we use $X \circ Y$ to denote the composition, as long as it is well defined. It is clear that θ_t are measurable with respect to $\mathcal{F}$, and $\theta_t(R^{L^*}, J) = (R^{L^*}, J) \circ \theta_t$. Let $\sigma_0 = 0$ and σ_n represent the time that the n th transition of the state of the process J occurs. Then we have

$$\sigma_{n+1} = \sigma_n + \sigma_1 \circ \theta_{\sigma_n}, \quad n = 0, 1, 2, \dots \quad (4.148)$$

Theorem 4.1. The dividend strategy L^* defined above is an optimal strategy.

Proof. It follows by setting $f = V$ in Lemma 3.6 and using (4.147) that

$$V(R_{\sigma_n}^{L^*}, J_{\sigma_n}) = W_V(R_{\sigma_n}^{L^*}, J_{\sigma_n}) = \mathcal{J}_V(L^{V, J_{\sigma_n}})(R_{\sigma_n}^{L^*}, J_{\sigma_n}). \quad (4.149)$$

From the structure of L^* we can see that given the initial state $J_0 = i$, the strategy $L^{V,i}$ is equivalent to L^* before time σ_1 . Hence, given that the initial state is J_{σ_n} , the strategy $L^{V, J_{\sigma_n}}$ is equivalent to L^* before the next transition time. By noting that

the operator $\mathcal{J}_V$ is fully defined by the path of (R^{L^*}, J) up to the first transition time and using (4.149), we can obtain

$$V(R_{\sigma_n}^{L^*}, J_{\sigma_n}) = \mathcal{J}_V(L^*)(R_{\sigma_n}^{L^*}, J_{\sigma_n}). \quad (4.150)$$

By noting $V(R_{T^{L^*}}^{L^*}, J_{T^{L^*}}) = 0$ and using (3.13) and (4.150) it follows that for any $x \in \mathbb{R}$ and $i \in \mathbb{E}$ that

$$\begin{aligned} \mathcal{J}_V(L^*)(R_{\sigma_n}^{L^*}, J_{\sigma_n}) &= E_{(R_{\sigma_n}^{L^*}, J_{\sigma_n})} \left[D_{(R,J)}^{L^*}(\sigma_0, \sigma_1) + e^{-\Lambda_{\sigma_1}} V(R_{\sigma_1}^{L^*}, J_{\sigma_1}); \sigma_1 < T^{L^*} \right] + E_{(R_{\sigma_n}^{L^*}, J_{\sigma_n})} \left[D_{(R,J)}^{L^*}(\sigma_0, T^{L^*}); \sigma_1 \geq T^{L^*} \right] \\ &= E \left[D_{\theta_{\sigma_n}(R,J)}^{L^*}(\sigma_0 \circ \theta_{\sigma_n}, \sigma_1 \circ \theta_{\sigma_n}); \sigma_1 \circ \theta_{\sigma_n} < T^{L^*} \circ \theta_{\sigma_n} | \mathcal{F}_{\sigma_n} \right] \\ &\quad + E \left[e^{-\Lambda_{\sigma_1 \circ \theta_{\sigma_n}}} V(R_{\sigma_1 \circ \theta_{\sigma_n}}^{L^*}, J_{\sigma_1 \circ \theta_{\sigma_n}}); \sigma_1 \circ \theta_{\sigma_n} < T^{L^*} \circ \theta_{\sigma_n} | \mathcal{F}_{\sigma_n} \right] \\ &\quad + E \left[D_{\theta_{\sigma_n}(R,J)}^{L^*}(\sigma_0 \circ \theta_{\sigma_n}, T^{L^*} \circ \theta_{\sigma_n}); \sigma_1 \circ \theta_{\sigma_n} \geq T^{L^*} \circ \theta_{\sigma_n} | \mathcal{F}_{\sigma_n} \right] \quad P_{(x,i)}\text{-a.s.}, \end{aligned} \quad (4.151)$$

where the last equality follows by the strong Markov property of L^* and (R^{L^*}, J) .

