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On Schröder's families of root-finding methods[☆]

M.S. Petković[a,*], L.D. Petković[b], Đ. Herceg[c]

[a] Faculty of Electronic Engineering, University of Niš, 18000 Niš, Serbia
[b] Faculty of Mechanical Engineering, University of Niš, 18000 Niš, Serbia
[c] Department of Mathematics, University of Novi Sad, 21000, Serbia

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ABSTRACT

Schröder's methods of the first and second kind for solving a nonlinear equation $f(x) = 0$, originally derived in 1870, are of great importance in the theory and practice of iteration processes. They were rediscovered several times and expressed in different forms during the last 130 years. It was proved in the paper of Petković and Herceg (1999) [7] that even seven families of iteration methods for solving nonlinear equations are mutually equivalent. In this paper we show that these families are also equivalent to another four families of iteration methods and find that all of them have the origin in Schröder's generalized method (of the second kind) presented in 1870. In the continuation we consider Smale's open problem from 1994 about possible link between Schröder's methods of the first and second kind and state the link in a simple way.

1. Introduction

Rediscovering some old root-finding methods is not a rarity in numerical analysis. From time to time, the already known iteration method for solving nonlinear equations appears, derived most frequently using different ways and presented in various forms. Our attention is restricted to the rediscovered families of rational iteration methods for solving nonlinear equations. It is initiated by the results that appeared after the World War II (e.g., [1,2]) and recent papers (e.g., [3–6]).

We consider two families of root-finding methods for solving nonlinear equations. Both the families were derived by E. Schröder in 1870, but his methods (of the first and second kind, in Schröder's terminology) have been often forgotten (or neglected). Indeed, during the last 130 years, these families were rediscovered several times. Our goal is to prove the equivalence of the rediscovered methods with Schröder's methods and, in this way, to complete the study on this subject partially discussed in [7]. In the second part of this paper we are concerned with the link between the mentioned Schröder's methods of the first and second kind.

2. The chain of equivalence of rational iteration methods

In this paper we will deal with the functional determinant

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* Corresponding author.
E-mail address: msp@elfak.ni.ac.yu (M.S. Petković).

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$$\Delta_m(x) = \det \begin{pmatrix} f'(x) & \dfrac{f''(x)}{2!} & \dfrac{f'''(x)}{3!} & \cdots & \dfrac{f^{(m)}(x)}{m!} \\ f(x) & f'(x) & \dfrac{f''(x)}{2!} & \cdots & \dfrac{f^{m-1}(x)}{(m-1)!} \\ 0 & f(x) & f'(x) & \cdots & \dfrac{f^{m-2}(x)}{(m-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f'(x) \end{pmatrix}, \tag{1}$$

where f is an analytic function in the neighborhood of a (real or complex) zero α of f. The following recursive relation

$$\Delta_0(x) = 1, \qquad \Delta_m(x) = \sum_{\nu=1}^{m} (-1)^{\nu+1} \frac{f(x)^{\nu-1} f^{(\nu)}(x)}{\nu!} \Delta_{m-\nu}(x), \tag{2}$$

can be derived, see [8] and Kalantari [9] (see, also, Lemma 2.1 in [6]).

We give first the list of papers which present the families of root-finding methods with arbitrary order of convergence having the form of rational iteration functions:
Wang's class $W_m(x)$ [8] (1966);
Varjuhin–Kasjanjuk's class $V_m(x)$ [10] (1969);
Jovanović's class $J_m(x)$ [11] (1972);
Farmer–Loizou's class $L_m(x)$ [12] (1975);
Igarashi–Nagasaka's class $I_m(x)$ [13] (1991);
Gerlach's class $G_m(x)$ [4] (1994);
Ford–Pennline's class $F_m(x)$ [5] (1996).
The short description of these families can be found in [7].

