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Optimal accelerated SOR-like (ASOR) method for singular symmetric saddle point problems

Xue-Ping Guo[1] and Apostolos Hadjidimos[2]

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Abstract

In a recent paper a new iterative method for the solution of the nonsingular symmetric saddle point problem was proposed [P.N. Njeru, X.-P. Guo, *Accelerated SOR-like (ASOR) method for augmented systems,* BIT Numer. Math. 56 (2016), 557–571 (doi:10.1007/s10543-015-0571-z)]. The ASOR method belongs to the family of the SOR-like methods and uses two parameters α and ω. Convergence intervals for the parameters involved were found. In the present work we analyze and study an extension of the above problem to the singular case, and determine optimal values for the two parameters as well as for the semi-convergence factor of the ASOR method. Numerical results are presented to show the efficiency of the optimal singular ASOR method.

AMS (MOS) Subject Classifications: Primary 65F10. Secondary 65F08

Keywords: singular saddle point problem, singular augmented linear systems, accelerated SOR method, optimal parameters, optimal semi-convergence factor

Running Title: Optimal ASOR method for singular saddle point problem

1 Introduction

Let the singular symmetric saddle point problem be defined by the linear system

$$\mathcal{A}\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} A & B \\ -B^T & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ -q \end{bmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times m}$ is symmetric positive definite, $B \in \mathbb{R}^{m\times n}$, $m \geq n$, is rank deficient, with $\text{rank}(B) = r < n$, $(\cdot)^T$ denotes transpose, $x, p \in \mathbb{R}^m$, $y, q \in \mathbb{R}^n$, and let system (1.1) be consistent that is $[p^T, -q^T]^T \in \text{range}(\mathcal{A})$ or, equivalently, $q \in \text{range}(B^T)$. (*Note:* The right hand side matrix equation (1.1), where the matrix B is of deficient rank, is also widely known as the augmented linear system.)

Linear system (1.1), when $\mathcal{A}$ is symmetric and of full (or deficient) rank arises in various scientific and engineering applications as, e.g., in linearly constrained quadratic programming problems, in weighted least-squares problems, in mixed or hybrid finite element approximations of second-order elliptic PDEs, in elasticity problems (Stokes equations) (see, e.g., [9]) and in Lagrange multipliers methods (see, e.g., [15]), etc. It seems that the oldest methods for the solution of such problems are the Usawa and the preconditioned Uzawa methods (see, e.g., [1], [14], [8]). For an account of

[1](xpguo@math.ecnu.edu.cn) Department of Mathematics, East China Normal University, 500 Dongchuan RD, Shanghai 200241, P.R. China.

[2]Corresponding author (hadjidim@inf.uth.gr), Department of Electrical & Computer Engineering, University of Thessaly, GR-382 21 Volos, Greece.

these applications as well as for related references see, e.g., [25]. Solutions for special cases of the linear system (1.1) have been proposed by many researchers in the area. Here we mention some of the main works based on extensions and generalizations of the classical iterative methods like SOR, SSOR, MSOR, etc. (see, e.g, Varga [26] or Young [31]). However, in our opinion, the most influential works in the development of generalizations of the problem in (1.1) and of methods of solution were the works by Golub et al. [17] and by Bai et al. [5]. In the former work, the SOR-like method was introduced for the solution of the augmented linear system. In the same work an excellent account of the works for the solution of the augmented linear system which had appeared till then can be found. Also, an account of the works till 2009, generalizing the idea of the SOR-like method [17], can be found in Zheng et al. [37]. In what follows we mention some of the main works in the area in the last sixteen years: Golub et al. [17], Bai et al. [5], Darvishi and Hessari [13], Bai and Wang [6], Wu et al. [27], Zheng et al. [37], Zhang and Wei [35], Zhang et al. [32], Zhang and Shen [34], Zhou and Zhang [38], Cao et al. [11], Louka and Missirlis [20], Njeru and Guo [25], although the list is far from being exhausted.

2 Preliminary analysis

In this section we present in a condensed form some of the basic results of the accelerated SOR-like (ASOR) iterative method of [25], properly extended to accommodate the rank deficient matrix B and, therefore, the singular matrix $\mathcal{A}$. The ASOR method for the solution of linear system (1.1) is based on the splitting of the matrix coefficient as follows

$$\mathcal{A} := \begin{bmatrix} A & B \\ -B^T & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha A & 0 \\ 0 & Q \end{bmatrix}}_{D} - \underbrace{\begin{bmatrix} -A & 0 \\ B^T & \frac{1}{2}Q \end{bmatrix}}_{L} - \underbrace{\begin{bmatrix} \alpha A & -B \\ 0 & \frac{1}{2}Q \end{bmatrix}}_{U}, \tag{2.1}$$

where α is a positive real number and $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and at the same time an approximate to the Schur complement $B^T A^{-1} B$ of $\mathcal{A}$. Then, we apply the SOR iterative method based on the above splitting and assume that its relaxation factor $\omega \in \mathbb{R}\backslash\{0, 2, -\alpha\}$ we obtain

$$\begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = (D - \omega L)^{-1}\left((1-\omega)D + \omega U\right) \begin{bmatrix} x^{(k)} \\ y^{(k)} \end{bmatrix} + \omega\,(D-\omega L)^{-1} \begin{bmatrix} p \\ -q \end{bmatrix}. \tag{2.2}$$

In view of (2.1) the iteration matrix $M_{\alpha,\omega}$ of ASOR method in (2.2) becomes successively

$$\begin{aligned} M_{\alpha,\omega} &= (D-\omega L)^{-1}\left((1-\omega)D + \omega U\right) = \begin{bmatrix} (\alpha+\omega)A & 0 \\ -\omega B^T & \frac{2-\omega}{2}Q \end{bmatrix}^{-1} \begin{bmatrix} \alpha A & -\omega B \\ 0 & \frac{2-\omega}{2}Q \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\alpha+\omega}A^{-1} & 0 \\ \frac{2\omega}{(\alpha+\omega)(2-\omega)}Q^{-1}B^T A^{-1} & \frac{2}{2-\omega}Q^{-1} \end{bmatrix} \begin{bmatrix} \alpha A & -\omega B \\ 0 & \frac{2-\omega}{2}Q \end{bmatrix} \\ &= \begin{bmatrix} \frac{\alpha}{\alpha+\omega}I_m & -\frac{\omega}{\alpha+\omega}A^{-1}B \\ \frac{2\alpha\omega}{(\alpha+\omega)(2-\omega)}Q^{-1}B^T & I_n - \frac{2\omega^2}{(\alpha+\omega)(2-\omega)}Q^{-1}B^T A^{-1} B \end{bmatrix}. \end{aligned} \tag{2.3}$$

Considering (2.2) and (2.3), the ASOR method can be given by the simple two-sweep iterative scheme below

$$\begin{aligned} x^{(k+1)} &= \tfrac{\alpha}{\alpha+\omega}x^{(k)} - \tfrac{\omega}{\alpha+\omega}A^{-1}(By^{(k)} - p), \\ y^{(k+1)} &= y^{(k)} + \tfrac{2\omega}{2-\omega}Q^{-1}(B^T x^{(k+1)} - q). \end{aligned} \tag{2.4}$$

Lemma 2.1. *(Lemma 1 of [25] extended): Let the parameters α and ω satisfy $\alpha > 0$, $\omega \in \mathbb{R}\backslash\{0, 2, -\alpha\}$ and $\sigma(\cdot)$ denote "eigenvalue spectrum".*
a) If $\lambda \in \sigma(M_{\alpha,\omega}) \setminus \left\{1, \frac{\alpha}{\alpha+\omega}\right\}$ and $\mu \neq 0$ satisfies

$$(\lambda - 1)(2 - \omega)(\alpha - \alpha\lambda - \omega\lambda) = 2\lambda\omega^2\mu, \tag{2.5}$$

then $\mu \in \sigma\left(Q^{-1}B^TA^{-1}B\right)$ and vice versa.
b) Similarly, if $\lambda \in \left\{1, \frac{\alpha}{\alpha+\omega}\right\} \subset \sigma(M_{\alpha,\omega})$, then $\mu = 0 \in \sigma\left(Q^{-1}B^TA^{-1}B\right)$ and vice versa.

