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A two-grid method for finite volume element approximations of second-order nonlinear hyperbolic equations

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ABSTRACT

The two-grid method is studied for solving a two-dimensional second-order nonlinear hyperbolic equation using finite volume element method. The method is based on two different finite element spaces defined on one coarse grid with grid size H and one fine grid with grid size h, respectively. The nonsymmetric and nonlinear iterations are only executed on the coarse grid and the fine grid solution can be obtained in a single symmetric and linear step. It is proved that the coarse grid can be much coarser than the fine grid. A prior error estimate in the H^1-norm is proved to be $O(h+H^3|\ln H|)$ for the two-grid semidiscrete finite volume element method. With these proposed techniques, solving such a large class of second-order nonlinear hyperbolic equations will not be much more difficult than solving one single linearized equation. Finally, a numerical example is presented to validate the usefulness and efficiency of the method.

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1. Introduction

Consider the following second-order nonlinear hyperbolic equation

$$\begin{cases} u_{tt} - \nabla\cdot(a(x)\nabla u) = f(u), & \forall x\in\Omega, 0<t\le T,\\ u(x,t)=0, & \forall x\in\partial\Omega, 0<t\le T,\\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset R^2$ is a convex polygonal domain, $x=(x_1,x_2)$, $f(u)=f(u,x,t)$ is a given real-valued function on Ω. We assume that $a(x)=(a_{ij}(x))_{i,j=1}^2$ is a symmetric and uniformly positive definite matrix in Ω and

$$|f'(u)|+|f''(u)|\le M,\quad \forall u\in R, \tag{1.2}$$

where M is a positive constant. It is assumed that the functions f, u_0, u_1 have enough regularity and they satisfy appropriate compatibility conditions so that the boundary value problem (1.1) has a unique solution satisfying the regularity results as demanded by our subsequent analysis.

The finite volume element method (FVEM), as a type of important numerical tool for solving differential equations, has been widely used in several engineering fields, such as fluid mechanics, heat and mass transfer and petroleum engineering. Perhaps the most important property of FVEM is that it can preserve the conservation laws (mass, momentum and heat flux) on each computational cell. This important property, combined with adequate accuracy and ease of implementation, has attracted more people to do research in this field. The theoretical framework and the basic tools for the analysis of FVEM

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have been developed in the last two decades (see, e.g. [1–9]). For the second-order hyperbolic problems, Kumar, Nataraj and Pani [10] presented and discussed a semidiscrete piecewise linear finite volume approximation to a second-order wave equation and obtained optimal error estimates in L^2-, H^1- and L^∞-norms.

On the other hand, two-grid method is a discretization technique for nonlinear equations based on two grids of different sizes. The main idea is to use a coarse grid space to produce a rough approximation of the solution of nonlinear problems, and then use it as the initial guess for one Newton-like iteration on the fine grid. This method involves a nonlinear solve on the coarse grid with grid size H and a linear solve on the fine grid with grid size $h \ll H$. Two-grid method was first introduced by Xu [11,12] for linear (nonsymmetric or indefinite) and especially nonlinear elliptic partial equations. Later on, two-grid method was further investigated by many authors (see,e.g., [13–18]). Dawson and Wheeler [13,14] have applied this method combined with mixed finite element method and finite difference method to nonlinear parabolic equations; Li and Allen [15] have applied two-grid method combined with mixed finite element method to reaction–diffusion equations; Chen, Huang and Yu [16] have constructed a two-grid method for expanded mixed finite element solution of semilinear reaction–diffusion equations. Bi and Ginting [17] have studied two-grid finite volume element method for linear and nonlinear elliptic problems. Chen, Yang and Bi [18] have studied two-grid methods for nonlinear parabolic equations using finite volume element. However, As far as we know there is no two-grid method convergence analysis for the second-order hyperbolic equations in the literature that can be applied in finite volume element method.

In this paper, based on two linear conforming finite element spaces V_H and V_h on one coarse grid with grid size H and one fine grid with grid size h respectively, we consider the two-grid finite volume element discretization techniques for the second-order nonlinear hyperbolic problems. With the proposed techniques, solving the nonsymmetric and nonlinear problems on the fine space is reduced to solving a symmetric and linear problems on the fine space and the nonsymmetric and nonlinear problems on a much smaller space. This means that solving a nonlinear problem is not much more difficult than solving one linear problem, since $\dim V_H \ll \dim V_h$ and the work for solving the nonlinear problem is relatively negligible. A remarkable fact about this simple approach is, as shown in [11], that the coarse mesh can be quite coarse and still maintain a good accuracy approximation.

The rest of this paper is organized as follows. In Section 2, we describe the finite volume element scheme for the second-order hyperbolic problems. In Section 3, based on the standard Ritz projection the optimal error estimates in H^1- and L^2-norms are given for the finite volume element method. In Section 4, two algorithms for the two-grid FVEM of problem (1.1) are introduced. The error estimate in H^1-norm for the two-grid method is proved to be $O(h+H^3|\ln H|)$. A numerical example is presented in the last section.

Throughout this paper, C denotes a generic positive constant which does not depend on the mesh parameters and may be different at its different occurrences.