Particular iteration methods, generated by the above families, were compared using symbolic computation in the programming package *Mathematica*. It was found that all families produce the same methods for each $m = 2, 3, \dots, 20$. Further comparison was disabled due to very complicated expressions even for the powerful digital computers. However, this coincidence initiated the following natural question: *Are the considered families the new ones or they are mutually equivalent?*

Studying the above question, the following equivalence chain for the listed classes of iteration functions (given in chronological order) was stated in [7]:

$$W_m(x) = V_m(x) = J_m(x) = L_m(x) = I_m(x) = G_m(x) = F_m(x). \tag{3}$$

The above chain can be expanded by inserting Kalantari–Kalantari–Nahandi's class $B_m(x)$ [3], introduced by

$$B_m(x) = x - f(x)\frac{D_{m-2}(x)}{D_{m-1}(x)}, \quad (m \ge 2). \tag{4}$$

Actually, $D_m(x)$ in (4) is the functional determinant given by (1), that is, $D_m(x) \equiv \Delta_m(x),\ D_0(x) = 1$. Then the Basic Family (in the terminology of the authors of [3]) is defined by

$$x_{k+1} = B_m(x_k), \quad (k = 0, 1, \dots) \tag{5}$$

having the order m if the sought zero is simple, see [3]. Kalantari and Gerlach [6] proved the equivalence of the Gerlach family $\{G_m(x)\}_{m=2}^{\infty}$ to a family of iteration functions $\{B_m(x)\}_{m=2}^{\infty}$. Since $G_m(x)$ is included in the chain (3), it follows that $B_m(x)$ is also included.

In Concluding remark of the paper [3] it was written: "... *Although individual members of the family like Newton's and Halley's iteration functions have been known, no closed formula for the general member of the Basic Family, nor their asymptotic constant had been known previously* ...". However, we will show that the Basic Family (4) was derived in [1] about 50 years before Kalantari et al. [3], using functional determinants too.

In 1946 Hamilton [1] introduced functional iterations using functional determinants. Let α be the zero of an analytic function f and let x be its approximation. Hamilton determined the powers of error $w = x - \alpha$ in the form

$$w^r = (-1)^{r+1}\frac{\Delta_{m,r}(x)}{\Delta_m(x)} + O(w^{m+1}) \quad (r = 1, 2, \dots, m),$$

where $\Delta_m(x)$ is just the determinant given by (1). $\Delta_{m,r}(x)$ is the determinant which is obtained from Δ_m by replacing the rth column by the column $(f(x)\ 0\ 0 \cdots 0)^{\mathrm{T}}$. If $r = 1$ then obviously $\Delta_{m,1}(x) = f(x)\Delta_{m-1}(x),\ \Delta_0(x) = 1$. In this special case one obtains

$$w = \frac{\Delta_{m,1}(x)}{\Delta_m(x)} + O(w^{m+1}),$$

which yields

$$\alpha = x - w = x - \frac{\Delta_{m,1}(x)}{\Delta_m(x)} + O(w^{m+1}) = x - f(x)\frac{\Delta_{m-1}(x)}{\Delta_m(x)} + O(w^{m+1}).$$

Hence Hamilton stated the family of iterative methods

$$x_{k+1} = H_m(x_k) := x_k - f(x_k)\frac{\Delta_{m-2}(x_k)}{\Delta_{m-1}(x_k)}\ (= B_m(x_k)) \quad (m = 2, 3, \ldots; k = 0, 1, \ldots) \tag{6}$$

having the order m. Here we use the symbol H after Hamilton. Since $H_m(x) \equiv B_m(x)$, we conclude that the Basic Family (4) is, in fact, Hamilton's family of iteration functions (6). This means that Hamilton's class $H_m(x)$ also belongs to the equivalence chain (3).

In 1953 Householder [2] presented the family of iteration methods

$$x_{k+1} = K_m(x_k) := x_k + (m-1)\left[\frac{(1/f(x))^{(m-2)}}{\left(1/f(x)\right)^{(m-1)}}\right]_{x=x_k} \qquad (m = 2, 3, \ldots; k = 0, 1, \ldots) \tag{7}$$

of the order m. In what follows we will show that the class $K_m(x)$ also belongs to the equivalence chain (3). Before doing this, we give the following lemma.