Note that $\sigma_n + \sigma_1 \circ \theta_{\sigma_n} = \sigma_{n+1}$ and

$$\begin{aligned} \{\sigma_1 \circ \theta_{\sigma_n} < T^{L^*} \circ \theta_{\sigma_n}\} \cap \{\sigma_n < T^{L^*}\} &= \left\{ R_t^{L^*} > -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j} \text{ for all } t \in [\sigma_n, \sigma_{n+1}] \right\} \\ &\quad \cap \left\{ R_t^{L^*} > -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j} \text{ for all } t \in [0, \sigma_n] \right\} \\ &= \{\sigma_{n+1} < T^{L^*}\}. \end{aligned} \quad (4.152)$$

Therefore,

$$\begin{aligned} &E \left[D_{\theta_{\sigma_n}(R,J)}^{L^*}(\sigma_0 \circ \theta_{\sigma_n}, \sigma_1 \circ \theta_{\sigma_n}); \sigma_1 \circ \theta_{\sigma_n} < T^{L^*} \circ \theta_{\sigma_n} | \mathcal{F}_{\sigma_n} \right] I\{\sigma_n < T^{L^*}\} \\ &= E \left[D_{\theta_{\sigma_n}(R,J)}^{L^*}(\sigma_0 \circ \theta_{\sigma_n}, \sigma_1 \circ \theta_{\sigma_n}); \sigma_1 \circ \theta_{\sigma_n} < T^{L^*} \circ \theta_{\sigma_n}, \sigma_n < T^{L^*} | \mathcal{F}_{\sigma_n} \right] \\ &= E \left[e^{\Lambda_{\sigma_n}} D_{(R,J)}^{L^*}(\sigma_n, \sigma_{n+1}); \sigma_{n+1} < T^{L^*} | \mathcal{F}_{\sigma_n} \right]. \end{aligned} \quad (4.153)$$

Further noting $\Lambda_{\sigma_n} + \Lambda_{\sigma_1 \circ \theta_{\sigma_n}} = \Lambda_{\sigma_{n+1}}$, we have

$$\begin{aligned} &E \left[e^{-\Lambda_{\sigma_1 \circ \theta_{\sigma_n}}} V(R_{\sigma_1 \circ \theta_{\sigma_n}}^{L^*}, J_{\sigma_1 \circ \theta_{\sigma_n}}); \sigma_1 \circ \theta_{\sigma_n} < T^{L^*} \circ \theta_{\sigma_n} | \mathcal{F}_{\sigma_n} \right] I\{\sigma_n < T^{L^*}\} \\ &= E \left[e^{\Lambda_{\sigma_n}} e^{-\Lambda_{\sigma_{n+1}}} V(R_{\sigma_{n+1}}^{L^*}, J_{\sigma_{n+1}}); \sigma_1 \circ \theta_{\sigma_n} < T^{L^*} \circ \theta_{\sigma_n}, \sigma_n < T^{L^*} | \mathcal{F}_{\sigma_n} \right] \\ &= e^{\Lambda_{\sigma_n}} E \left[e^{-\Lambda_{\sigma_{n+1}}} V(R_{\sigma_{n+1}}^{L^*}, J_{\sigma_{n+1}}); \sigma_{n+1} < T^{L^*} | \mathcal{F}_{\sigma_n} \right]. \end{aligned} \quad (4.154)$$

It can also be seen that $\sigma_n + T^{L^*} \circ \theta_{\sigma_n} = T^{L^*}$ on $\{\sigma_n < T^{L^*}\}$ and that

$$\begin{aligned} \{\sigma_1 \circ \theta_{\sigma_n} \geq T^{L^*} \circ \theta_{\sigma_n}\} \cap \{\sigma_n < T^{L^*}\} &= \left\{ R_t^{L^*} \leq -\max_{j \in \mathbb{E}} \frac{p_j}{\alpha_j} \text{ for some } t \in [\sigma_n, \sigma_{n+1}] \right\} \cap \{\sigma_n < T^{L^*}\} \\ &= \{\sigma_n < T^{L^*} \leq \sigma_{n+1}\}. \end{aligned} \quad (4.155)$$

Then, we can obtain

$$\begin{aligned} &E \left[D_{\theta_{\sigma_n}(R,J)}^{L^*}(\sigma_0 \circ \theta_{\sigma_n}, T^{L^*} \circ \theta_{\sigma_n}); \sigma_1 \circ \theta_{\sigma_n} \geq T^{L^*} \circ \theta_{\sigma_n} | \mathcal{F}_{\sigma_n} \right] I\{\sigma_n < T^{L^*}\} \\ &= E \left[e^{\Lambda_{\sigma_n}} D_{(R,J)}^{L^*}(\sigma_n, T^{L^*}); \sigma_n < T^{L^*} \leq \sigma_{n+1} | \mathcal{F}_{\sigma_n} \right]. \end{aligned} \quad (4.156)$$

