

Some remarks on weighted Lupaş  $q$ -Bézier curves<sup>☆</sup>Lizheng Lu<sup>\*</sup>, Chengkai Jiang, Xueyan Xiang

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## ABSTRACT

Recently, Han et al. (2016) introduced weighted Lupaş  $q$ -Bézier curves as a novel generalization of the classical rational Bézier curves. In this note, we make some remarks on their article. In particular, we revisit the rational quadratic case for an illustrative exposition.

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## 1. Introduction

Recently, Han et al. [1] introduced *weighted Lupaş  $q$ -Bézier curves*, which are a novel generalization of the classical rational Bézier curves and share many properties with rational Bézier curves. Given the control points  $\mathbf{P}_i \in \mathbb{R}^d$  and positive weights  $\omega_i \in \mathbb{R}$ , a weighted Lupaş  $q$ -Bézier curve of degree  $n$  is defined by

$$\mathbf{R}(t; q) = \frac{\sum_{i=0}^n \omega_i \mathbf{P}_i a_{n,i}(t; q)}{\sum_{i=0}^n \omega_i a_{n,i}(t; q)}, \quad 0 \leq t \leq 1, \quad (1)$$

where  $q > 0$ ,  $r_{n,i}(t; q) = \frac{\omega_i a_{n,i}(t; q)}{\sum_{i=0}^n \omega_i a_{n,i}(t; q)}$  are called the weighted Lupaş  $q$ -analogue of Bernstein functions and

$$a_{n,i}(t; q) := \begin{bmatrix} n \\ i \end{bmatrix} q^{i(i-1)/2} t^i (1-t)^{n-i}, \quad i = 0, 1, \dots, n. \quad (2)$$

If all weights are the same, it reduces to a Lupaş  $q$ -Bézier curve [2].

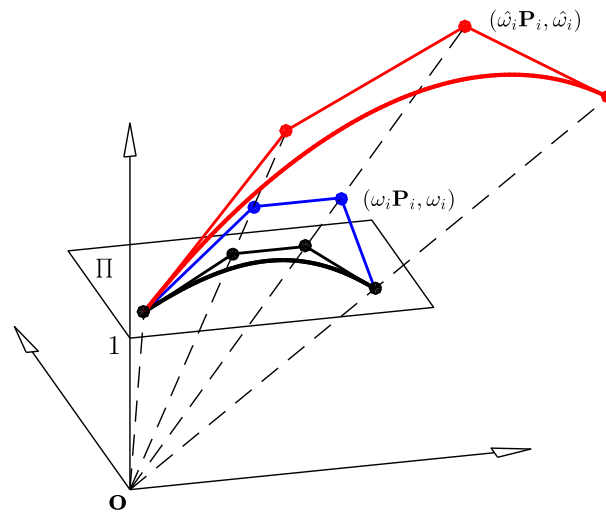
When  $q = 1$ , the weighted Lupaş  $q$ -Bézier curve reduces to the classical rational Bézier curve; and  $a_{n,i}(t; q)$  are identically equal to the classical Bernstein polynomials  $B_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$ , i.e.,  $a_{n,i}(t; q) \equiv B_{n,i}(t)$ , for  $i = 0, 1, \dots, n$ . For the general case  $q \in (0, 1) \cup (1, +\infty)$ , the situation is different.

This note is devoted to provide some remarks on weighted Lupaş  $q$ -Bézier curves [1]. Throughout the rest of this note,  $q \neq 1$  is always assumed.

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**Fig. 1.** A weighted Lupaş  $q$ -Bézier curve (displayed by heavy black lines) as the projection of its homogeneous form (displayed by heavy red lines). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

## 2. Remarks

First of all, we point out the nature of weighted Lupaş  $q$ -Bézier curves in the following.

**Remark 1.** A weighted Lupaş  $q$ -Bézier curve (1) is a rational Bézier curve in nature, but with different weights. More precisely, it can be equivalently formulated in rational Bernstein–Bézier form as

$$\mathbf{R}(t) = \frac{\sum_{i=0}^n \hat{\omega}_i \mathbf{P}_i B_{n,i}(t)}{\sum_{i=0}^n \hat{\omega}_i B_{n,i}(t)}, \quad 0 \leq t \leq 1, \quad (3)$$

with all the control points  $\mathbf{P}_i$  remaining unchanged. But each weight  $\hat{\omega}_i$  associated to  $\mathbf{P}_i$  is changed by the relation

$$\hat{\omega}_i = \omega_i \delta_i, \quad \delta_i := \delta_i(q) = \frac{\begin{bmatrix} n \\ i \end{bmatrix}}{\binom{n}{i}} q^{i(i-1)/2}, \quad i = 0, 1, \dots, n. \quad (4)$$