Lemma 1. *The following relation is valid*

$$\Delta_n(x) = \frac{(-1)^n f(x)^{n+1}}{n!}\left(\frac{1}{f(x)}\right)^{(n)} \quad (n = 1, 2, \ldots). \tag{8}$$

Proof. The proof goes by induction using the recursive relation (2). The relation (8) is true for $n = 1$:

$$\Delta_1(x) = f'(x) = \frac{(-1)f(x)^2}{1}\frac{(-f'(x))}{f(x)^2} = \frac{(-1)f(x)^2}{1}\left(\frac{1}{f(x)}\right)^{(1)}.$$

Let us assume that the relation (8) is true for $n = 1, 2, \ldots, k$. Then we use the recursive relation (2) and obtain for $n = k + 1$

$$\begin{aligned}
\Delta_{k+1}(x) &= \sum_{r=1}^{k+1}(-1)^{r+1}\frac{f(x)^{r-1}f^{(r)}(x)}{r!}\Delta_{k+1-r}(x)\\
&= \sum_{r=1}^{k+1}(-1)^{r+1}\frac{f(x)^{r-1}f^{(r)}(x)}{r!}\frac{(-1)^{k+1-r}}{(k+1-r)!}f(x)^{k+2-r}\left(\frac{1}{f(x)}\right)^{(k+1-r)}\\
&= \frac{(-1)^k f(x)^{k+1}}{(k+1)!}\sum_{r=1}^{k+1}\binom{k+1}{r}f^{(r)}(x)\left(\frac{1}{f(x)}\right)^{(k+1-r)}\\
&= \frac{(-1)^k f(x)^{k+1}}{(k+1)!}\left(\sum_{r=0}^{k+1}\binom{k+1}{r}f^{(r)}(x)\left(\frac{1}{f(x)}\right)^{(k+1-r)} - f(x)\left(\frac{1}{f(x)}\right)^{(k+1)}\right)\\
&= \frac{(-1)^k f(x)^{k+1}}{(k+1)!}\left(\left(f(x)\left(\frac{1}{f(x)}\right)\right)^{(k+1)} - f(x)\left(\frac{1}{f(x)}\right)^{(k+1)}\right)\\
&= \frac{(-1)^k f(x)^{k+1}}{(k+1)!}\left(0 - f(x)\left(\frac{1}{f(x)}\right)^{(k+1)}\right)\\
&= \frac{(-1)^{k+1} f(x)^{k+2}}{(k+1)!}\left(\frac{1}{f(x)}\right)^{(k+1)}.
\end{aligned}$$

Therefore, the relation (8) also holds for $n = k + 1$ and, according to induction, it is valid for each $n = 1, 2, \ldots$. □

Now we return to Householder's iteration formula (7) and using (8) we obtain

$$K_m(x) = x + (m-1)\frac{(-1)^{m-2}(m-2)!\Delta_{m-2}(x)/f(x)^{m-1}}{(-1)^{m-1}(m-1)!\Delta_{m-1}(x)/f(x)^m} = x - f(x)\frac{\Delta_{m-2}(x)}{\Delta_{m-1}(x)} = H_m(x),$$

which proves that Householder's iteration functions $K_m(x)$ are the same as Hamilton's iteration functions $H_m(x)$. Hence, Householder's class $K_m(x)$ also belongs to the equivalence chain (3).

We have shown above that the families of iteration functions, developed from 1946 (Hamilton [1]) to 1997 (Kalantari et al. [3]) are mutually equivalent. However, the priority of the considered family of rational iterations goes back to Schröder in 1870 yet. In his remarkable paper [14] (see, also, [15]) he proposed two general algorithms with arbitrary order of convergence, referred to as the methods of the first and second kind.