It follows by (4.150), (4.151), (4.153), (4.154) and (4.156) that for any $x \in \mathbb{R}$ and $i \in \mathbb{E}$,

$$\begin{aligned} e^{-\Lambda_{\sigma_n}} V(R_{\sigma_n}^{L^*}, J_{\sigma_n}) I\{\sigma_n < T^{L^*}\} &= E \left[D_{(R,J)}^{L^*}(\sigma_n, \sigma_{n+1}); \sigma_{n+1} < T^{L^*} | \mathcal{F}_{\sigma_n} \right] + E \left[e^{-\Lambda_{\sigma_{n+1}}} V(R_{\sigma_{n+1}}^{L^*}, J_{\sigma_{n+1}}); \sigma_{n+1} < T^{L^*} | \mathcal{F}_{\sigma_n} \right] \\ &\quad + E \left[D_{(R,J)}^{L^*}(\sigma_n, T^{L^*}); \sigma_n < T^{L^*} \leq \sigma_{n+1} | \mathcal{F}_{\sigma_n} \right] \\ &= E \left[D_{(R,J)}^{L^*}(\sigma_n, \sigma_{n+1} \wedge T^{L^*}) I\{\sigma_n < T^{L^*}\} | \mathcal{F}_{\sigma_n} \right] \\ &\quad + E \left[e^{-\Lambda_{\sigma_{n+1}}} V(R_{\sigma_{n+1}}^{L^*}, J_{\sigma_{n+1}}) I\{\sigma_{n+1} < T^{L^*}\} | \mathcal{F}_{\sigma_n} \right], \quad P_{(x,i)}\text{-a.s.} \end{aligned} \quad (4.157)$$

We will show in the following that for $i = 1, 2, \dots$,

$$V(x, i) = E_{(x,i)} \left[D_{(R,J)}^{L^*}(0, T^{L^*} \wedge \sigma_k) + e^{-\Lambda \sigma_k} V(R_{\sigma_k}^{L^*}, J_{\sigma_k}) I\{\sigma_k < T^{L^*}\} \right]. \quad (4.158)$$

We prove this by induction. Using the same argument as in the proof of (4.150), we have

$$\begin{aligned} V(x, i) &= \mathcal{J}_V(L^*)(x, i) \\ &= E_{(x,i)} \left[D_{(R,J)}^{L^*}(0, T^{L^*} \wedge \sigma_1) + e^{-\Lambda \sigma_1} V(R_{\sigma_1}^{L^*}, J_{\sigma_1}) I\{\sigma_1 < T^{L^*}\} \right], \end{aligned} \quad (4.159)$$

where the last inequality follows by the definition of $\mathcal{J}$ in (3.10). Therefore, the Eq. (4.158) holds for $k = 1$. Now suppose the Eq. (4.158) holds for $k = n$. Then,

$$\begin{aligned} V(x, i) &= E_{(x,i)} \left[D_{(R,J)}^{L^*}(0, T^{L^*} \wedge \sigma_n) + e^{-\Lambda \sigma_n} V(R_{\sigma_n}^{L^*}, J_{\sigma_n}) I\{\sigma_n < T^{L^*}\} \right] \\ &= E_{(x,i)} \left[D_{(R,J)}^{L^*}(0, T^{L^*} \wedge \sigma_n) \right] + E_{(x,i)} \left[E \left[D_{(R,J)}^{L^*}(\sigma_n, \sigma_{n+1} \wedge T^{L^*}) I\{\sigma_n < T^{L^*}\} \middle| \mathcal{F}_{\sigma_n} \right] \right] \\ &\quad + E_{(x,i)} \left[E \left[e^{-\Lambda \sigma_{n+1}} V(R_{\sigma_{n+1}}^{L^*}, J_{\sigma_{n+1}}) I\{\sigma_{n+1} < T^{L^*}\} \middle| \mathcal{F}_{\sigma_n} \right] \right] \end{aligned} \quad (4.160)$$

where the last equality follows by (4.157). Consequently, by double expectation formula it follows

$$V(x, i) = E_{(x,i)} \left[D_{(R,J)}^{L^*}(0, T^{L^*} \wedge \sigma_{n+1}) + e^{-\Lambda \sigma_{n+1}} V(R_{\sigma_{n+1}}^{L^*}, J_{\sigma_{n+1}}) I\{\sigma_{n+1} < T^{L^*}\} \right]. \quad (4.161)$$