2. Finite volume element method

We will use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ with $1 \le p \le \infty$ consisting of functions that have generalized derivatives of order s in the space $L^p(\Omega)$. The norm of $W^{s,p}(\Omega)$ is defined by

$$\|u\|_{s,p,\Omega} = \|u\|_{s,p} = \left(\int_\Omega \sum_{|\alpha| \le s} |D^\alpha u|^p \mathrm{d}x \right)^{\frac{1}{p}},$$

with the standard modification for $p = \infty$. In order to simplify the notation, we denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$ and omit the index $p = 2$ and Ω whenever possible; i.e., $\|u\|_{s,2,\Omega} = \|u\|_{s,2} = \|u\|_s$. We define $L^2(H^2) = L^2(0,T;H^2(\Omega)) = \{u(\cdot,t) \in H^2(\Omega), 0 < t \le T\}$, $L^2(W^{1,\infty}) = L^2(0,T;W^{1,\infty}(\Omega)) = \{u(\cdot,t) \in W^{1,\infty}(\Omega), 0 < t \le T\}$ and $L^2(H^1) = L^2(0,T;H^1(\Omega)) = \{u(\cdot,t) \in H^1(\Omega), 0 < t \le T\}$ with norms $\|\cdot\|_{L^2(H^2)}$, $\|\cdot\|_{L^2(W^{1,\infty})}$ and $\|\cdot\|_{L^2(H^1)}$, respectively. We denote by $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ of functions vanishing on the boundary $\partial\Omega$.

The weak formulation of the problem (1.1) is to find $u(\cdot,t) \in H_0^1(\Omega)$, $0 < t \le T$ such that

$$\begin{cases} (u_{tt}, v) + a(u,v) = (f,v), & \forall v \in H_0^1(\Omega), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$-inner product and the bilinear form $a(\cdot,\cdot)$ is defined by

$$a(u,v) = \int_\Omega a \nabla u \cdot \nabla v \mathrm{d}x, \quad \forall u, v \in H_0^1(\Omega). \tag{2.2}$$

Let T_h be a quasi-uniform triangulation of Ω with $h = \max h_K$, where h_K is the diameter of the triangle $K \in T_h$. Based on this triangulation, let V_h be the standard conforming finite element space of piecewise linear functions,

$$V_h = \{v \in C(\bar{\Omega}) : v|_K \text{ is linear}, \forall K \in T_h; v|_{\partial\Omega} = 0\}.$$

In order to describe the finite volume element method we shall introduce a dual partition T_h^* based upon the original partition T_h whose elements are called control volumes. We construct the control volumes in the same way as in [4,8,9]. Let z_K be the

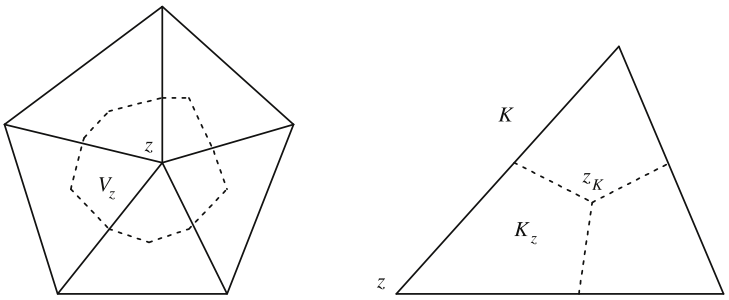


Fig. 1. *Left*: The dotted line shows the boundary of the corresponding control volume V_z with z a common vertex. *Right*: A triangle K partitioned into three subregions K_z.

barycenter of $K \in T_h$. We connect z_K with line segments to the midpoints of the edges of K, thus partitioning K into three quadrilaterals K_z, $z \in Z_h(K)$, where $Z_h(K)$ are the vertices of K. Then with each vertex $z \in Z_h = \cup_{K\in T_h} Z_h(K)$ we associate a control volume V_z, which consists of the union of the subregions K_z, sharing the vertex z (See Fig. 1). Thus we finally obtain a group of control volumes covering the domain Ω, which is called the dual partition T_h^* of the triangulation T_h. We denote the set of interior vertices of Z_h by Z_h^0.

We call the partition T_h^* regular or quasi-uniform, if there exists a positive constant $C > 0$ such that

$$C^{-1}h^2 \le \text{meas}(V_z) \le Ch^2, \quad \forall V_z \in T_h^*.$$

The barycenter-type dual partition can be introduced for any finite element triangulation T_h and leads to relatively simple calculations. Besides, if the finite element triangulation T_h is quasi-uniform, then the dual partition T_h^* is also quasi-uniform.

We formulate the finite volume element method for the problem (1.1) as follows. Given a vertex $z \in Z_h$, integrating (1.1) over the associated control volume V_z and applying Green's formula, we obtain

$$\begin{cases} \displaystyle\int_{V_z} u_{tt}\mathrm{d}x - \int_{\partial V_z} (a\nabla u)\cdot \mathbf{n}\mathrm{d}s = \int_{V_z} f(u)\mathrm{d}x, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in V_z, \end{cases} \tag{2.3}$$

where $\mathbf{n}$ denotes the unit outer-normal vector to ∂V_z. It should be noted that the above formulation is a way of stating that we have an integral conservation form on the control volume.