Schröder [14] defined the **method of the second kind** of the order m by the iteration function

$$S_m(x) = x - \frac{R_{m-2}(x)}{R_{m-1}(x)}, \tag{9}$$

where $R_m(x)$ is calculated from the recursive relation

$$R_0(x) = 1/f(x), \qquad R_k(x) = \sum_{r=1}^{k} (-1)^{r-1} \frac{f^{(r)}(x)}{r!f(x)} R_{k-r}(x) \quad (k = 1, 2, \ldots). \tag{10}$$

We note that, putting

$$R_k(x) = \frac{\Delta_k(x)}{f(x)^{k+1}} \tag{11}$$

in (10), the recursive relation (10) reduces to (2), while (9) becomes

$$S_m(x) = x - \frac{\Delta_{m-2}(x)/f(x)^{m-1}}{\Delta_{m-1}(x)/f(x)^m} = x - f(x)\frac{\Delta_{m-2}(x)}{\Delta_{m-1}(x)} = H_m(x).$$

In this way we further extend the equivalence chain by adding $S_m(x)$,

$$\begin{aligned} S_m(x) &= H_m(x) = K_m(x) = W_m(x) = V_m(x) = J_m(x) \\ &= L_m(x) = I_m(x) = G_m(x) = F_m(x) = B_m(x). \end{aligned} \tag{12}$$

Therefore, we have found that even eleven rational iteration functions are mutually equivalent, having their origin in Schröder's classical method [14] stated in 1870.

Remark 1. According to (1), (2) and (11), the function $R_k(x)$ involved in Schröder's method of the second kind (9) can be evaluated by the functional determinant

$$R_0 = 1/f(x), \qquad R_k(x) = \frac{1}{f(x)} \det \begin{pmatrix} B_1 & B_2 & B_3 & \ldots & B_k \\ 1 & B_1 & B_2 & \ldots & B_{k-1} \\ 0 & 1 & B_1 & \ldots & B_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & B_1 \end{pmatrix} \quad (k \geq 1), \tag{13}$$

where $B_k(x) = f^{(k)}(x)/(k!f(x))$.

Schröder derived the iteration formula (9) using suitable development to partial fractions and restricting himself to a rational function whose roots are sought. Today the natural approach to Schröder's formula (9) would be through König's theorem [16].

Theorem 1 (*König [16]*). *Let $v(z) = c_0(x)+c_1(x)(z-x)+c_2(x)(z-x)^2+\cdots$ be analytic function in the disk $|z-x| < \rho$ centered at x and v has a single pole at the point ζ belonging to this disk. If $|\zeta - x| < \sigma\rho < \rho$, then $c_k(x)/c_{k+1}(x) = \zeta - x + O(\sigma^{k+1})$.*

Let us apply König's theorem to root-finding methods. If α is the root of the equation $f(x) = 0$ nearest to x, then the function

$$\frac{1}{f(x-\varepsilon)} = R_0(x) + R_1(x)\varepsilon + R_2(x)\varepsilon^2 + \cdots$$

has a pole at $\varepsilon = x - \alpha$. If α is unique and simple, then

$$\frac{R_{m-2}(x)}{R_{m-1}(x)} \to x - \alpha.$$

Hence the iteration method (9) immediately follows. Since this approach is quite natural, many authors refer the functional iteration (9) to as König's method (see, e.g., [17,18]).