Proof: a) For the proof of the first part of the statement, let λ be an eigenvalue of $M_{\alpha,\omega}$ and $[x'^T, y'^T]^T$ be the corresponding eigenvector, we will have

$$M_{\alpha,\omega}\begin{bmatrix} x' \\ y' \end{bmatrix} = \lambda \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Using the most suitable expression for $M_{\alpha,\omega}$ from (2.3), the above equation becomes

$$\begin{aligned} &\begin{bmatrix} (\alpha+\omega)A & 0 \\ -\omega B^T & \frac{2-\omega}{2}Q \end{bmatrix}^{-1}\begin{bmatrix} \alpha A & -\omega B \\ 0 & \frac{2-\omega}{2}Q \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = \lambda\begin{bmatrix} x' \\ y' \end{bmatrix} \iff \\ &\begin{bmatrix} \alpha A & -\omega B \\ 0 & \frac{2-\omega}{2}Q \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = \lambda\begin{bmatrix} (\alpha+\omega)A & 0 \\ -\omega B^T & \frac{2-\omega}{2}Q \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}. \end{aligned} \tag{2.6}$$

Then, equating the components of the vectors of the two sides in the second equation of (2.6) and rearranging terms we obtain

$$(\alpha - \alpha\lambda - \omega\lambda)Ax' = \omega By' \quad \text{and} \quad (\lambda - 1)(2 - \omega)Qy' = 2\lambda\omega B^Tx'. \tag{2.7}$$

In view of the assumptions on α and ω, then even if we assume that $\lambda = 0$, something that does not satisfy (2.5), from the second equation of (2.7) we obtain $y' = 0$ and then from the first one we have $x' = 0$, which is not possible. Hence $\lambda \neq 0$. Note that to the same non possible result we arrive if we consider $y' = 0$ and follow the same reasoning as before. Now, if $x' = 0$ the second equation of (2.7) gives $y' = 0$ which also satisfies the first equation; hence, this is again not possible. From the first relation in (2.7) we get $x = \frac{\omega}{\alpha-\alpha\lambda-\omega\lambda}A^{-1}By'$, which when it is replaced into the second equation of (2.7) gives, after a little manipulation,

$$Q^{-1}B^TA^{-1}By' = \frac{(\lambda-1)(2-\omega)(\alpha-\alpha\lambda-\omega\lambda)}{2\lambda\omega^2}y'.$$

Since, $y' \neq 0$, it is implied that $\mu = \frac{(\lambda-1)(2-\omega)(\alpha-\alpha\lambda-\omega\lambda)}{2\lambda\omega^2}$ is a nonzero eigenvalue of $Q^{-1}B^TA^{-1}B$ which satisfies relation (2.5).
Conversely, let $\mu \in \sigma(Q^{-1}B^TA^{-1}B)\backslash\{0\}$ and y'' be the corresponding eigenvector. Then, (2.5) gives

$$\frac{(\lambda-1)(2-\omega)(\alpha-\alpha\lambda-\omega\lambda)}{2\lambda\omega^2}y'' = Q^{-1}B^TA^{-1}By''$$

which can be rewritten as

$$\frac{(\lambda-1)(2-\omega)}{2}Qy'' = \frac{\omega^2\lambda}{\alpha-\alpha\lambda-\omega\lambda}B^TA^{-1}By''.$$

Setting $x'' = \frac{\omega}{\alpha-\alpha\lambda-\omega\lambda}A^{-1}By''$ and using the previous relation we can have

$$(\alpha-\alpha\lambda-\omega\lambda)Ax'' = \omega By'' \quad \text{and} \quad \frac{(\lambda-1)(2-\omega)}{2}Qy'' = \omega\lambda B^T x''. \tag{2.8}$$

The equality on the left and the equality on the right of (2.8) can be rewritten, respectively, as

$$\alpha Ax'' - \omega By'' = \lambda(\alpha+\omega)Ax'' \;\text{ and }\; \tfrac{2-\omega}{2}Qy'' = \lambda(-\omega B^T x'' + \tfrac{2-\omega}{2}Qy''). \tag{2.9}$$

Writing relations (2.9) in matrix form we take

$$\begin{bmatrix} \alpha A & -\omega B \\ 0 & \frac{2-\omega}{2}Q \end{bmatrix} \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \lambda \begin{bmatrix} (\alpha+\omega)A & 0 \\ -\omega B^T & \frac{2-\omega}{2}Q \end{bmatrix} \begin{bmatrix} x'' \\ y'' \end{bmatrix}$$

or

$$\begin{bmatrix} (\alpha+\omega)A & 0 \\ -\omega B^T & \frac{2-\omega}{2}Q \end{bmatrix}^{-1} \begin{bmatrix} \alpha A & -\omega B \\ 0 & \frac{2-\omega}{2}Q \end{bmatrix} \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \lambda \begin{bmatrix} x'' \\ y'' \end{bmatrix} \iff M_{\alpha,\omega} \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \lambda \begin{bmatrix} x'' \\ y'' \end{bmatrix}$$

and the converse of the first part has just been proved.

b) For the proof of the second part we readily see from (2.5) that if $\lambda = 1$ or $\lambda = \frac{\alpha}{\alpha+\omega}$ then $\mu = 0$ while if $\mu = 0$ then either $\lambda = 1$ or $\lambda = \frac{\alpha}{\alpha+\omega}$. Furthermore, if $\lambda = 1$ then from the second equation of (2.7) we obtain $x' \in \mathcal{N}(B^T)$, with $\mathcal{N}(\cdot)$ denoting "*nullspace*", while from the first equation of (2.7) we have for $A^{-1}By' \in \mathcal{N}(B^T)$ (or $By' \in \mathcal{N}(B^T)$). Therefore, as an eigenvector corresponding to the eigenvalue $\lambda = 1$ we can take $\left[x'^T, y'^T\right]^T$, with $x' \in \mathcal{N}(B^T)\backslash\{0\}$ and $y' = 0$. On the other hand, if $\lambda = \frac{\alpha}{\alpha+\omega}$ then from the first equation of (2.7) it is $y' \in \mathcal{N}(B)$ in which case from the second equation of (2.7) we can have, similarly $Q^{-1}B^T x' \in \mathcal{N}(B)$ (or $B^T x' \in \mathcal{N}(B)$). In this case we can take as an eigenvector corresponding to the eigenvalue $\lambda = \frac{\alpha}{\alpha+\omega}$, the vector $\left[x'^T, y'^T\right]^T$, where $x' = 0$ and $y' \in \mathcal{N}(B)\backslash\{0\}$.
The proof of the converse is straightforward and is therefore omitted. □

Next we find the conditions under which, except 1 and $\frac{\alpha}{\alpha+\omega}$, all other eigenvalues of the iteration matrix of ASOR method are strictly less than 1 in modulus. For this we need the following lemma.