The finite volume element approximation of (1.1) is defined as a solution $u_h(t) \in V_h$, $0 < t \le T$, such that

$$\int_{V_z} u_{h,tt}\mathrm{d}x - \int_{\partial V_z} (a\nabla u_h)\cdot \mathbf{n}\mathrm{d}s = \int_{V_z} f(u_h)\mathrm{d}x. \tag{2.4}$$

For any $v_h \in H_0^1(\Omega) \cap H^2(\Omega)$, we define an interpolation operator $I_h : H_0^1(\Omega) \cap H^2(\Omega) \to V_h$, such that

$$I_h v_h = \sum_{z\in Z_h^0} v_h(z)\Phi_z,$$

where Φ_z is the standard nodal basis function associated with the node z. By the interpolation theorem in Sobolev space, we have

$$\|v - I_h v\|_m \le Ch^{2-m}\|v\|_2, \ m = 0, 1. \tag{2.5}$$

It is easy to see that

$$\|I_h v_h\|_1 \le C\|v_h\|_1. \tag{2.6}$$

The FVEM (2.4) can be rewritten in a variational form similar to the finite element method with the help of an interpolation operator $I_h^* : V_h \to V_h^*$, defined by

$$I_h^* v_h = \sum_{z\in Z_h^0} v_h(z)\Psi_z,$$

where

$$V_h^* = \{v \in L^2(\Omega) : v|_{V_z} \text{ is constant for all } V_z \in T_h^*;\ v|_{V_z} = 0, \text{ if } z \in \partial\Omega\},$$

and Ψ_z is the characteristic function of the control volume V_z. It was shown in [19] that

$$\|v_h - I_h^* v_h\|_{0,p} \le Ch^s\|v_h\|_{s,p}, \quad s = 0, 1, \tag{2.7}$$

and in [9] that

$$\|I_h^* v_h\| \le C\|v_h\|, \tag{2.8}$$

for $p > 1$.

For any arbitrary $I_h^* v_h$, we multiply the integral relation in (2.4) by $v_h(z)$ and sum over all $z \in Z_h^0$ to obtain the semidiscrete finite volume element formulation, to find $u_h(t) \in V_h$ for $0 < t \le T$ such that

$$(u_{h,tt}, I_h^* v_h) + a_h(u_h, I_h^* v_h) = (f(u_h), I_h^* v_h), \quad \forall v_h \in V_h, \tag{2.9}$$

where the bilinear form $a_h(\cdot, I_h^* \cdot)$ is defined by, for any $u_h, v_h \in V_h$,

$$a_h(u_h, I_h^* v_h) = -\sum_{z \in Z_h^0} v_h(z) \int_{\partial V_z} (a\nabla u_h) \cdot \mathbf{n} ds. \tag{2.10}$$

The initial conditions of (2.9) will be defined appropriately later.

3. Error analysis for finite volume element method

3.1. Some useful lemmas

To describe error estimates for the finite volume element scheme (2.9), we will give some useful lemmas. As shown in [8], we have

$$(u_h, I_h^* v_h) = (v_h, I_h^* u_h), \quad \forall u_h, v_h \in V_h. \tag{3.1}$$

Further we define $|\|u_h\|\|_0 = (u_h, I_h^* u_h)^{1/2}$, $\forall u_h \in V_h$, which is positive definite and the corresponding discrete norm is equivalent to the L^2-norm, i.e., that there exist two positive constants C_*, $C^* > 0$, independent of h such that

$$C_*\|u_h\| \le |\|u_h\|\|_0 \le C^*\|u_h\|, \quad \forall u_h \in V_h. \tag{3.2}$$

The following two lemmas have been proved in [8,20], where Lemma 1 indicates that the bilinear form $a_h(\cdot, I_h^* \cdot)$ is continuous and coercive on V_h, while Lemma 2 shows that $a_h(\cdot, I_h^* \cdot)$ is generally not symmetric and how far it is from being symmetric.

Lemma 1. *For h sufficiently small, there exist two positive constants $C_1, C_2 > 0$ such that, for all $u_h, v_h \in V_h$, the coercive property*

$$a_h(u_h, I_h^* u_h) \ge C_1\|u_h\|_1^2$$

and the boundedness property

$$|a_h(u_h, I_h^* v_h)| \le C_2\|u_h\|_1\|v_h\|_1$$

hold true.

Lemma 2. *For h sufficiently small, there exists a positive constant $C > 0$ such that*

$$|a_h(u_h, I_h^* v_h) - a_h(v_h, I_h^* u_h)| \le Ch\|u_h\|_1\|v_h\|_1, \quad \forall u_h, v_h \in V_h. \tag{3.3}$$

Let $R_h : H^2(\Omega) \cap H_0^1(\Omega) \to V_h$ be the standard Ritz projection such that

$$a(R_h u, v_h) = a(u, v_h), \quad \forall v_h \in V_h, \tag{3.4}$$

where $a(\cdot, \cdot)$ is the bilinear form related to the finite element scheme defined by (2.2). It is well known [21] that

$$\|u - R_h u\|_s \le Ch^{2-s}\|u\|_2, \quad s = 0, 1. \tag{3.5}$$

$$\|u - R_h u\|_{0,\infty} \le Ch|\ln h|\|u\|_{1,\infty}. \tag{3.6}$$

For error analysis we introduce two error functions

$$\varepsilon_h(f, \chi) = (f, \chi) - (f, I_h^* \chi), \quad \forall \chi \in V_h, \tag{3.7}$$

$$\varepsilon_a(\chi, \psi) = a(\chi, \psi) - a_h(\chi, I_h^* \psi), \quad \forall \chi, \psi \in V_h. \tag{3.8}$$

The two error functions are defined in [9] and the bounds for (3.7) and (3.8) are shown as the following lemma.

Lemma 3. *Let $\chi \in V_h$, then*

$$|\varepsilon_h(f, \chi)| \le Ch^{i+j}\|f\|_i\|\chi\|_j, \quad f \in H^i(\Omega), i, j = 0, 1, \tag{3.9}$$

$$|\varepsilon_a(R_h v, \chi)| \le Ch^{i+j}\|v\|_{1+i}\|\chi\|_j, \quad v \in H^{1+i}(\Omega) \cap H_0^1(\Omega), i, j = 0, 1. \tag{3.10}$$

$$|\varepsilon_a(u_h, \chi)| \le Ch\|u_h\|_1\|\chi\|_1, \quad u_h \in V_h. \tag{3.11}$$

3.2. Error estimate in the H^1-norm

Now we turn to error estimates for the finite volume element scheme (2.9). First, we will give the optimal error estimate in H^1-norm.