Note that $D_{(R,J)}^{L^*}(0, t)$ is increasing in t and $\lim_{k \rightarrow \infty} \sigma_k = \infty$ a.s. Then it follows by the monotone convergence that

$$\lim_{k \rightarrow \infty} E_{(x,i)} \left[D_{(R,J)}^{L^*}(0, T^{L^*} \wedge \sigma_k) \right] = E_{(x,i)} \left[D_{(R,J)}^{L^*}(0, T^{L^*}) \right]. \quad (4.162)$$

Consider the stochastic processes $\{Y_t; t \geq 0\}$ given by

$$Y_0 = R_0 \quad \text{and} \quad dY_t = (p_{J_t} + r_{J_t} Y_{t-}) dt. \quad (4.163)$$

Comparing (2.2) and (4.163) shows

$$R_t^{L^*} \leq Y_t \quad \text{for all } t \geq 0. \quad (4.164)$$

Since for any fixed $k, J_t = J_{\sigma_{k-1}}$ for all $\sigma_{k-1} \leq t < \sigma_k$, We can see from (4.163) that

$$dY_t = (p_{J_{\sigma_{k-1}}} + r_{J_{\sigma_{k-1}}} Y_{t-}) dt, \quad \sigma_{k-1} \leq t < \sigma_k. \quad (4.165)$$

Solving the above gives

$$Y_{\sigma_k} = \left(Y_{\sigma_{k-1}} + \frac{p_{J_{\sigma_{k-1}}}}{r_{J_{\sigma_{k-1}}}} \right) e^{r_{J_{\sigma_{k-1}}} (\sigma_k - \sigma_{k-1})} - \frac{p_{J_{\sigma_{k-1}}}}{r_{J_{\sigma_{k-1}}}}. \quad (4.166)$$

Therefore,

$$Y_{\sigma_k} + \frac{p_{J_{\sigma_{k-1}}}}{r_{J_{\sigma_{k-1}}}} = \left(Y_{\sigma_{k-1}} + \frac{p_{J_{\sigma_{k-1}}}}{r_{J_{\sigma_{k-1}}}} \right) e^{r_{J_{\sigma_{k-1}}} (\sigma_k - \sigma_{k-1})} \quad (4.167)$$

$$= \left(Y_{\sigma_{k-1}} + \frac{p_{J_{\sigma_{k-2}}}}{r_{J_{\sigma_{k-2}}}} \right) e^{r_{J_{\sigma_{k-1}}} (\sigma_k - \sigma_{k-1})} + \left(\frac{p_{J_{\sigma_{k-1}}}}{r_{J_{\sigma_{k-1}}}} - \frac{p_{J_{\sigma_{k-2}}}}{r_{J_{\sigma_{k-2}}}} \right) e^{r_{J_{\sigma_{k-1}}} (\sigma_k - \sigma_{k-1})} \quad (4.168)$$