Lemma 2.2. *(Lemma 6.2.1 of Young [31]): If b and c are real, then both roots of the quadratic equation*

$$x^2 - bx + c = 0 \tag{2.10}$$

are strictly less than one in modulus if and only if

$$|c| < 1, \quad |b| < 1 + c. \tag{2.11}$$

Theorem 2.1. *(Theorem 1 of [25] extended): Let the parameters α and ω satisfy the assumptions of Lemma 2.1, then equation (2.5) can be rewritten as*

$$\lambda^2 - \frac{\left((2\alpha+\omega)(2-\omega) - 2\omega^2\mu\right)}{(2-\omega)(\alpha+\omega)}\lambda + \frac{\alpha}{\alpha+\omega} = 0. \tag{2.12}$$

Except 1 *and* $\frac{\alpha}{\alpha+\omega}$ *yielded for* $\mu = 0$*, all other eigenvalues* λ *of* $M_{\alpha,\omega}$ *are strictly less than one in modulus if and only if*

$$\alpha > 0 \ (by \ \ assumption), \quad 0 < \omega < 2, \quad \mu_{\max} < \frac{(2\alpha+\omega)(2-\omega)}{\omega^2}, \tag{2.13}$$

where $\mu_{\max}$ *is the maximum eigenvalue of* $Q^{-1}B^TA^{-1}B$.

Proof: The proof is identical to that in [25] except that the statement of the theorem refers to the eigenvalues 1 and $\frac{\alpha}{\alpha+\omega}$ of $M_{\alpha,\omega}$ as well, while conditions (2.13) are presented in a little different form. Note, however, that due to conditions (2.13) the eigenvalue $\lambda = \frac{\alpha}{\alpha+\omega}$ of $M_{\alpha,\omega}$ corresponding to $\mu = 0$ also satisfies $\lambda = \frac{\alpha}{\alpha+\omega} \in (0,1)$. In addition, we would like to make a point regarding the condition $\alpha > 0$ and the third condition of (2.13). Solving the last condition for α we get $\frac{\omega(\omega\mu_{\max}-2+\omega)}{2(2-\omega)} < \alpha$. So the last relation holds for all $\alpha > 0$ if and only if $\omega\mu_{\max} - 2 + \omega \leq 0$ or $\mu_{\max} \leq \frac{2-\omega}{\omega}$. However, since $\lim_{\omega\to 2^-}\left(\frac{2-\omega}{\omega}\right) = 0$ and $\lim_{\omega\to 0^+}\left(\frac{2-\omega}{\omega}\right) = +\infty$ it is implied that the third relation in (2.13) holds for any possible $\mu_{\max} \in (0,+\infty)$ and for any $\alpha > 0$ provided ω satisfies $\mu_{\max} \leq \frac{2-\omega}{\omega} \Leftrightarrow \omega \leq \frac{2}{1+\mu_{\max}}$ or, equivalently, for any $\omega \in (\ 0,\ \frac{2}{1+\mu_{\max}}\ ] \subset (0,2)$. □

3 Semi-convergence of the ASOR method

Since $M_{\alpha,\omega}$ has an eigenvalue equal to 1 our ASOR iterative method cannot converge in the classical sense (see, e.g., Varga [26], Young [31], Berman and Plemmons [7], et al.), i.e., for all right-hand sides and all initial guesses. However, if certain conditions hold our method may semi-converge. For semi-convergence to take place some basic conditions must hold. These are given below taken from Berman and Plemmons [7].

Lemma 3.1. *(Definition (4.8) and Exercise (4.9) on Page 152 of Berman and Plemmons [7]): Let* $T \in \mathbb{R}^{s\times s}$*. Then* T *is semi-convergent if and only if each of the following conditions holds:*

1. $\rho(T) \leq 1$*, where* $\rho(\cdot)$ *denotes spectral radius.*
2. *If* $\rho(T) = 1$ *then* $index(I_s - T) = 1$ $\big(index(I_s - T) = 1 \ \Leftrightarrow \ rank\big((I_s - T)^2\big) = rank(I_s - T)\big)$.
3. *If* $\rho(T) = 1$ *then* $\lambda \in \sigma(T)$ *with* $|\lambda| = 1$ *implies* $\lambda = 1$.

A lemma equivalent to Lemma 3.1 is the following.

Lemma 3.2. *(Lemma 2.2 of [37]) Let* $H \in \mathbb{C}^{l\times l}$ *and* $I_{s-l} \in \mathbb{C}^{(s-l)\times(s-l)}$ *be the identity matrix, then the block partitioned matrix*

$$T = \begin{bmatrix} H & 0_{l,s-l} \\ L & I_{l_{s-l}} \end{bmatrix} \tag{3.1}$$

is semi-convergent if either $L = 0$ *and* H *is semi-convergent or if* $\rho(H) < 1$.

Definition 3.1. *If* T *of Lemma 3.1 is semi-convergent then the quantity*

$$\gamma(T) = \max\{|\lambda| \big| \lambda \in \sigma(T), \lambda \neq 1\} \tag{3.2}$$

is called semi-convergence factor.

Lemma 3.3. *Let $T \in \mathbb{R}^{s\times s}$ be semi-convergent. Then, the iterative scheme*

$$z^{(k+1)} = Tz^{(k)} + c, \;\; k = 0, 1, 2, \cdots , \;\; z^{(0)} \in \mathbb{R}^s,$$

semi-converges, namely

$$\lim_{k\to\infty} z^{(k)} = (I_s - T)^D c + (I_s - E)z^{(0)}, \quad E = (I_s - T)(I_s - T)^D, \tag{3.3}$$

(see Berman and Plemmons [7], formula (6.14) on page 199), where $(\cdot)^D$ denotes Drazin inverse (see same reference).

Next we present a statement which will be of help in the sequel.

Lemma 3.4. *Under the assumptions on its factors, the matrix $Q^{-1}B^TA^{-1}B \in \mathbb{R}^{n\times n}t$ is semi-positive definite with index$\big(Q^{-1}B^TA^{-1}B\big) = 1$ and a zero eigenvalue of multiplicity $n - r$.*