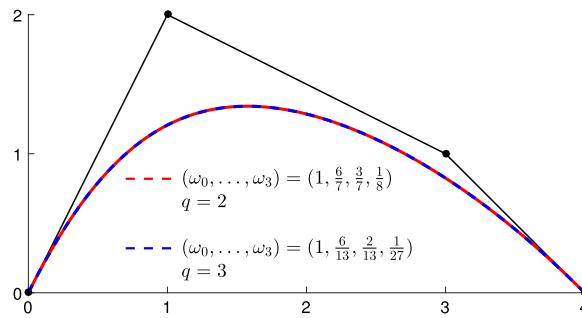
As is well known, the rational Bernstein–Bézier representation has been widely used and investigated in the field of computer aided geometric design (CAGD); see e.g. the monograph by Farin [3]. Based on Remark 1, we can straightforwardly obtain those properties derived in [1] for weighted Lupaş  $q$ -Bézier curves, since they are also rational Bézier curves. For example, the first derivative of  $\mathbf{R}(t; q)$  at the endpoint  $t = 0$  is equal to (using the formula in [3, Page 233] and (4))

$$\mathbf{R}'(0) = \frac{n\hat{\omega}_1}{\hat{\omega}_0} (\mathbf{P}_1 - \mathbf{P}_0) = \frac{n\omega_1}{\omega_0} \frac{\begin{bmatrix} n \\ 1 \end{bmatrix} / \binom{n}{1}}{\begin{bmatrix} n \\ 0 \end{bmatrix} / \binom{n}{0}} (\mathbf{P}_1 - \mathbf{P}_0) = \frac{[n]\omega_1}{\omega_0} (\mathbf{P}_1 - \mathbf{P}_0)$$

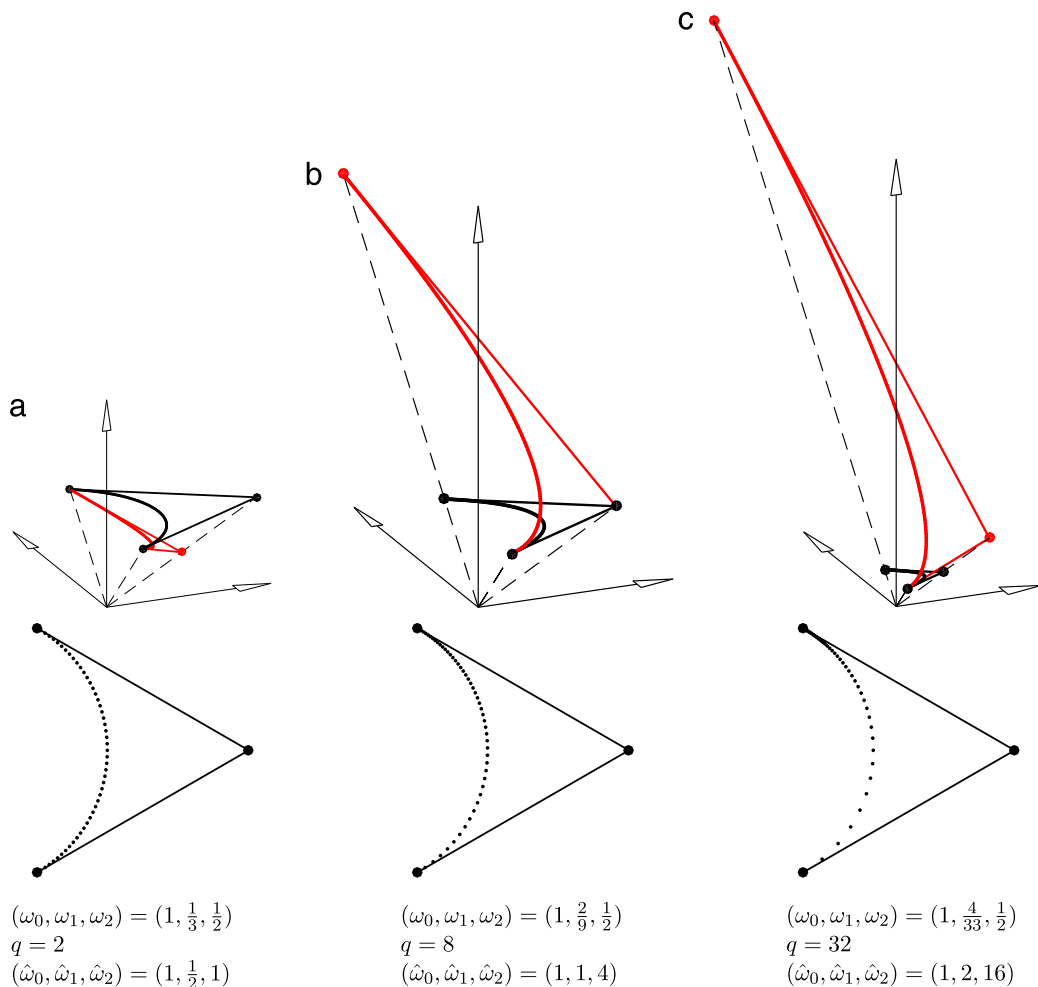
which coincides with the result proved in Theorem 3.1 of [1].