Theorem 1. *Let u and u_h be the solutions of problem (1.1) and the semidiscrete finite volume element scheme (2.9), respectively. Assume that (1.2) is satisfied. If $u_h(0) = R_h u_0$ and $u_{h,t}(0) = R_h u_1$ with R_h defined by (3.4), for $0 < t \le T$, we have*

$$\|u(t) - u_h(t)\|_1 \le \mathcal{C}h, \tag{3.12}$$

where $\mathcal{C} = C(\|u\|_{L^2(H^2)}, \|u_{tt}\|_{L^2(H^2)}, \|f\|_{L^2(H^1)})$ is independent of h.

Proof. For convenience, let $u - u_h = (u - R_h u) + (R_h u - u_h) =: \eta + \xi$. Then from (1.1) and (2.9), we get the following error equation

$$(\xi_{tt}, I_h^* v_h) + a_h(\xi, I_h^* v_h) = -(\eta_{tt}, I_h^* v_h) - a_h(\eta, I_h^* v_h) + (f(u) - f(u_h), I_h^* v_h), \quad \forall v_h \in V_h. \tag{3.13}$$

Using (3.4) and (3.8), it follows that, $\forall v_h \in V_h$,

$$(\xi_{tt}, I_h^* v_h) + a(\xi, v_h) = -(\eta_{tt}, I_h^* v_h) + \varepsilon_a(\xi, v_h) + \varepsilon_a(\eta, v_h) + (f(u) - f(u_h), I_h^* v_h). \tag{3.14}$$

Choosing $v_h = \xi_t$ in (3.13) to get

$$(\xi_{tt}, I_h^* \xi_t) + a(\xi, \xi_t) = -(\eta_{tt}, I_h^* \xi_t) + \varepsilon_a(\xi, \xi_t) + \varepsilon_a(\eta, \xi_t) + (f(u) - f(u_h), I_h^* \xi_t). \tag{3.15}$$

By (3.1), (3.15) is equivalent to

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\xi_t, I_h^* \xi_t) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}a(\xi, \xi) &= -(\eta_{tt}, I_h^* \xi_t) + \varepsilon_a(\xi, \xi_t) + \varepsilon_a(\eta, \xi_t) + (f(u) - f(u_h), I_h^* \xi_t) \\ &= -(\eta_{tt}, I_h^* \xi_t) + \varepsilon_a(\xi, \xi_t) - a_h(\eta, I_h^* \xi_t) + (f(u) - f(u_h), I_h^* \xi_t). \end{aligned} \tag{3.16}$$

By (3.4), (3.7) and (3.8), we have

$$\begin{aligned} a_h(\eta, I_h^* v_h) &= a_h(u, I_h^* v_h) - a_h(R_h u, I_h^* v_h) \\ &= (f(u) - u_{tt}, I_h^* v_h) - a(R_h u, v_h) + [a(R_h u, v_h) - a_h(R_h u, I_h^* v_h)] \\ &= (f(u) - u_{tt}, I_h^* v_h) - a(u, v_h) + [a(R_h u, v_h) - a_h(R_h u, I_h^* v_h)] \\ &= (f(u) - u_{tt}, I_h^* v_h - v_h) + [a(R_h u, v_h) - a_h(R_h u, I_h^* v_h)] \\ &= \varepsilon_h(u_{tt} - f(u), v_h) + \varepsilon_a(R_h u, v_h). \end{aligned} \tag{3.17}$$

Then we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\xi_t, I_h^* \xi_t) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}a(\xi, \xi) &= -(\eta_{tt}, I_h^* \xi_t) + \varepsilon_a(\xi, \xi_t) - \varepsilon_h(u_{tt} - f(u), \xi_t) \\ &\quad - \varepsilon_a(R_h u, \xi_t) + (f(u) - f(u_h), I_h^* \xi_t). \end{aligned} \tag{3.18}$$

Integrating (3.18) from 0 to t, noting that $\xi(0) = 0$ and $\xi_t(0) = 0$, we have

$$\begin{aligned} |\|\xi_t|\|_0^2 + a(\xi, \xi) &= -2\int_0^t (\eta_{tt}, I_h^* \xi_t)\mathrm{d}t + 2\int_0^t \varepsilon_a(\xi, \xi_t)\mathrm{d}t - 2\int_0^t \varepsilon_h(u_{tt} - f(u), \xi_t)\mathrm{d}t - 2\int_0^t \varepsilon_a(R_h u, \xi_t)\mathrm{d}t \\ &\quad + 2\int_0^t (f(u) - f(u_h), I_h^* \xi_t)\mathrm{d}t \equiv \sum_{i=1}^{5} T_i. \end{aligned} \tag{3.19}$$

Now let us estimate the right-hand side of (3.19), for T_1, by (3.5) and (2.8), there is

$$\begin{aligned} |T_1| &\le C\int_0^t \|\eta_{tt}\|\|I_h^* \xi_t\|\mathrm{d}t \le C\int_0^t (\|\eta_{tt}\|^2 + \|I_h^* \xi_t\|^2)\mathrm{d}t \\ &\le C\int_0^t (h^2\|u_{tt}\|_1^2 + \|\xi_t\|^2)\mathrm{d}t. \end{aligned} \tag{3.20}$$