Proof: Let $\operatorname{rank}(B) = r$ and consider the singular value decomposition of B, following a similar analysis to the corresponding part of Theorem 3.1 of [37] (see [16]), and let it be as follows

$$B = U \underbrace{\begin{bmatrix} \Sigma & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}}_{S} V^T,$$

where

$$U \in \mathbb{R}^{m\times m}, \; V^T \in \mathbb{R}^{n\times n}, \; \Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \cdots , \sigma_r), \tag{3.4}$$

with U and V unitary matrices and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ the singular values of B. Partitioning $V^TQ^{-1}V$ and $U^TA^{-1}U$ in conformity with the partition of S^T and S, respectively, we obtain

$$\begin{aligned}
V^TQ^{-1}B^TA^{-1}BV &= (V^TQ^{-1}V)(V^TB^TU)(U^TA^{-1}U)(U^TBV) \\
&= (V^TQ^{-1}V) \underbrace{\begin{bmatrix} \Sigma & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix}}_{S^T} (U^TA^{-1}U) \underbrace{\begin{bmatrix} \Sigma & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}}_{S} \\
&= \begin{bmatrix} (V^TQ^{-1}V)_{r,r} & (V^TQ^{-1}V)_{r,n-r} \\ (V^TQ^{-1}V)_{n-r,r} & (V^TQ^{-1}V)_{n-r,n-r} \end{bmatrix} \begin{bmatrix} \Sigma & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix} \\
&\times \begin{bmatrix} (U^TA^{-1}U)_{r,r} & (U^TA^{-1}U)_{r,m-r} \\ (U^TA^{-1}U)_{m-r,r} & (U^TA^{-1}U)_{m-r,m-r} \end{bmatrix} \begin{bmatrix} \Sigma & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} \\
&= \begin{bmatrix} (V^TQ^{-1}V)_{r,r}\Sigma(U^TA^{-1}U)_{r,r}\Sigma & 0_{r,n-r} \\ (V^TQ^{-1}V)_{n-r,r}\Sigma(U^TA^{-1}U)_{r,r}\Sigma & 0_{n-r,n-r} \end{bmatrix}.
\end{aligned} \tag{3.5}$$

The last matrix above is block diagonal with its $(1, 1)$ block $(V^TQ^{-1}V)_{r,r}\Sigma(U^TA^{-1}U)_{r,r}\Sigma$ positive definite, since it is similar to the matrix

$$(V^TQ^{-1}V)_{r,r}^{-\frac{1}{2}}\Sigma(U^TA^{-1}U)_{r,r}\Sigma(V^TQ^{-1}V)_{r,r}^{-\frac{1}{2}}$$

which is symmetric positive definite, while its $(2, 2)$ block is an $(n - r) \times (n - r)$ zero matrix. These results prove both assertions of our statement which complete the proof. □

Note: A symmetric positive definite matrix $C \in \mathbb{R}^{r\times r}$ has a symmetric positive definite square root denoted by $C^{\frac{1}{2}}$ (see, e.g., Theorem 2.2.7 of Young [31]).

Theorem 3.1. *Let the parameters α and ω of the ASOR method satisfy the conditions (2.13). Then, the iteration matrix $M_{\alpha,\omega}$ of the ASOR method (2.2) is semi-convergent; namely, there exists a similarity transformation that transforms it in the form of matrix T of Lemma 3.2 whose assumptions are satisfied. In addition, rank$(M_{\alpha,\omega}) = m + r$.*

Proof: The matrix $M_{\alpha,\omega}$ in (2.3) can successively be written as

$$\begin{aligned} M_{\alpha,\omega} &= \begin{bmatrix} \frac{\alpha}{\alpha+\omega} I_m & -\frac{\omega}{\alpha+\omega} A^{-1}B \\ \frac{2\alpha\omega}{(\alpha+\omega)(2-\omega)} Q^{-1}B^T & I_n - \frac{2\omega^2}{(\alpha+\omega)(2-\omega)} Q^{-1}B^T A^{-1}B \end{bmatrix} \\ &= \begin{bmatrix} \left(1-\frac{\omega}{\alpha+\omega}\right) I_m & -\frac{\omega}{\alpha+\omega} A^{-1}B \\ (1-\frac{\omega}{\alpha+\omega}) \cdot \frac{2\omega}{\alpha+\omega} Q^{-1}B^T & I_n - \frac{\omega}{\alpha+\omega} \cdot \frac{2\omega}{2-\omega} Q^{-1}B^T A^{-1}B \end{bmatrix} \\ &= \begin{bmatrix} (1-\phi) I_m & -\phi A^{-1}B \\ (1-\phi)\psi Q^{-1}B^T & I_n - \phi\psi Q^{-1}B^T A^{-1}B \end{bmatrix}, \end{aligned} \tag{3.6}$$

where we have set

$$\phi := \frac{\omega}{\alpha+\omega}, \quad \psi := \frac{2\omega}{2-\omega}. \tag{3.7}$$

Consider the block diagonal matrices diag(U^T, V^T) and diag(U, V) where U and V are the unitary matrices of the singular value decomposition of $U^T BV$ of Lemma 3.4. Then, from the last matrix in (3.6) we form its similar matrix

$$\begin{aligned} &\begin{bmatrix} U^T & \\ & V^T \end{bmatrix} \begin{bmatrix} (1-\phi) I_m & -\phi A^{-1}B \\ (1-\phi)\psi Q^{-1}B^T & I_n - \phi\psi Q^{-1}B^T A^{-1}B \end{bmatrix} \begin{bmatrix} U & \\ & V \end{bmatrix} = \\ &\begin{bmatrix} (1-\phi) I_m & -\phi(U^T A^{-1}U)(U^T BV) \\ (1-\phi)\psi(V^T Q^{-1}V)(U^T BV)^T & I_n - \phi\psi(V^T Q^{-1}V)(U^T BV)^T(U^T A^{-1}U)(U^T BV) \end{bmatrix}. \end{aligned} \tag{3.8}$$

Next, we analyze each block element, except the $(1,1)$ block, of the rightmost matrix in (3.8) using Lemma 3.4. So we have:

Block element $(1,2)$ apart from its coefficient:

$$\begin{aligned} (U^T A^{-1}U)(U^T BV) &= \begin{bmatrix} (U^T A^{-1}U)_{r,r} & (U^T A^{-1}U)_{r,m-r} \\ (U^T A^{-1}U)_{m-r,r} & (U^T A^{-1}U)_{m-r,m-r} \end{bmatrix} \begin{bmatrix} \Sigma & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} \\ &= \begin{bmatrix} (U^T A^{-1}U)_{r,r}\Sigma & 0_{r,n-r} \\ (U^T A^{-1}U)_{m-r,r}\Sigma & 0_{m-r,n-r} \end{bmatrix}. \end{aligned}$$

Block element $(2,1)$ apart from its coefficient:

$$\begin{aligned} (V^T Q^{-1}V)(U^T BV)^T &= \begin{bmatrix} (V^T Q^{-1}V)_{r,r} & (V^T Q^{-1}V)_{r,n-r} \\ (V^T Q^{-1}V)_{n-r,r} & (V^T Q^{-1}V)_{n-r,n-r} \end{bmatrix} \begin{bmatrix} \Sigma & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix} \\ &= \begin{bmatrix} (V^T Q^{-1}V)_{r,r}\Sigma & 0_{r,m-r} \\ (V^T Q^{-1}V)_{n-r,r}\Sigma & 0_{n-r,m-r} \end{bmatrix}. \end{aligned}$$

Block element $(2,2)$ apart from the unit matrix I_n and the coefficient of the following product of matrices has already been found in Lemma 3.4 and is the rightmost matrix in (3.5). Reconstructing the rightmost matrix in (3.8) by using its four blocks and making a new repartitioning we have the following

$$\left[\begin{array}{ccc|c} (1-\phi)I_r & 0_{r,m-r} & -\phi(U^TA^{-1}U)_{r,r}\Sigma & 0_{r,n-r} \\ 0_{m-r,r} & (1-\phi)I_{m-r} & -\phi(U^TA^{-1}U)_{m-r,r}\Sigma & 0_{m-r,n-r} \\ (1-\phi)\psi(V^TQ^{-1}V)_{r,r}\Sigma & 0_{r,m-r} & I_r-\phi\psi(V^TQ^{-1}V)_{r,r}\Sigma(U^TA^{-1}U)_{r,r}\Sigma & 0_{r,n-r} \\ \hline (1-\phi)\psi(V^TQ^{-1}V)_{n-r,r}\Sigma & 0_{n-r,m-r} & I_{n-r}-\phi\psi(V^TQ^{-1}V)_{n-r,r}\Sigma(U^TA^{-1}U)_{r,r} & I_{n-r} \end{array}\right]. \tag{3.9}$$