In CAGD, rational curves are usually studied as the projection of higher-dimensional polynomial curves. For a weighted Lupaş  $q$ -Bézier curve (1), its homogeneous form is  $\mathbf{R}(t; q) = \sum_{i=0}^n (\omega_i \mathbf{P}_i, \omega_i) a_{n,i}(t; q)$ . Unfortunately, since the blending functions  $a_{n,i}(t; q)$  do not form a partition of unity when  $q \neq 1$ ,  $\mathbf{R}(t; q)$  may lie outside the convex hull of the  $(\omega_i \mathbf{P}_i, \omega_i)$ . Conversely, it is advantageous to convert  $\mathbf{R}(t; q)$  to its rational Bernstein–Bézier representation (3), whose homogeneous form,  $\mathbf{R}(t) = \sum_{i=0}^n (\hat{\omega}_i \mathbf{P}_i, \hat{\omega}_i) B_{n,i}(t)$ , has the convex hull property. See Fig. 1 for an illustration.

**Remark 2.** When all the weights  $\omega_i$  are fixed,  $q$  in (1) is interpreted as a shape parameter of  $\mathbf{R}(t; q)$ . However, when the weights  $\omega_i$  are considered as shape parameters (in CAGD, for shape design through rational Bézier curves (3),  $\hat{\omega}_i$  are shape parameters),  $q$  is not a shape parameter of  $\mathbf{R}(t; q)$  any more. This is because the weights  $\omega_i$  and  $q$  contribute jointly to the final values of  $\hat{\omega}_i$ , referring to the relation (4). For any configuration  $(\hat{\omega}_0^*, \hat{\omega}_1^*, \dots, \hat{\omega}_n^*)$ , it can always be reached for arbitrarily specified  $q$  by assigning  $\omega_i = \hat{\omega}_i^* / \delta_i$ ,  $i = 0, 1, \dots, n$ . An example is demonstrated in Fig. 2.



**Fig. 2.** Two different groups of weights  $(\omega_0, \dots, \omega_3)$  and  $q$  generating the same cubic weighted Lupaş  $q$ -Bézier curve. The same  $(\hat{\omega}_0, \dots, \hat{\omega}_3) = (1, 2, 2, 1)$  follows by (4).



**Fig. 3.** Representing a circular arc in the second row by three quadratic weighted Lupaş  $q$ -Bézier curves, with the same control points but different weights  $(\omega_0, \omega_1, \omega_2)$  and  $q$ . The homogeneous forms of quadratic weighted Lupaş  $q$ -Bézier curves are displayed by heavy red lines. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

**Remark 3.** The distribution of weights in  $(\hat{\omega}_0, \hat{\omega}_1, \dots, \hat{\omega}_n)$  will become highly nonuniform if  $q$  is not close to 1. Let us take for example the case when  $q > 1$ :  $\begin{bmatrix} n \\ i \end{bmatrix} \geq \binom{n}{i} \geq 1$ , and  $\begin{bmatrix} n \\ i \end{bmatrix}$  are monotonically increasing in  $q$ ; thus, only  $\hat{\omega}_0 = \omega_0$  keeps unchanged, but the other  $\hat{\omega}_i$  (e.g.,  $\hat{\omega}_n = \omega_n q^{n(n-1)/2}$ ) increase quickly as  $q$  increases. Clearly, the nonuniformity will be even exaggerated if  $q$  is large or the degree  $n$  is high. A direct consequence is that the homogeneous points  $(\hat{\omega}_i \mathbf{P}_i, \hat{\omega}_i)$  locate far away from the origin, leading to poor shape of the homogeneous curve  $\mathbf{R}(t)$  (i.e.,  $\mathbf{R}(t; q)$ ); see Fig. 3 for an example. Furthermore, we want to mention that weight distribution has significant influence on the inherent characteristics of