Let us introduce the abbreviations

$$u(x) = \frac{f(x)}{f'(x)}, \qquad C_\nu(x) = \frac{f^{(\nu)}(x)}{\nu!f'(x)} \quad (\nu = 1, 2, \ldots),$$

and define

$$P_0(x) = 1, \qquad P_m(x) = \det \begin{pmatrix} 1 & C_2 & C_3 & \dots & C_m \\ u & 1 & C_2 & \dots & C_{m-1} \\ 0 & u & 1 & \dots & C_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{14}$$

Comparing the determinants (13) and (14) we find $P_k(x) = f(x)u(x)^k R_k(x)$. According to this we start from (9) and obtain the following form of the Schröder method of the second kind

$$S_m(x) = x - u(x)\frac{P_{m-2}(x)}{P_{m-1}(x)}, \quad (m \geq 2). \tag{15}$$

In addition, having in mind (10), we conclude that the following recursive relation is valid

$$P_m(x) = \sum_{\nu=1}^{m} (-1)^{\nu+1} u(x)^{\nu-1} C_\nu(x) P_{m-\nu}(x), \qquad C_1(x) = 1, \quad (m \geq 1). \tag{16}$$

This relation, together with (15), is convenient to generate an array of iteration methods. For example, suppressing the argument of functions for brevity, we obtain for $m = 2, \dots, 7$:

Newton's method of the order 2:

$$S_2 = x - u.$$

Halley's method [19] *of the order* 3:

$$S_3 = x - \frac{u}{1 - C_2 u}.$$

Kiss' method [20] *of the order* 4:

$$S_4 = x - \frac{u(1 - C_2 u)}{1 - 2C_2 u + C_3 u^2}.$$

Kiss' method [20] *of the order* 5:

$$S_5 = x - \frac{u(1 - 2C_2 u + C_3 u^2)}{1 - 3C_2 u + (2C_3 + C_2^2)u^2 - C_4 u^3}.$$

Method of the order 6:

$$S_6 = x - \frac{u\left[1 - 3C_2 u + (2C_3 + C_2^2)u^2 - C_4 u^3\right]}{1 - 4C_2 u + (3C_2^2 + 3C_3)u^2 - (2C_2C_3 + 2C_4)u^3 + C_5 u^4}.$$

Method of the order 7:

$$S_7 = x - \frac{u\left[1 - 4C_2 u + (3C_2^2 + 3C_3)u^2 - (2C_2C_3 + 2C_4)u^3 + C_5 u^4\right]}{1 - 5C_2 u + (6C_2^2 + 4C_3)u^2 - (C_2^3 + 6C_2C_3 + 3C_4)u^3 + (C_3^2 + 2C_2C_4 + 2C_5)u^4 - C_6 u^5}$$

and so on. Let us note that the methods S_4 and S_5, often attributed to Kiss [20], can be found in Schröder's paper [14].

3. The link between the methods of first and second kind

Schröder's **method of the first kind** [14] is often presented in the form

$$E_m(x) = x + \sum_{r=1}^{m-1} (-1)^r \frac{f(x)^r}{r!} (f^{-1})^{(r)}(f(x)), \tag{17}$$

where f^{-1} is the inverse of f. The order of convergence of the method (17) is m ($m \geq 2$). Let us note that in the Russian literature the above sequence E_m is attributed to Chebyshev (1837 or 1838), see Traub [23, p. 81]. However, some authors (e.g., [21,22]) ascribes E_m to Euler quoting his *Opera Omnia*, Ser. I, Vol. X, pp. 422–455.

The following formula is useful for the evaluation of the derivative of f^{-1}:

$$(f^{-1})^{(r)}(f(x)) = \frac{Z_r}{(f')^{2r-1}}, \qquad Z_r = f'Z_r' - (2r-1)Z_r f'', \quad (Z_1 = 1; r = 2, 3, \dots),$$

where Z_r is a polynomial in $f', f'', \dots, f^{(r)}$, $f^{(j)} \equiv f^{(j)}(x)$.

Theorem 2 (*Schröder [14]*). *Any root-finding algorithm F_m of the order m can be presented in the form*

$$F_m(x) = E_m(x) + f(x)^m \varphi_m(x), \tag{18}$$

where φ_m is a function bounded in α which depends on f and its derivatives.