It is observed that the matrix in (3.9) is a lower block triangular matrix whose upper diagonal block is an $(m+r)\times(m+r)$ matrix and the lower block is the I_{n-r} unit matrix. By virtue of Theorem 2.1 and its proof, all the eigenvalues of the iteration matrix $M_{\alpha,\omega}$ except 1, or equivalently, of its similar matrix in (3.9) are strictly less than 1 in modulus. In other words, to complete our proof we have to show that rank $(M_{\alpha,\omega}) = m+r$ or by virtue of Lemma 3.2 that the upper diagonal block of the matrix (3.9) is convergent or, equivalently, that the difference of this block from the unit matrix I_{m+r} has **no** zero eigenvalue. For this, we form the difference

$$\begin{array}{l} I_{m+r} - \begin{bmatrix} (1-\phi)I_r & 0_{r,m-r} & -\phi(U^TA^{-1}U)_{r,r}\Sigma \\ 0_{m-r,r} & (1-\phi)I_{m-r} & -\phi(U^TA^{-1}U)_{m-r,r}\Sigma \\ (1-\phi)\psi(V^TQ^{-1}V)_{r,r}\Sigma & 0_{r,m-r} & I_r-\phi\psi(V^TQ^{-1}V)_{r,r}\Sigma(U^TA^{-1}U)_{r,r}\Sigma \end{bmatrix} \\ = \begin{bmatrix} \phi I_r & 0_{r,m-r} & \phi(U^TA^{-1}U)_{r,r}\Sigma \\ 0_{m-r,r} & \phi I_{m-r} & \phi(U^TA^{-1}U)_{m-r,r}\Sigma \\ -(1-\phi)\psi(V^TQ^{-1}V)_{r,r}\Sigma & 0_{r,m-r} & \phi\psi(V^TQ^{-1}V)_{r,r}\Sigma(U^TA^{-1}U)_{r,r}\Sigma \end{bmatrix}, \end{array} \tag{3.10}$$

next we consider the determinant of the matrix in (3.10), then we multiply the first block row by the nonsingular block matrix $\frac{(1-\phi)\psi}{\phi}(V^TQ^{-1}V)_{r,r}\Sigma$, and finally add it to its third block row to successively obtain

$$\begin{array}{l} \det\left(\begin{bmatrix} \phi I_r & 0_{r,m-r} & \phi(U^TA^{-1}U)_{r,r}\Sigma \\ 0_{m-r,r} & \phi I_{m-r} & \phi(U^TA^{-1}U)_{m-r,r}\Sigma \\ -(1-\phi)\psi(V^TQ^{-1}V)_{r,r}\Sigma & 0_{r,m-r} & \phi\psi(V^TQ^{-1}V)_{r,r}\Sigma(U^TA^{-1}U)_{r,r}\Sigma \end{bmatrix}\right) = \\ \det\left(\begin{bmatrix} \phi I_r & 0_{r,m-r} & \phi(U^TA^{-1}U)_{r,r}\Sigma \\ 0_{m-r,r} & \phi I_{m-r} & \phi(U^TA^{-1}U)_{m-r,r}\Sigma \\ 0_{r,r} & 0_{r,m-r} & \psi(V^TQ^{-1}V)_{r,r}\Sigma(U^TA^{-1}U)_{r,r}\Sigma \end{bmatrix}\right) = \\ \phi^m\psi^r \det\big((V^TQ^{-1}V)_{r,r}\big)\det\big((\Sigma(U^TA^{-1}U)_{r,r}\Sigma\big) > 0. \end{array} \tag{3.11}$$

The last result effectively proves that the difference we considered has **no** zero eigenvalue or that the upper block of the matrix in (3.10) has **no** eigenvalue 1 and Lemma 3.2 applies. Hence, rank $(M_{\alpha,\omega}) = m+r$, which completes the proof of the present theorem. □

4 Optimal parameters and optimal semi-convergence factor

Since as was proved index$(I_{m+n}-M_{\alpha,\omega}) = 1$ the ASOR method is semi-convergent. So, to determine the parameters α and ω that minimize the semi-convergence factor $\gamma(M_{\alpha,\omega})$ we base our analysis on the restrictions (2.13).

Note that (2.12) has the product of its roots $c = \frac{\alpha}{\alpha+\omega} > 0$; c is also an eigenvalue of $M_{\alpha,\omega}$ corresponding to $\mu = 0$ as was noted in the second part of the proof of Lemma 2.1. The positiveness of c means that both roots λ_1, λ_2 are either real of the same sign or complex conjugate numbers.

In the former case if they are not equal one of them, say λ_1, will be in modulus $|\lambda_1| > \sqrt{c}$ $(> |\lambda_2|)$. On the other hand, in case they are equal or complex conjugate, both roots will have modulus $|\lambda_1| = |\lambda_2| = \sqrt{c} \left(> c = \frac{\alpha}{\alpha+\omega}\right)$. This means that for any fixed pair of values for α and ω the smallest modulus of λ_1, λ_2 is achieved if and only if the latter case is satisfied. For this to hold the discriminant D of (2.12) for all possible $\mu > 0$ must be ≤ 0. Hence, we require that

$$D := \left(\frac{2\omega^2\mu}{(2-\omega)(\alpha+\omega)} - \frac{2\alpha+\omega}{\alpha+\omega}\right)^2 - 4\frac{\alpha}{\alpha+\omega} \leq 0. \tag{4.1}$$

The right relation in (4.1) is equivalent to

$$4\omega^2\mu^2 - 4(2\alpha+\omega)(2-\omega)\mu + (2-\omega)^2 \leq 0, \tag{4.2}$$

whose discriminant D_μ is given by

$$D_\mu = 64(2-\omega)^2\alpha(\alpha+\omega) > 0. \tag{4.3}$$

Since the last discriminant is positive and the coefficient of the highest power of μ in (4.2), $4\omega^2$, is also positive it is implied that the inequality in (4.2) holds true for all μ satisfying

$$\frac{(2-\omega)\left[(2\alpha+\omega) - 2\sqrt{\alpha(\alpha+\omega)}\right]}{2\omega^2} \leq \mu \leq \frac{(2-\omega)\left[(2\alpha+\omega) + 2\sqrt{\alpha(\alpha+\omega)}\right]}{2\omega^2}, \tag{4.4}$$

provided the fraction on the right satisfies the restriction on μ in (2.13), namely

$$\frac{(2-\omega)\left[(2\alpha+\omega) + 2\sqrt{\alpha(\alpha+\omega)}\right]}{2\omega^2} < \frac{(2-\omega)(2\alpha+\omega)}{\omega^2}. \tag{4.5}$$

The inequality in (4.5) is equivalent to

$$2\sqrt{\alpha(\alpha+\omega)} < (2\alpha+\omega) \quad \text{or} \quad 0 < \omega^2,$$

which always hold.