rational curves, such as optimality of parameterization [3,4], bounds on the magnitude of derivatives [5], etc. In order to improve the nonuniformity of weights, reparameterization methods are required for weighted Lupaş  $q$ -Bézier curves.

### 3. A revisit to quadratic weighted Lupaş $q$ -Bézier curves

According to Remark 1, for a quadratic weighted Lupaş  $q$ -Bézier curve

$$\mathbf{R}_2(t; q) = \frac{\omega_0 \mathbf{P}_0(1-t)^2 + \omega_1(1+q)\mathbf{P}_1 t(1-t) + \omega_2 q \mathbf{P}_2 t^2}{\omega_0(1-t)^2 + \omega_1(1+q)t(1-t) + \omega_2 q t^2}, \quad (5)$$

it can be rewritten as a rational quadratic Bézier curve (or a conic section)

$$\mathbf{R}_2(t) = \frac{\hat{\omega}_0 \mathbf{P}_0 B_{2,0}(t) + \hat{\omega}_1 \mathbf{P}_1 B_{2,1}(t) + \hat{\omega}_2 \mathbf{P}_2 B_{2,2}(t)}{\hat{\omega}_0 B_{2,0}(t) + \hat{\omega}_1 B_{2,1}(t) + \hat{\omega}_2 B_{2,2}(t)} \quad (6)$$

where

$$\hat{\omega}_0 = \omega_0, \quad \hat{\omega}_1 = \omega_1 \frac{1+q}{2}, \quad \hat{\omega}_2 = \omega_2 q. \quad (7)$$

Fig. 3 shows the results of representing a circular arc,  $\mathbf{C}(\theta) = (\cos \theta, \sin \theta)$ ,  $\theta \in [-\frac{\pi}{3}, \frac{\pi}{3}]$ , by three quadratic weighted Lupaş  $q$ -Bézier curves. Although the curve shape is the same, the curve representation in Fig. 3(a) outperforms the other two from the viewpoints of optimality of parameterization and shape of the homogeneous curve. For optimal parameterization of rational quadratic curves, we refer the reader to [4].

Although we are able to modify the shape of weighted Lupaş  $q$ -Bézier curves by adjusting the values of weights  $\omega_i$  and  $q$ , it seems difficult to develop an intuitive scheme to understand what happens if many values are adjusted simultaneously. But fortunately, reparameterization is an available method for improving the inherent quality of the resulting curves [3]; see also Remark 3. By using the Möbius transformation  $t \mapsto \frac{\gamma t}{\gamma t + (1-t)}$  with  $\gamma = \sqrt{\frac{\omega_0}{\omega_2 q}}$ ,  $\mathbf{R}_2(t; q)$  in (5) and  $\mathbf{R}_2(t)$  in (6) are rewritten in standard form:

$$\begin{aligned} \bar{\mathbf{R}}_2(t; q) &= \frac{\mathbf{P}_0(1-t)^2 + \bar{\omega}_1 \mathbf{P}_1 t(1-t) + \mathbf{P}_2 t^2}{(1-t)^2 + \bar{\omega}_1 t(1-t) + t^2} \\ &= \frac{\mathbf{P}_0 B_{2,0}(t) + \hat{\omega}_1 \mathbf{P}_1 B_{2,1}(t) + \mathbf{P}_2 B_{2,2}(t)}{B_{2,0}(t) + \hat{\omega}_1 B_{2,1}(t) + B_{2,2}(t)} \end{aligned} \quad (8)$$

where

$$\bar{\omega}_1 = \frac{1+q}{\sqrt{q}} \sqrt{\frac{\omega_1^2}{\omega_0 \omega_2}}, \quad \hat{\omega}_1 = \frac{\bar{\omega}_1}{2}.$$

This is just what we have done in Fig. 3. By using standard form and assuming  $\omega_0 = \omega_2 = 1$ , shape modification via  $\omega_1$  and/or  $q$  becomes meaningful.

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