For T_2, by (3.11) and the inverse estimate, we obtain

$$|T_2| \le C\int_0^t h\|\xi\|_1\|\xi_t\|_1\mathrm{d}t \le C\int_0^t \|\xi\|_1\|\xi_t\|\mathrm{d}t \le C\int_0^t (\|\xi\|_1^2 + \|\xi_t\|^2)\mathrm{d}t. \tag{3.21}$$

For T_3 and T_4, by (3.7)–(3.10), we have

$$|T_3| \le C\int_0^t h\|u_{tt} - f(u)\|_1\|\xi_t\|\mathrm{d}t$$
$$\le C\int_0^t (h^2\|u_{tt}\|_1^2 + h^2\|f\|_1^2 + \|\xi_t\|^2)\mathrm{d}t, \tag{3.22}$$

and

$$|T_4| \le C\int_0^t h\|u\|_2\|\xi_t\|\mathrm{d}t \le C\int_0^t (h^2\|u\|_2^2 + \|\xi_t\|^2)\mathrm{d}t. \tag{3.23}$$

For T_5, at any point $x \in \Omega$, by the Taylor expansion, we have

$$f(u) - f(u_h) = f'(\tilde{u})(u - u_h) = f'(\tilde{u})(\eta + \xi),$$

for some value $\tilde{u}$ between u and u_h. From (1.2) and (3.5), we have

$$|T_5| \le C\int_0^t \|\eta + \xi\|\|I_h^*\xi_t\|\mathrm{d}t \le C\int_0^t (h^4\|u\|_2^2 + \|\xi\|^2 + \|\xi_t\|^2)\mathrm{d}t. \tag{3.24}$$

Substituting the estimates (3.20)–(3.24) in (3.19), applying (3.2) and Lemma 1, we obtain

$$\|\xi_t\|^2 + \|\xi\|_1^2 \le Ch^2\int_0^t (\|u_{tt}\|_1^2 + \|u\|_2^2 + \|f\|_1^2)\mathrm{d}t + C\int_0^t (\|\xi\|_1^2 + \|\xi_t\|^2)\mathrm{d}t. \tag{3.25}$$

So it follows from the Gronwall's inequality that

$$\|\xi_t\|^2 + \|\xi\|_1^2 \le Ch^2\int_0^T (\|u_{tt}\|_1^2 + \|u\|_2^2 + \|f\|_1^2)\mathrm{d}t. \tag{3.26}$$

Together with (3.5) this yields (3.12). □

3.3. Error estimate in the L^2-norm

Now we state and show the optimal error estimate in the L^2-norm for the finite volume element scheme (2.9).

Theorem 2. *Let u and u_h be the solutions of problem (1.1) and the semidiscrete finite volume element scheme (2.9), respectively. Assume that (1.2) is satisfied. If $u_h(0) = R_h u_0$ and $u_{h,t}(0) = R_h u_1$ with R_h defined by (3.4), for $t \le T$, we have*

$$\|u(t) - u_h(t)\| \le \mathcal{C}h^2, \tag{3.27}$$

where $\mathcal{C} = C(\|u\|_{L^2(H^2)}, \|u_{tt}\|_{L^2(H^1)}, \|u_t\|_{L^2(H^2)}, \|f\|_{L^2(H^1)})$ is independent of h.

Proof. Integrating (3.14) from 0 to t and setting $v_h = \hat{\xi}_t = \xi$, we have

$$(\xi_t, I_h^*\xi) + a(\hat{\xi}, \xi) = -(\eta_t, I_h^*\xi) + \varepsilon_a(\hat{\xi}, \xi) + \varepsilon_a(\hat{\eta}, \xi) + (\hat{f}(u) - \hat{f}(u_h), I_h^*\xi), \tag{3.28}$$

where $\hat{\theta} = \int_0^t \theta(\tau)\mathrm{d}\tau$.

Using (3.1) and symmetry of the bilinear form $a(\cdot,\cdot)$, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\xi, I_h^*\xi) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}a(\hat{\xi}, \hat{\xi}) = -(\eta_t, I_h^*\xi) + \varepsilon_a(\hat{\xi}, \xi) + \varepsilon_a(\hat{\eta}, \xi) + (\hat{f}(u) - \hat{f}(u_h), I_h^*\xi). \tag{3.29}$$

Integrating (3.30) from 0 to t, noting that $\xi(0) = 0$ and $\hat{\xi}(0) = 0$ we obtain

$$|||\xi|||_0^2 + a(\hat{\xi}, \hat{\xi}) = -2\int_0^t (\eta_t, I_h^*\xi)\mathrm{d}\tau + 2\int_0^t \varepsilon_a(\hat{\xi}, \xi)\mathrm{d}\tau + 2\int_0^t \varepsilon_a(\hat{\eta}, \xi)\mathrm{d}\tau + 2\int_0^t (\hat{f}(u) - \hat{f}(u_h), I_h^*\xi)\mathrm{d}\tau$$
$$\equiv \sum_{i=1}^4 Q_i. \tag{3.30}$$

Now we will estimate the right-hand terms of (3.30), for Q_1, by (2.8) and (3.5), we have

$$|Q_1| \le 2\int_0^t |(\eta_t, I_h^*\xi)|\mathrm{d}\tau \le C\int_0^t \|\eta_t\|\|\xi\|\mathrm{d}\tau \le C\int_0^t (h^4\|u_t\|_2^2 + \|\xi\|^2)\mathrm{d}\tau. \tag{3.31}$$

By (3.11) and the inverse estimate we find that

$$|Q_2| = 2\int_0^t |\varepsilon_a(\hat{\xi},\xi)|\mathrm{d}\tau \le Ch\int_0^t \|\hat{\xi}\|_1\|\xi\|_1\mathrm{d}\tau$$
$$\le C\int_0^t \|\hat{\xi}\|_1\|\xi\|\mathrm{d}\tau \le C\int_0^t (\|\hat{\xi}\|_1^2 + \|\xi\|^2)\mathrm{d}\tau. \tag{3.32}$$