Next, let $\mu_{\min}$ and $\mu_{\max}$ be the minimum and the maximum positive eigenvalues of $Q^{-1}B^TA^{-1}B$, respectively, and then set

$$\begin{aligned} \nu_{\min} &:= \tfrac{(2-\omega)\left[(2\alpha+\omega)-2\sqrt{\alpha(\alpha+\omega)}\right]}{2\omega^2}, \quad \nu_{\min} \in (0, \mu_{\min}], \\ \nu_{\max} &:= \tfrac{(2-\omega)\left[(2\alpha+\omega)+2\sqrt{\alpha(\alpha+\omega)}\right]}{2\omega^2}, \quad \nu_{\max} \in [\mu_{\max}, +\infty). \end{aligned} \tag{4.6}$$

Relations (4.6) guarantee that the discriminant D in (4.1) will be ≤ 0 for all $\mu > 0$. From (4.6) we can obtain that

$$\begin{aligned} s &:= \nu_{\max} + \nu_{\min} = \tfrac{(2-\omega)(2\alpha+\omega)}{\omega^2}, \\ d &:= \nu_{\max} - \nu_{\min} = \tfrac{2(2-\omega)\sqrt{\alpha(\alpha+\omega)}}{\omega^2} \end{aligned} \tag{4.7}$$

or

$$s = \left(\frac{2}{\omega} - 1\right)\left(2\frac{\alpha}{\omega} + 1\right) \quad \text{and} \quad d = 2\left(\frac{2}{\omega} - 1\right)\sqrt{\left(\frac{\alpha}{\omega}\right)^2 + \frac{\alpha}{\omega}}. \tag{4.8}$$

Setting

$$\gamma := \frac{2}{\omega} \quad \text{and} \quad \delta := \frac{\alpha}{\omega} \tag{4.9}$$

and substituting in (4.8) we have that

$$s = (\gamma - 1)(2\delta + 1) \quad \text{and} \quad d = 2(\gamma - 1)\sqrt{\delta^2 + \delta}. \tag{4.10}$$

Then, dividing relations (4.10) by each other we obtain $\frac{2\delta+1}{2\sqrt{\delta^2+\delta}} = \frac{s}{d}$, from which we get

$$4(s^2 - d^2)\delta^2 + 4(s^2 - d^2)\delta - d^2 = 0$$

whose positive root, the other is negative, is

$$\delta = \frac{s - \sqrt{s^2 - d^2}}{2\sqrt{s^2 - d^2}}. \tag{4.11}$$

Substituting the value of δ into any of the two relations in (4.10), say the first one, we take

$$\gamma = \frac{s}{2\delta + 1} + 1 = \frac{s}{\frac{s - \sqrt{s^2 - d^2}}{\sqrt{s^2 - d^2}} + 1} + 1 = \sqrt{s^2 - d^2} + 1. \tag{4.12}$$

Hence, from (4.9) we obtain that

$$\begin{aligned} \omega &= \tfrac{2}{\gamma} = \tfrac{2}{1+\sqrt{s^2-d^2}}, \\ \alpha = \delta\omega &= \tfrac{s-\sqrt{s^2-d^2}}{2\sqrt{s^2-d^2}} \tfrac{2}{1+\sqrt{s^2-d^2}} = \tfrac{s-\sqrt{s^2-d^2}}{\sqrt{s^2-d^2}(1+\sqrt{s^2-d^2})}. \end{aligned} \tag{4.13}$$

Having found the values for α and ω we can readily determine these values in terms of $\nu_{\min}$ and $\nu_{\max}$ from (4.6). Then, we eventually have that

$$\alpha = \frac{(\sqrt{\nu_{\max}} - \sqrt{\nu_{\min}})^2}{2\sqrt{\nu_{\max}\nu_{\min}}\left(1 + 2\sqrt{\nu_{\max}\nu_{\min}}\right)}, \quad \omega = \frac{2}{1 + 2\sqrt{\nu_{\max}\nu_{\min}}}. \tag{4.14}$$

Using the value of $c = \frac{\alpha}{\alpha+\omega} \in (0,1)$ and the values for α and ω from (4.14), the semi-convergence factor of the iteration matrix $M_{\alpha,\omega}$ is found to be

$$\gamma\left(M_{\alpha,\omega}\right) = \sqrt{c} = \sqrt{\frac{\alpha}{\alpha+\omega}} = \left(\frac{\frac{(\sqrt{\nu_{\max}}-\sqrt{\nu_{\min}})^2}{2\sqrt{\nu_{\max}\nu_{\min}}(1+2\sqrt{\nu_{\max}\nu_{\min}})}}{\frac{(\sqrt{\nu_{\max}}-\sqrt{\nu_{\min}})^2}{2\sqrt{\nu_{\max}\nu_{\min}}(1+2\sqrt{\nu_{\max}\nu_{\min}})} + \frac{2}{1+2\sqrt{\nu_{\max}\nu_{\min}}}} \right)^{\frac{1}{2}} = \frac{\sqrt{\nu_{\max}}-\sqrt{\nu_{\min}}}{\sqrt{\nu_{\max}}+\sqrt{\nu_{\min}}}. \tag{4.15}$$

Note that the semi-convergence factor just found in the rightmost side of (4.15) can be written as

$$\frac{\sqrt{\frac{\nu_{\max}}{\nu_{\min}}} - 1}{\sqrt{\frac{\nu_{\max}}{\nu_{\min}}} + 1}.$$

The latter expression is a strictly increasing function of the condition number $\frac{\nu_{\max}}{\nu_{\min}}$. However, in view of the definitions of $\nu_{\max}$ and $\nu_{\min}$ in (4.6) it is concluded that the best condition number will be the smallest value of the aforementioned ratio that is $\frac{\mu_{\max}}{\mu_{\min}}$ which is assumed for $\nu_{\max} = \mu_{\max}$ and $\nu_{\min} = \mu_{\min}$. Consequently, from (4.14) and (4.15) the optimal parameters α_{opt} and ω_{opt} and the optimal semi-convergence factor $\gamma\left(M_{\alpha_{opt},\omega_{opt}}\right)$, respectively, are given in the following statement.

Theorem 4.1. *Under the assumptions on the parameters α and ω given in (2.13) and the detailed preceding analysis, the optimal parameters α_{opt}, ω_{opt} and the optimal semi-convergence factor $\gamma\left(M_{\alpha_{opt},\omega_{opt}}\right)$ are*

$$\alpha_{opt}=\frac{(\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}})^2}{2\sqrt{\mu_{\max}\mu_{\min}}\left(1+2\sqrt{\mu_{\max}\mu_{\min}}\right)},\quad \omega_{opt}=\frac{2}{1+2\sqrt{\mu_{\max}\mu_{\min}}},\quad \gamma\left(M_{\alpha_{opt},\omega_{opt}}\right)=\frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}}. \tag{4.16}$$

Remark 4.1. *Recently a paper by one of the present authors has appeared [18], where the singular nonsymmetric saddle point problem is "best", and in some cases "optimally", solved by means of an extension of Manteuffel's algorithm to accommodate the singular case. From its solution many previous "best" or "optimal" results based on generalizations and extensions of SOR-like methods can be recovered. Further work in this direction has led to fruitful results.*

Remark 4.2. *It seems that if we set $\phi := \frac{\omega}{\alpha+\omega}$ and $\psi := \frac{2\omega}{2-\omega}$ as we did in (3.7), we can recover from (4.16) the optimal results of Bai et al.'s [5] and especially those of [37] via a quite different splitting and a quite different analysis regarding semi-convergence. This suggested that a unification of various iterative methods for the solution of the more general singular nonsymmetric saddle point problem was possible. Work in this direction was fruitful and some of the results have appeared recently in [19]. (Specifically, see Remark 3.5 of [19] referring to the present work and the one in [37]).*