$$\begin{aligned} &= \left(Y_{\sigma_{k-2}} + \frac{p_{J_{\sigma_{k-3}}}}{r_{J_{\sigma_{k-3}}}} \right) e^{r_{J_{\sigma_{k-2}}} (\sigma_{k-1} - \sigma_{k-2}) + r_{J_{\sigma_{k-1}}} (\sigma_k - \sigma_{k-1})} \\ &\quad + \left(\frac{p_{J_{\sigma_{k-2}}}}{r_{J_{\sigma_{k-2}}}} - \frac{p_{J_{\sigma_{k-3}}}}{r_{J_{\sigma_{k-3}}}} \right) e^{r_{J_{\sigma_{k-2}}} (\sigma_{k-1} - \sigma_{k-2}) + r_{J_{\sigma_{k-1}}} (\sigma_k - \sigma_{k-1})} + \left(\frac{p_{J_{\sigma_{k-1}}}}{r_{J_{\sigma_{k-1}}}} - \frac{p_{J_{\sigma_{k-2}}}}{r_{J_{\sigma_{k-2}}}} \right) e^{r_{J_{\sigma_{k-1}}} (\sigma_k - \sigma_{k-1})} \\ &= \left(Y_{\sigma_1} + \frac{p_{J_{\sigma_0}}}{r_{J_{\sigma_0}}} \right) e^{\sum_{j=2}^k r_{J_{\sigma_{j-1}}} (\sigma_j - \sigma_{j-1})} + \sum_{l=2}^k \left(\frac{p_{J_{\sigma_{l-1}}}}{r_{J_{\sigma_{l-1}}}} - \frac{p_{J_{\sigma_{l-2}}}}{r_{J_{\sigma_{l-2}}}} \right) e^{\sum_{j=l}^k r_{J_{\sigma_{j-1}}} (\sigma_j - \sigma_{j-1})} \\ &= \left(Y_{\sigma_0} + \frac{p_{J_{\sigma_0}}}{r_{J_{\sigma_0}}} \right) e^{\sum_{j=1}^k r_{J_{\sigma_{j-1}}} (\sigma_j - \sigma_{j-1})} + \sum_{l=2}^k \left(\frac{p_{J_{\sigma_{l-1}}}}{r_{J_{\sigma_{l-1}}}} - \frac{p_{J_{\sigma_{l-2}}}}{r_{J_{\sigma_{l-2}}}} \right) e^{\sum_{j=l}^k r_{J_{\sigma_{j-1}}} (\sigma_j - \sigma_{j-1})}, \end{aligned} \quad (4.169)$$

where the third and fourth equalities follow by using (4.168) and the last equality follows by using (4.167). Note that $e^{-\Lambda\sigma_k} = e^{-\sum_{j=1}^k \delta_{J\sigma_{j-1}}(\sigma_j - \sigma_{j-1})}$. Hence,

$$\begin{aligned} \left| e^{-\Lambda\sigma_k} \left(Y_{\sigma_0} + \frac{p_{J\sigma_0}}{r_{J\sigma_0}} \right) e^{\sum_{j=1}^k r_{J\sigma_{j-1}}(\sigma_j - \sigma_{j-1})} \right| &= \left| Y_{\sigma_0} + \frac{p_{J\sigma_0}}{r_{J\sigma_0}} \right| e^{-\sum_{j=1}^k (\delta_{J\sigma_{j-1}} - r_{J\sigma_{j-1}})(\sigma_j - \sigma_{j-1})} \\ &\leq \left| Y_{\sigma_0} + \frac{p_{J\sigma_0}}{r_{J\sigma_0}} \right| e^{-\min_{l \in \mathbb{E}} (\delta_l - r_l) \sum_{j=1}^k (\sigma_j - \sigma_{j-1})} \\ &= \left| Y_{\sigma_0} + \frac{p_{J\sigma_0}}{r_{J\sigma_0}} \right| e^{-\min_{l \in \mathbb{E}} (\delta_l - r_l) \sigma_k}, \end{aligned} \quad (4.170)$$

and

$$\begin{aligned} \left| e^{-\Lambda\sigma_k} \sum_{l=2}^k \left(\frac{p_{J\sigma_{l-1}}}{r_{J\sigma_{l-1}}} - \frac{p_{J\sigma_{l-2}}}{r_{J\sigma_{l-2}}} \right) e^{\sum_{j=l}^k r_{J\sigma_{j-1}}(\sigma_j - \sigma_{j-1})} \right| &< \left| \sum_{l=2}^k \left(\frac{p_{J\sigma_{l-1}}}{r_{J\sigma_{l-1}}} - \frac{p_{J\sigma_{l-2}}}{r_{J\sigma_{l-2}}} \right) \right| e^{-\sum_{j=1}^k (\delta_{J\sigma_{j-1}} - r_{J\sigma_{j-1}})(\sigma_j - \sigma_{j-1})} \\ &\leq \left| \frac{p_{J\sigma_{k-1}}}{r_{J\sigma_{k-1}}} - \frac{p_{J\sigma_0}}{r_{J\sigma_0}} \right| e^{-\min_{l \in \mathbb{E}} (\delta_l - r_l) \sigma_k}. \end{aligned} \quad (4.171)$$