For Q_3, we can estimate it as in [10],

$$\varepsilon_a(\hat{\eta},\hat{\xi}_t) = \varepsilon_h(\hat{f}-\hat{u}_{tt},\hat{\xi}_t) - \varepsilon_a(R_h\hat{u},\hat{\xi}_t)$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon_h(\hat{f}-\hat{u}_{tt},\hat{\xi}) - \varepsilon_a(R_h\hat{u},\hat{\xi})) - (\varepsilon_h(f-u_{tt},\hat{\xi}) - \varepsilon_a(R_hu,\hat{\xi})),$$

and hence,

$$Q_3 = 2(\varepsilon_h(\hat{f}-\hat{u}_{tt},\hat{\xi}) - \varepsilon_a(R_h\hat{u},\hat{\xi})) - 2\int_0^t (\varepsilon_h(f-u_{tt},\hat{\xi}) - \varepsilon_a(R_hu,\hat{\xi}))\mathrm{d}\tau.$$

From (3.9) and (3.10), we have

$$|Q_3| \le 2|\varepsilon_h(\hat{f}-\hat{u}_{tt},\hat{\xi})| + 2|\varepsilon_a(R_h\hat{u},\hat{\xi})| + 2\int_0^t (|\varepsilon_h(f-u_{tt},\hat{\xi})| + |\varepsilon_a(R_hu,\hat{\xi})|)\mathrm{d}\tau$$
$$\le Ch^2\left[\int_0^t (\|f\|_1 + \|u_{tt}\|_1 + \|u\|_2)\mathrm{d}\tau\right]\|\hat{\xi}\|_1 + Ch^2\int_0^t (\|f\|_1 + \|u_{tt}\|_1 + \|u\|_2)\|\hat{\xi}\|_1\mathrm{d}\tau$$
$$\le Ch^4\int_0^t (\|f\|_1^2 + \|u_{tt}\|_1^2 + \|u\|_2^2)\mathrm{d}\tau + C\int_0^t \|\hat{\xi}\|_1^2\mathrm{d}\tau. \tag{3.33}$$

For Q_4, by (1.2) and (3.5), we obtain

$$|Q_4| \le 2\int_0^t |(\hat{f}(u) - \hat{f}(u_h), I_h^*\xi)|\mathrm{d}\tau \le C\int_0^t \|u-u_h\|\|\xi\|\mathrm{d}\tau$$
$$\le C\int_0^t (\|\xi\|^2 + \|\eta\|^2)\mathrm{d}\tau \le C\int_0^t (\|\xi\|^2 + h^4\|u\|_2^2)\mathrm{d}\tau. \tag{3.34}$$

Substituting the estimates of Q_i in (3.30) and by (3.2), we obtain

$$\|\xi\|^2 + \|\hat{\xi}\|_1^2 \le Ch^4\int_0^t (\|f\|_1^2 + \|u_{tt}\|_1^2 + \|u\|_2^2 + \|u_t\|_2^2)\mathrm{d}\tau + C\int_0^t (\|\hat{\xi}\|_1^2 + \|\xi\|^2)\mathrm{d}\tau. \tag{3.35}$$

Then the Gronwall's lemma yields for $t \le T$,

$$\|\xi\|^2 + \|\hat{\xi}\|_1^2 \le Ch^4\int_0^T (\|f\|_1^2 + \|u_{tt}\|_1^2 + \|u\|_2^2 + \|u_t\|_2^2)\mathrm{d}t. \tag{3.36}$$

Together with (3.5) this yields (3.27). □

4. Two-grid finite volume element method

In this section, we shall present two-grid finite volume element algorithm for problem (1.1) based on two finite element spaces. The idea of the two-grid method is to reduce the nonlinear and nonsymmetric problem on a fine grid into a linear and symmetric system on a fine grid by solving a nonlinear and nonsymmetric problem on a coarse grid. The basic mechanisms are two quasi-uniform triangulations of Ω, T_H and T_h, with two different mesh sizes H and h ($H > h$), and the corresponding finite element spaces V_H and V_h which satisfies $V_H \subset V_h$ and will be called the coarse grid and the fine grid spaces, respectively.

To solve problem (1.1), we introduce two-grid algorithms into the finite volume element method. The idea is to use a coarse space to produce a rough approximation of the solution, and then use it as the initial guess for one Newton-like iteration on the fine grid. This method involves a nonlinear solve on the coarse space and a linear solve on the fine grid space. We present the two-grid finite volume element method as two steps:

Algorithm 1. *Step* 1: On the coarse grid T_H, find $u_H \in V_H$, such that

$$\begin{cases} (u_{tt}, I_H^*v_H) + a_H(u_H, I_H^*v_H) = (f(u_H), I_H^*v_H), & \forall v_H \in V_H, \\ u_H(0) = R_Hu_0, \qquad u_{H,t}(0) = R_Hu_1, \end{cases} \tag{4.1}$$

where R_H is defined in the same way as R_h defined by (3.4).