A convenient technique for generating basic sequences E_m is based on Traub's difference–differential relation (see [23, Lemma 5-3])

$$E_{k+1}(x) = E_k(x) - \frac{u(x)}{k} E'(x), \qquad E_2(x) = x - u(x), \quad (k \geq 2). \tag{19}$$

According to (19) we obtain the first few E_k (omitting the argument x):

$$E_3 = E_2 - C_2 u^2, \quad \text{(Chebyshev's method)},$$
$$E_4 = E_3 - (2C_2^2 - C_3)u^3,$$
$$E_5 = E_4 - (5C_2^3 - 5C_2C_3 + C_4)u^4,$$
$$E_6 = E_5 - (14C_2^4 - 21C_2^2C_3 + 6C_2C_4 + 3C_3^2 - C_5)u^5,$$
$$E_7 = E_6 - (42C_2^5 - 84C_2^3C_3 + 28C_2^2C_4 + 28C_2C_3^2 - 7C_5C_2 - 7C_3C_4 + C_6)u^6.$$

From the paper [3] we learn that Steven Smale posed a question of finding possible link between the Schröder method of the second kind $S_m(x)$ given by (9) (or the Basic Family $B_m(x)$) and the Schröder method of the first kind $E_m(x)$ given by (17). Investigating this problem using experimentations, we came to the conjecture which can be expressed in the following symbolic form:

Conjecture 1. $x - truncation_{m-1}\Big[u * P_{m-2}(u) * Series\big[1/P_{m-1}(u), \{u, 0, m-1\}\big]\Big] = E_m.$

Here $*$ denotes the multiplication, `Series` executes the development into the power series at the point $u = 0$ taking $m-1$ members, and t*runcation*$_{m-1}$ means that the terms in the bracket containing the powers of u higher than $m-1$ should be neglected.

We have started to investigate this conjecture using symbolic computation in the programming package *Mathematica* 6 in three steps:

1°. We note that the denominator $P_{m-1}(x)$ in (15) is a polynomial in u of degree $m-2$,

$$P_{m-1}(x; u) = 1 - \phi_1 u + \phi_2 u^2 + \cdots + (-1)^{m-2}\phi_{m-2}u^{m-2},$$

where $\phi_k = \phi_k(C_2, \ldots, C_{k+1})$ $(k = 1, \ldots, m-3)$, $\phi_{m-2} = C_{m-1}$. Developing the function $T(x; u) = 1/P_{m-1}(x; u)$ into the power series (about the point $u = 0$) and taking $m-1$ members, we obtain

$$T(x; u) = 1 + \lambda_1 u + \lambda_2 u^2 + \cdots + \lambda_{m-1}u^{m-2} + O(u^{m-1}),$$

where $\lambda_k = \lambda_k(C_2, \ldots, C_{k+1})$.

2°. We multiply

$$uP_{m-2}(x; u)T(x; u) = Q_m(x; u)$$

and neglect (in $Q_m(x; u)$) the terms containing the powers of u higher than $m-1$ to obtain the truncated $\widetilde{Q}_m(x; u) = \sum_{k=1}^{m-1} h_k(x)u^k$.

3°. Since the Schröder sequence is of the form $E_m(x; u) = x - \sum_{k=1}^{m-1} Y_k(x)u^k$, where $Y_1 = 1$ and Y_k depends of $C_2, \ldots, C_k$ $(k \geq 2)$ (see Traub [23, p. 83]), we check the identity

$$\widetilde{Q}_m(x; u) = \sum_{k=1}^{m-1} h_k(x)u^k = \sum_{k=1}^{m-1} Y_k(x)u^k = E_m(x; u) \tag{20}$$

by comparing the corresponding functional coefficients h_k and Y_k.