Remark 4.3. *Work on other real problems using the results of this work and of [18] and [19] can also extend the present one, as in [2] and [3], in the direction of the solution of non-Hermitian positive definite matrix coefficients problems which has already begun.*

5 Numerical examples

Consider the singular symmetric saddle point problem in [37] with the matrix coefficient blocks as in (1.1) of the following specific form:

$$\begin{gathered}
A=\begin{bmatrix} I_l\otimes T+T\otimes I_l & 0\\ 0 & I_l\otimes T+T\otimes I_l\end{bmatrix}\in\mathbb{R}^{2l^2\times 2l^2},\quad l\quad \text{even},\\
T=\frac{1}{h^2}\text{tridiag}(-1,2,-1)\in\mathbb{R}^{l\times l},\\
B=[B_1\, b_1\, b_2]\equiv\left[B_1,\widetilde{B}\right]\in\mathbb{R}^{2l^2\times(l^2+2)},\\
B_1=\begin{bmatrix} I_l\otimes F\\ F\otimes I_l\end{bmatrix},\ b_1=B_1\begin{bmatrix}\mathbf{1}\\ \mathbf{0}\end{bmatrix},\ b_2=B_1\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix},\\
F=\frac{1}{h}\text{tridiag}(-1,1,0)\in\mathbb{R}^{l\times l},\quad \mathbf{0}=[0,0,\cdots,0]^T\in\mathbb{R}^{l^2/2},\quad \mathbf{1}=[1,1,\cdots,1]^T\in\mathbb{R}^{l^2/2}.
\end{gathered} \tag{5.1}$$

Here $\otimes$ denotes the Kronecker (tensor) product symbol, $h=\frac{1}{l+1}$ is the discretization mesh size, and $J=\text{tridiag}(a,b,c)$ denotes a tridiagonal matrix with $J_{i-1,i}=a$, $J_{i,i}=b$, $J_{i+1,i}=c$. In problem (5.1) for appropriate even l, we have $m=2l^2$ and $n=l^2+2$. Thus the total number of components of a solution $\left[x^T,y^T\right]^T\in\mathbb{R}^{3l^2+2}$ is $m+n=3l^2+2$.

Remark 5.1. *To the best of our knowledge, Example 5.1 restricted to its nonsingular symmetric part, with rank$(B)=n$, and without the two additional vectors b_1 and b_2 that make $\mathcal{A}$ be singular, is Example 5.1 of [5] and was taken from [4]. The same nonsingular symmetric example was also used in [13, 27, 32, 25, 18] and in others. Technical modification of the aforementioned example to* *make $\mathcal{A}$ be singular, namely by adding the two vectors b_1 and b_2, as in Example 5.1 above, was first appeared in [37], subsequently in [24, 22, 29, 34, 36, 21, 38, 33, 12, 30, 19] and maybe in others. Mainly for comparison reasons and especially because of its "popularity" made us use it too.*

The preconditioning matrix Q, as an approximation to the matrix $B^TA^{-1}B$, was chosen in four cases as is indicated in Table 1. These cases were previously used in the parameterized Uzawa (PU) method [37].

Table 1: Choices of the matrix Q, with $\widetilde{Q} = \text{diag}(B_1^T\widetilde{A}^{-1}B_1, \widetilde{B}^T\widetilde{B})$

Case no.	**Matrix Q**	**Description**
1	$\widetilde{Q}$	$\widetilde{A} = \text{tridiag}(A)$
2	$\widetilde{Q}$	$\widetilde{A} = \text{diag}(A)$
3	$\text{tridiag}(\widetilde{Q})$	$\widetilde{A} = \text{tridiag}(A)$
4	$\text{tridiag}(\widetilde{Q})$	$\widetilde{A} = A$

All experiments were implemented in MATLAB (version 7.8.0.347 (R2009a)) with a machine precision 10^{-16}, on a personal computer with 2.39 GHz central processing unit (Intel(R) Core(TM) i7-4500U 1.80GHz), 8.00G memory and Windows 8.1 operating system. For the ASOR method, all numerical examples were started with an initial vector $\left[x^{(0)^T}, y^{(0)^T}\right]^T$ and terminated when the current iteration satisfied ERR $\leq \varepsilon$, where ε is a small positive number, or when a prescribed maximum iteration number was exceeded. ERR denotes the ratio of the norm of the residual of the iteration vector at hand over that of the initial vector. Both ERR and RES are defined by

$$\text{ERR} := \frac{\sqrt{\|p - Ax^{(k)} - By^{(k)}\|_2^2 + \|q - B^Tx^{(k)}\|_2^2}}{\sqrt{\|p - Ax^{(0)} - By^{(0)}\|_2^2 + \|q - B^Tx^{(0)}\|_2^2}} \leq \varepsilon. \tag{5.2}$$

(Note that if and only if the initial vector $\left[x^{(0)^T}, y^{(0)^T}\right]^T$ is the zero vector then the relation for the ERR is simplified to

$$\text{ERR} := \frac{\sqrt{\|p - Ax^{(k)} - By^{(k)}\|_2^2 + \|q - B^Tx^{(k)}\|_2^2}}{\sqrt{\|p\|_2^2 + \|q\|_2^2}} \leq \varepsilon. \;) \tag{5.3}$$

The norm of the residual vector RES is given by

$$\text{RES} = \sqrt{\|p - x^{(k)} - By^{(k)}\|_2^2 + \|q - B^Tx^{(k)}\|_2^2}. \tag{5.4}$$

It is pointed out that in the examples we ran ε was taken to be 10^{-6}.

The right hand side vector $\left[p^T, q^T\right]^T \in \mathbf{R}^{m+n}$ is chosen such that the exact solution of the augmented linear system (1.1) is $\left[x_*^T, y_*^T\right]^T = [1, 1, \cdots, 1]^T \in \mathbf{R}^{m+n}$.

The two optimal parameters α and ω, the iteration numbers (IT), the relative absolute errors (ERR), and the residuals (RES) of the ASOR iterative method are listed with various sizes of m and n, respectively.

Table 2: Optimal parameters: α_{opt}, ω_{opt}, IT, ERR, RES

m			128	512	1152	2048
n			66	258	578	1026
m+n			194	770	1730	3074
Case 1	ASOR	α_{opt}	0.2027	0.3041	0.3654	0.3947
		ω_{opt}	0.3994	0.2498	0.1805	0.1412
		IT	31	60	89	119
		ERR	9.2054e-7	9.5871e-7	9.7243e-7	8.6745e-7
		RES	7.1471e-4	3.4522e-3	8.9944e-3	1.5888e-2
Case 2	ASOR	α_{opt}	0.2652	3605	0.4009	0.4232
		ω_{opt}	0.3158	0.1873	0.1328	0.1029
		IT	45	88	130	173
		ERR	9.7984e-7	8.8686e-7	9.9007e-7	9.7045e-7
		RES	7.6075e-4	3.1935e-3	9.1576e-3	1.7775e-2
Case 3	ASOR	α_{opt}	0.2975	0.6331	0.9274	1.1932
		ω_{opt}	0.9759	1.1100	1.1910	1.2491
		IT	23	35	43	51
		ERR	7.4070e-7	6.3140e-7	9.5419e-7	8.0967e-7
		RES	5.7508e-4	2.2736e-3	8.8257e-3	1.4830e-2
Case 4	ASOR	α_{opt}	0.2413	0.5223	0.7700	0.9941
		ω_{opt}	1.0122	1.1675	1.2556	1.3163
		IT	20	29	36	42
		ERR	6.3064e-7	7.9550e-7	8.6442e-7	9.0583e-7
		RES	4.8963e-4	2.8645e-3	7.9953e-3	1.6591e-2