By combining (4.164) and (4.169)–(4.171), we have

$$\begin{aligned} e^{-\Lambda\sigma_k} |R_{\sigma_k}^{L^*}| &\leq e^{-\Lambda\sigma_k} \left(\left| Y_{\sigma_k} + \frac{p_{J\sigma_{k-1}}}{r_{J\sigma_{k-1}}} \right| + \frac{p_{J\sigma_{k-1}}}{r_{J\sigma_{k-1}}} \right) \\ &\leq \left(\left| Y_{\sigma_0} + \frac{p_{J\sigma_0}}{r_{J\sigma_0}} \right| + \left| \frac{p_{J\sigma_{k-1}}}{r_{J\sigma_{k-1}}} - \frac{p_{J\sigma_0}}{r_{J\sigma_0}} \right| + \max_{j \in \mathbb{E}} \frac{p_j}{r_j} \right) e^{-\min_{l \in \mathbb{E}} (\delta_l - r_l) \sigma_k} \\ &\leq \left(\left| Y_{\sigma_0} + \frac{p_{J\sigma_0}}{r_{J\sigma_0}} \right| + 2 \max_{j \in \mathbb{E}} \frac{p_j}{r_j} \right) e^{-\min_{l \in \mathbb{E}} (\delta_l - r_l) \sigma_k} \\ &\xrightarrow{k \rightarrow \infty} 0 \quad P_{(x,i)}\text{-a.s., } x \in \mathbb{R}, i \in \mathbb{E} \end{aligned} \quad (4.172)$$

where the last one follows by noticing $\min_{l \in \mathbb{E}} (\delta_l - r_l) > 0$ and $\sigma_k \rightarrow \infty$ as $k \rightarrow \infty$.

Consequently, by Remark 4.147(b) we know that there exists a $y_0 > 0$ such that $(y_0, \infty) \subset \mathcal{B}_j$ for all $j \in \mathbb{E}$ and hence by Definition 4.1 that $R_{\sigma_k}^{L^*} \leq y_0$ if $\sigma_k < T^{L^*}$. It follows by Lemma 3.1(ii) and Fatou's lemma that

$$\begin{aligned} \limsup_{k \rightarrow \infty} E_{(x,i)}[e^{-\Lambda\sigma_k} V(R_{\sigma_k}^{L^*}, J_{\sigma_k}) I\{\sigma_k < T^{L^*}\}] &\leq \limsup_{k \rightarrow \infty} E_{(x,i)}[e^{-\Lambda\sigma_k} (\text{constant} \cdot R_{\sigma_k}^{L^*} + \text{constant})] \\ &\leq E_{(x,i)} \left[\limsup_{k \rightarrow \infty} e^{-\Lambda\sigma_k} (\text{constant} \cdot R_{\sigma_k}^{L^*} + \text{constant}) \right] = 0. \end{aligned} \quad (4.173)$$

By letting $k \rightarrow \infty$ in (4.158) and using (4.162) and (4.173) we derive

$$V(x, i) = E_{(x,i)}[D_{(R,J)}^{L^*}(0, T^{L^*})],$$

which implies that L^* is an optimal strategy. $\square$

The optimal strategy L^* has a band structure with regime switching. According to the optimal strategy L^* , the decision of how to pay out dividends at any time depends only on the current surplus level and the environment regime.

If we let the state space consist of only one state, say $\mathbb{E} = \{1\}$. This reduces to the compound Poisson case. Note now the transition intensity $q_1 = 0$. Then we can see that the optimal strategy obtained here is a band strategy (with no regime switching) with all the crucial sets $\mathcal{A}_1$, $\mathcal{B}_1$ and $\mathcal{C}_1$ defined exactly same as the compound Poisson case. Hence, the optimal strategy here by assuming only one regime is same as the one obtained in the literature for the compound Poisson model with credit and debit interest [10].

5. Conclusion

We solved the singular dividend optimization problem for the regime-switching compound Poisson process with interest. Our results show that it is optimal to pay dividends according to a regime-switching band strategy: when the current surplus at time t is in the set $\mathcal{A}_{j_t}$, dividends should be paid out continuously at the same rate as the surplus incoming rate, when

the current surplus is in the set $\mathcal{B}_{j_t}$, a positive lump sum of dividends should be paid out, and when the current surplus is in the set $\mathcal{C}_{j_t}$, no dividends should be paid out. The decision of how to pay out dividends at any time depends not only on the current surplus level and also the environment regime at the time. This is supported by the empirical study that shows dividend policies behave according to macroeconomic conditions (see for example, [26, p. 296]).

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