We performed the above procedure and found that the identity (20) holds true for $m = 3, 4, \ldots, 13$. For example, taking $m = 6$ we find by (16)

$$P_4(x; u) = 1 - 3C_2u + (C_2^2 + 2C_3)u^2 - C_4u^3,$$
$$P_5(x; u) = 1 - 4C_2u + (3C_2^2 + 3C_3)u^2 - (2C_2C_3 + 2C_4)u^3 + C_5u^4.$$

In the programming package *Mathematica* 6 we obtain:

Step **1**°

$$T(x; u) = \frac{1}{P_5(x; u)} = 1 + 4C_2u + (13C_2^2 - 3C_3)u^2 + (40C_2^3 - 22C_2C_3 + 2C_4)u^3$$
$$+ (121C_2^4 - 110C_2^2C_3 + 9C_3^2 + 16C_2C_4 - C_5)u^4 + O(u^5).$$

Step **2°**

$$\widetilde{Q}_6(x;u) = \mathtt{truncation}_5\big[uP_4(x;u)T(x;u)\big] = u + C_2u^2 + (2C_2^2 - C_3)u^3 + (5C_2^3 - 5C_2C_3 + C_4)u^4 + (14C_2^4 - 21C_2^2C_3 + 3C_3^2 + 6C_2C_4 - C_5)u^5.$$

Step **3°**

We check the validity of the equality $\widetilde{Q}_6(x;u) = E_6(x;u)$ and conclude that it holds.

Theoretically, it is possible to verify the validity of Conjecture 1 for any specific m. However, the exponentially growing complexity of the checking procedure for large m kept us in practice to work for large m. For this reason, we were forced to search for a theoretical proof. This proof is given in what follows.

The sketch of the proof of Conjecture 1

Let

$$x_{k+1} = g_m(x_k) \quad (k = 0, 1, \ldots) \tag{21}$$

define an iteration method of the order m for finding a simple zero α of a given function f (sufficiently many times differentiable), that is,

$$g_m(x_k) - \alpha = O((x_k - \alpha)^m) = O(\varepsilon_k^m), \tag{22}$$

where we put $\varepsilon_k = x_k - \alpha$. According to Theorem 2.2 of Traub [23, p. 20], then

$$g_m(\alpha) = \alpha, \qquad g_m'(\alpha) = \cdots = g_m^{(m-1)}(\alpha) = 0, \qquad g_m^{(m)}(\alpha) \neq 0. \tag{23}$$

Using the relations (23), we find by Taylor's series

$$g_m(x_k) = \alpha + \frac{1}{m!}g_m^{(m)}(\alpha)\varepsilon_k^m + O(\varepsilon_k^{m+1}), \tag{24}$$

$$g_m'(x_k) = \frac{1}{(m-1)!}g_m^{(m)}(\alpha)\varepsilon_k^{m-1} + O(\varepsilon_k^m). \tag{25}$$

By (22), (24) and (25) we obtain

$$\begin{aligned}\frac{1}{m}g_m'(x_k)(x_k - g_m(x_k)) &= \frac{1}{m}g_m'(x_k)(x_k - \alpha - (g_m(x_k) - \alpha)) \\ &= \frac{1}{m!}g_m^{(m)}(\alpha)\varepsilon_k^m - \frac{1}{m!}g_m^{(m)}(\alpha)\varepsilon_k^{2m-1} + O(\varepsilon_k^{m+1}),\end{aligned}$$

wherefrom

$$\frac{1}{m}g_m'(x_k)(x_k - g_m(x_k)) = \frac{1}{m!}g_m^{(m)}(\alpha)\varepsilon_k^m + O(\varepsilon_k^{m+1}). \tag{26}$$

Combining (24) and (26), we get

$$g_m(x_k) - \frac{1}{m}g_m'(x_k)(x_k - g_m(x_k)) = \alpha + O(\varepsilon_k^{m+1}). \tag{27}$$

The relation (27) suggests the following iteration method

$$x_{k+1} = \Psi_{m+1}(x_k) := g_m(x_k) - \frac{1}{m}g_m'(x_k)(x_k - g_m(x_k)). \tag{28}$$

According to the relation (27), it follows immediately that the order of convergence of the iteration method (28) is $m+1$. Let us note that the iteration method (28) was previously derived in [11,24].