Table 3: Optimal parameters: α_{opt}, ω_{opt}, IT, ERR, RES

m			128	512	1152	2048
n			66	258	578	1026
m+n			194	770	1730	3074
Case 1	ASOR	α_{opt}	0.2027	0.3041	0.3654	0.3947
		ω_{opt}	0.3994	0.2498	0.1805	0.1412
		IT	28	53	78	102
		ERR	8.3875e-7	8.8004e-7	8.3319e-7	9.2187e-7
		RES	1.6287e-3	1.1860e-2	3.6072e-2	4.2264e-2
Case 2	ASOR	α_{opt}	0.2652	3605	0.4009	0.4232
		ω_{opt}	0.3158	0.1873	0.1328	0.1029
		IT	41	78	114	150
		ERR	8.6141e-7	8.5660e-7	9.2067e-7	9.4488e-7
		RES	1.6727e-3	1.1544e-2	3.9860e-2	9.4567e-2
Case 3	ASOR	α_{opt}	0.2975	0.6331	0.9274	1.1932
		ω_{opt}	0.9759	1.1100	1.1910	1.2491
		IT	21	30	38	44
		ERR	5.6732e-7	9.3456e-7	7.25498e-7	7.9332e-7
		RES	1.1016e-3	1.2446e-2	3.1409e-2	7.9398e-2
Case 4	ASOR	α_{opt}	0.2413	0.5223	0.7700	0.9941
		ω_{opt}	1.0122	1.1675	1.2556	1.3163
		IT	18	26	31	36
		ERR	6.6097e-7	6.4547e-7	9.9802e-7	9.9032e-7
		RES	1.2835e-3	8.6987e-2	4.3208e-2	9.9115e-2

Two numerical examples are worked out. One with initial vector $[0, 0, 0, 0, \cdots, 0, 0]^T \in \mathbb{R}^{3l^2+2}$ and the other with initial vector $[1, 0, 1, 0, \cdots, 1, 0]^T \in \mathbb{R}^{3l^2+2}$. The results obtained as described above are depicted in Tables 2 and 3, respectively.

Comparing the results of the present Table 2 and Table 2 of [37] one will see that the number of iterations IT and the residuals RES obtained are identical to the accuracy used. However, if we choose Q as in Table 3 of [37], then in view of Remark 4.2 and the reported CPU times for the same accuracy of the residuals RES obtained in the same paper [37] are better than the corresponding ones given by the MINRES and PMINRES methods. The former observation must not be surprising since as is stated in Remark 4.2 our method and the one by Bai et al.'s or by Zeng et al.'s, [5] and [37], respectively, are equivalent for the solution of the singular symmetric saddle point problem (1.1) by the corresponding methods although our method started from a quite different splitting of the matrix coefficient $\mathcal{A}$ and the subsequently analysis differs considerably from that in [37].

The results of Table 3 are a little less accurate than those of Table 2. However, we should not forget that the results of Table 2 come from an initial zero vector and, therefore, its theoretical solution is given by the expression in (3.3), where the term $(I_{m+n} - E)\left[x^{(0)T}, y^{(0)T}\right]^T$ is zero while the initial vector in the example of Table 3 is different from zero. In the latter case the initial vector $z^{(0)}$ is **not** known and equation (3.3) has to be solved for it, after we determine $\lim_{k\to\infty} z^{(k)}$, in order to find the actual $z^{(0)}$ used.

6 Concluding remarks

The method of the present paper for the solution of the singular symmetric saddle point problem (1.1) is based on the ASOR iterative method introduced by one of the present authors, as coauthor, [25] for the solution of the nonsingular symmetric saddle point problem. We were able not only to obtain "best" and "optimal" theoretical values for the parameters involved by the simple analysis of Section 4 but also for the corresponding semi-convergence factor. The method of this paper, equivalent to the one by Zheng et al. [37], as this is pointed out in Remark 4.2, started from a quite different splitting of the matrix coefficient and the analysis to obtain the optimal results, in many cases, follows quite different and simpler routes. The results exhibited in Tables 2 and 3 together with the ones in Tables 3 and 4 of [37] verify the theory developed in the present work.

It should be mentioned that the preconditioners of the present work (Table 1) and especially those of Table 3 of [37] can be used in connection with the MINRES and other Krylov subspace methods. Especially, the extension of the present method to the solution of the singular nonsymmetric saddle point problem (see [19]) may be more effective. In all possible directions we have also been working.

Last but not least we would like to raise one more issue. From Tables 2 and 3, we can observe that the number of iterations IT in Case 1 is less than that of Case 3, while IT in Case 3 is less than that of Case 4. Although IT in Case 4 seems to be the best, $\tilde{A}$ is chosen to be A which is neither a tridiagonal nor a diagonal matrix of A. So Case 3 may be a good "practical" choice for obtaining the optimal parameters α and ω. Two tridiagonal matrices are only needed for Q. Even so, in order to compute the optimal parameters (4.16), the minimum and the maximum positive eigenvalues of $Q^{-1}B^TA^{-1}B$ are needed. This may be costly for large size problems. The choice of "best" (or "optimal") parameters in practice still needs further investigation and this will be one of our future research interests, see also [23]. But when we choose some particular values for the parameters, such as $\alpha = 0.1$, $\omega = 1$, $\alpha = 0.1$, $\omega = 10$ in Case 1, IT decreases as the dimension of the problem increases, see Figure 1, which may constitute a suggestion as regards the direction we should follow in our future work.

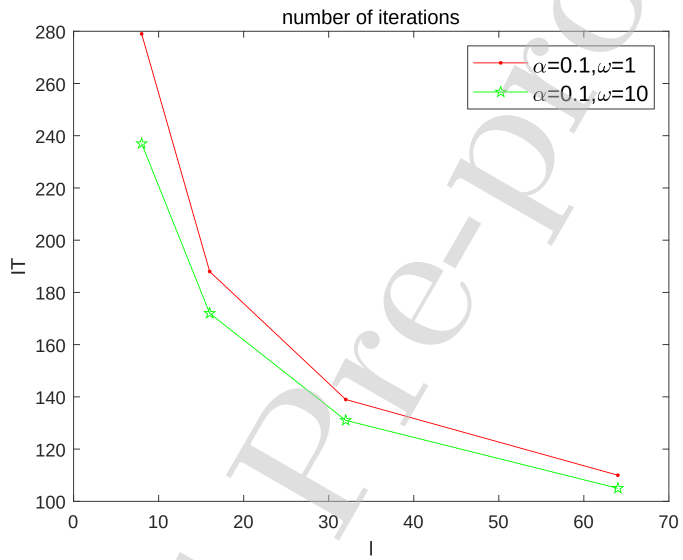

Figure 1: *Performance of the ASOR method regarding the number of iterations versus the size of the problem for two arbitrary chosen pairs of values for α and ω.*

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