Step 2: On the fine grid T_h, find $u_h \in V_h (n = 1, 2, \ldots)$, such that

$$\begin{cases} (u_{tt}, I_h^* v_h) + a_h(u_h, I_h^* v_h) = (f(u_H) + f'(u_H)(u_h - u_H), I_h^* v_h), \quad \forall v_h \in V_h, \\ u_h(0) = R_h u_0, \qquad u_{h,t}(0) = R_h u_1. \end{cases} \tag{4.2}$$

From Lemma 2 we note that the system in the second step of Algorithm 1 is a linear problem but still nonsymmetric. In order to get a symmetric system, we introduce the following bilinear forms

$$a_c(u_h, v_h) = \int_\Omega \bar{a} \nabla u_h \cdot \nabla v_h \mathrm{d}x, \quad \forall u_h, v_h \in V_h, \tag{4.3}$$

$$\begin{aligned} a_{h,c}(u_h, I_h^* v_h) &= -\sum_{z \in Z_h^0} \int_{\partial V_z} (\bar{a} \nabla u_h) \cdot \mathbf{n} I_h^* v_h \mathrm{d}s \\ &= -\sum_{z \in Z_h^0} v_h(z) \int_{\partial V_z} (\bar{a} \nabla u_h) \cdot \mathbf{n} \mathrm{d}s, \quad \forall u_h, v_h \in V_h, \end{aligned} \tag{4.4}$$

where $\bar{a} = \bar{a}|_K = a_K$ and

$$a_K = \frac{1}{\mathrm{meas}(K)} \int_K a(x) \mathrm{d}x, \quad \forall K \in T_h.$$

Then from [4,20], we have the following lemma

Lemma 4. *For any $u_h, v_h \in V_h$, we have*

$$a_{h,c}(u_h, I_h^* v_h) = a_c(u_h, v_h).$$

From this lemma we can see that $a_{h,c}(u_h, I_h^* v_h)$ is symmetric. Then we obtain the second algorithm.

Algorithm 2. *Step* 1: The same as in Algorithm 1.

Step 2: On the fine grid T_h, find $u_h \in V_h$, such that

$$\begin{cases} (u_{tt}, I_h^* v_h) + a_{h,c}(u_h, I_h^* v_h) = (f(u_H) + f'(u_H)(u_h - u_H), I_h^* v_h), \quad \forall v_h \in V_h, \\ u_h(0) = R_h u_0, \qquad u_{h,t}(0) = R_h u_1. \end{cases} \tag{4.5}$$

We note that the system in the first step of Algorithm 2 is the same as in Algorithm 1. But on the fine grid in the second step the coefficient matrix of the system is symmetric. So the system is easier to solve(e.g. conjugate-gradient-like methods can be applied effectively). We call these algorithms two-grid finite volume element methods.

Now we consider the error estimates in H^1-norm for the two-grid finite volume element method Algorithms 1 and 2. For Algorithm 1, we have

Theorem 3. *Let u and u_h be the solutions of (1.1) and the two-grid finite volume element method Algorithm 1, respectively. Assume that (1.2) is satisfied. Then for $t \le T$, we have*

$$\|u(t) - u_h(t)\|_1 \le \mathcal{C}(h + H^3 |\ln H|), \tag{4.6}$$

where $\mathcal{C} = C(\|u\|_{L^2(H^2)}, \|u_{tt}\|_{L^2(H^2)}, \|u_t\|_{L^2(H^2)}, \|u\|_{L^2(W^{1,\infty})}, \|f\|_{L^2(H^1)})$ is independent of h and H.

Proof. Once again, we set $u - u_h = (u - R_h u) + (R_h u - u_h) =: \eta + \xi$ and choose $v_h = \xi_t$. Then for Algorithm 1, we get the error equation

$$(\xi_{tt}, I_h^* \xi_t) + a_h(\xi, I_h^* \xi_t) = -(\eta_{tt}, I_h^* \xi_t) - a_h(\eta, I_h^* \xi_t) + (f(u) - f(u_H) - f'(u_H)(u_h - u_H), I_h^* \xi_t). \tag{4.7}$$

Similarly as in Theorem 1, we have

$$\begin{aligned} |\|\xi_t|\|_0^2 + a(\xi, I_h^* \xi) &= -2 \int_0^t (\eta_{tt}, I_h^* \xi_t) \mathrm{d}t + 2 \int_0^t \varepsilon_a(\xi, \xi_t) \mathrm{d}t - 2 \int_0^t \varepsilon_h(u_{tt} - f(u), \xi_t) \mathrm{d}t \\ &\quad - 2 \int_0^t \varepsilon_a(R_h u, \xi_t) \mathrm{d}t + 2 \int_0^t (f(u) - f(u_H) - f'(u_H)(u_h - u_H), I_h^* \xi_t) \mathrm{d}t. \end{aligned} \tag{4.8}$$

For the first four terms of the right-hand side of (4.8), we can estimate them similarly as in Theorem 1. So our main task is to deal with the last term of the right-hand side of (4.8). A Taylor expansion about u_H yields

$$f(u) = f(u_H) + f'(u_H)(u - u_H) + \frac{1}{2} f''(\tilde{u})(u - u_H)^2,$$

for some function $\tilde{u}$ between u and u_H. Then

$$f(u) - f(u_H) - f'(u_H)(u_h - u_H) = f'(u_H)(u - u_h) + \frac{1}{2}f''(\tilde{u})(u - u_H)^2$$
$$= f'(u_H)(\eta + \xi) + \frac{1}{2}f''(\tilde{u})(u - u_H)^2.$$

By (1.2) and (3.5), we have

$$\left|2\int_0^t (f(u) - f(u_H) - f'(u_H)(u_h - u_H), I_h^*\xi_t)\right| \le C\int_0^t (\|\xi\|^2 + \|\eta\|^2) + \|(u - u_H)^2\|^2 + \|\xi_t\|^2 \mathrm{d}t$$
$$\le C\int_0^t (\|\xi\|^2 + h^4\|u\|_2^2) + \|(u - u_H)^2\|^2 + \|\xi_t\|^2 \mathrm{d}t. \tag{4.9}$$