By virtue of Theorem 2, we have

$$\Psi_{m+1} = E_{m+1}(x) + f(x)^{m+1}\varphi_{m+1}(x). \tag{29}$$

Taking $g_2(x) = E_2(x) = x - u(x)$ and neglecting the term of higher order $f(x)^{m+1}\varphi_{m+1}(x)$ in (29), we conclude that the iteration formula (28) generates the same sequence of iteration methods as the Schröder family of the first kind (17). Regarding (29) we have particular cases

$$\varphi_3(x) = 0, \qquad \varphi_4(x) = \frac{f''(x)f'''(x)}{12f'(x)^6} - \frac{f''(x)^3}{4f'(x)^7},$$

$$\varphi_5(x) = \frac{23f''(x)^2f'''(x)}{48f'(x)^8} - \frac{5f''(x)^4}{8f'(x)^9} - \frac{f'''(x)^2}{48f'(x)^7} - \frac{f''(x)f^{(4)}(x)}{24f'(x)^7}, \text{ etc.}$$

The expressions of φ_m for $m \geq 6$ become more and more complicated. For this reason and having in mind that the iteration formula (28) produces not only the basic sequence but also unnecessary "parasite" terms (members of higher order), it is clear that the Schröder method of the first kind (17) is considerably simpler and thus preferable in theory and practice in comparison to the method (28).

Substituting $g_m(x_k) = x_{k+1}$ in the second term of (28) and solving the equation

$$x_{k+1} = g_m(x_k) - \frac{1}{m} g_m'(x_k)(x_k - x_{k+1})$$

in x_{k+1}, the following iteration method was derived in [11]:

$$x_{k+1} = x_k - \frac{x_k - g_m(x_k)}{1 - \frac{1}{m} g_m'(x_k)}. \tag{30}$$

This method also has the order $m+1$, see [11]. The methods (28) and (30) were referred in [11,24] to as the methods for accelerating convergence. Indeed, if g_m is the method of the order m, then the methods (28) and (30) are of the order $m+1$.

According to the equivalence chain (12), we have

$$S_m(x) = x - \frac{R_{m-2}(x)}{R_{m-1}(x)} = x - f(x)\frac{\Delta_{m-2}(x)}{\Delta_{m-1}(x)} = x - u(x)\frac{P_{m-2}(x)}{P_{m-1}(x)} \equiv x - \frac{x - g_{m-1}(x)}{1 - \frac{1}{m-1} g_{m-1}'(x)}. \tag{31}$$

Searching for a link between the Schröder methods of the first and second kind, in view of (31) and Theorem 2 it is sufficient to consider the connection between the iteration formulas (28) and (30) (or (31)). In our analysis we assume that x_k is sufficiently close to the zero α, meaning that $|\varepsilon_k| = |x_k - \alpha|$ is sufficiently small. According to (25) we conclude that $|g_m'(x_k)| = O(|\varepsilon_k^{m-1}|)$ is also very small quantity so that we can apply the approximation

$$\frac{1}{1 - \frac{1}{m} g_m'(x_k)} \approx 1 + \frac{1}{m} g_m'(x_k)$$

in (30). In this way we obtain the iteration formula (28). Furthermore, neglecting the higher order member $f(x)^{m+1}\varphi_{m+1}(x)$ in (28) (compare with (29)), we obtain the Schröder method of the first kind $E_{m+1}(x)$. Therefore, we have shown Conjecture 1:

The method of the first kind (17) (or (15)) *is obtained from the method of the second kind* (31) *by the development of the reciprocal of the denominator of* (31) *into the power series and constructing a polynomial in* u *of degree* $m-1$ *by neglecting the terms containing the powers of* u *higher than* u^{m-1} (*or* f^{m-1}, *according to* (29)).

In fact, the construction of E_m (given by (17)) from (31) or (15) is performed using the steps **1**°–**3**° presented above.

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