By (4.8), similarly as in Theorem 1, we obtain

$$\|\xi_t\|^2 + \|\xi\|_1^2 \le Ch^2\int_0^T (\|u_{tt}\|_2^2 + \|u_{tt}\|_1^2 + \|u\|_2^2 + \|f\|_1^2)\mathrm{d}t + \int_0^T \|(u - u_H)^2\|^2 \mathrm{d}t. \tag{4.10}$$

For the last term of (4.10), there is

$$\|(u - u_H)^2\| \le \|u - u_H\|_{0,\infty}\|u - u_H\|$$
$$\le (\|u - R_H u\|_{0,\infty} + \|R_H u - u_H\|_{0,\infty})\|u - u_H\|. \tag{4.11}$$

By Theorem 1, Theorem 2, (3.6) and the inverse estimate, we get

$$\|(u - u_H)^2\| \le \mathcal{C}(H|\ln H| + H^{-1}H^2)H^2 \le \mathcal{C}H^3|\ln H|. \tag{4.12}$$

Combining with (4.10), we can easily get

$$\|\xi_t\|^2 + \|\xi\|_1^2 \le \mathcal{C}(h^2 + (H^3|\ln H|)^2), \tag{4.13}$$

where $\mathcal{C} = C(\|u\|_{L^2(H^2)}, \|u_{tt}\|_{L^2(H^2)}, \|u_t\|_{L^2(H^2)}, \|u\|_{L^2(W^{1,\infty})}, \|f\|_{L^2(H^1)})$. By (3.5) and the triangular inequality it completes the proof. □

Remark 4.1. Under the same condition of Theorem 3 we can obtain the same result for Algorithm 2. From the results we can see that the two-grid method can achieve asymptotically optimal approximation in the H^1-norm error estimate as long as the mesh sizes satisfy $h = O(H^3|\ln H|)$.

Remark 4.2. In order to give the numerical experiments, we further discretize time t of the semidiscrete two-grid finite volume element method in this section. We consider a time step Δt and approximate the solutions at $t^n = n\Delta t$, $\Delta t = T/N$, $n = 0, 1, \ldots, N$. Denote $u_h^n = u_h(t^n)$, $u_{h,tt}^n = \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{(\Delta t)^2}$, $u_{h,t}^n = \frac{u_h^{n+1} - u_h^n}{\Delta t}$, we can get the fully discrete two-grid finite volume element scheme for (1.1). For simple and convenient, we only give the fully discrete scheme for Algorithm 2.

Algorithm 2′. *Step* 1: On the coarse grid T_H, find $u_H^n \in V_H$ $(n = 1, 2, \ldots)$, such that

$$\begin{cases} (u_{H,tt}^n, I_H^* v_H) + a_H(u_H^{n+1}, I_H^* v_H) = (f(u_H^{n+1}), I_H^* v_H), \quad \forall v_H \in V_H, \\ u_H^0 = R_H u_0, \qquad u_{H,t}^0 = R_H u_1. \end{cases}$$

Step 2: On the fine grid T_h, find $u_h^n \in V_h$ $(n = 1, 2, \ldots)$, such that

$$\begin{cases} (u_{h,tt}^n, I_h^* v_h) + a_{h,c}(u_h^{n+1}, I_h^* v_h) = (f(u_H^{n+1}) + f'(u_H^{n+1})(u_h^{n+1} - u_H^{n+1}), I_h^* v_h), \quad \forall v_h \in V_h, \\ u_h^0 = R_h u_0, \qquad u_{h,t}^0 = R_h u_1. \end{cases}$$

We can get the same kind of estimate as Theorem 3 with the result $\|u^n - u_h^n\|_1 \le \mathcal{C}((\Delta t)^2 + h + H^3|\ln H|)$.

5. Numerical example

We consider the following second-order hyperbolic problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \nabla\cdot(A\nabla u) + u^2 = g(x,t), \quad x \in \Omega = [0,1]^2, \quad t > 0, \\ &u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \\ &u(x,0) = 0, \end{aligned} \tag{5.1}$$

Table 1
H^1 error and computational time of the finite volume element scheme.

h	H_1 error	Computational time (s)
1/9	2.6419×10^{-5}	33.0310
1/16	2.6401×10^{-5}	457.2190
1/25	2.6070×10^{-5}	4604.7250
1/36	2.5832×10^{-5}	35407.9140

Table 2
H^1 error and computational time of the two-grid finite volume element method.

h	H	H_1 error	Computational time (s)
1/9	1/3	2.6482×10^{-5}	2.5460
1/16	1/4	2.6462×10^{-5}	3.1090
1/25	1/5	2.6122×10^{-5}	6.3280
1/36	1/6	2.5882×10^{-5}	13.5310

where $x = (x_1, x_2), A = 1, g(x, t)$ is decided by the exact solution of (5.1).

Let the exact solution of (5.1) be

$$u(x, t) = t^2 x_1 x_2 (1 - x_1)(1 - x_2) e^{x_1 + x_2}. \tag{5.2}$$

Choose the space step H and then obtain the coarse grids. Let $h = H^2$ and then obtain the fine grids. In order to prove the efficiency of the two-grid finite volume element method, we use two different method to obtain numerical solutions of (5.1): the usual finite volume element scheme and the two-grid finite volume element scheme. Computational results are shown in Tables 1 and 2.

By comparison of Tables 1 and 2, we can see that the two-grid finite volume element method is more efficient than the usual finite volume element scheme.

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