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On approximate solutions in set-valued optimization problems

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ABSTRACT

In this paper we introduce several concepts of approximate solutions of set-valued optimization problems with vector and set optimization. We prove existence results and necessary and sufficient conditions by using limit sets.

1. Introduction and notation

Set-valued optimization problems have been investigated during the last decade usually with the criterion of vector optimization (see for example [1–4]). In the last years new criteria have been given. These criteria are based on set-relations defined on the space of subsets (see [5]). They are independent of the vector case and are known as set optimization criteria. Recently they have been of increasing interest [5–8]. With vector optimization the character of the set image for each element x is not considered. In this sense set optimization constitutes a more natural criterion for studying set-valued optimization problems and finding a solution.

However in a lot of cases the set of solutions is empty with both criteria: vector and set optimization. This is the reason for introducing approximately efficient solutions. Concepts of approximate solutions have been introduced and applied by several authors in the literature (see for example [9–11]). In this work we define some notions of approximate solution or ε-efficient element for vector and set optimization. These concepts are, in some sense, more flexible than that ones defined in the mentioned works. We establish necessary and sufficient conditions for the existence of these elements by means of limit sets.

This paper is organized as follows. In Section 2 we define the concept of weak ε-efficient element of a multifunction F. We give necessary and sufficient conditions using the limit set of F. Section 3 is devoted to the case of set optimization. We define the concept of ε-lower weak minimal element and give conditions on F and its domain for its existence. Also we study necessary and sufficient conditions using limit sets.

We consider X, Y real normed spaces with Y partially ordered by a convex closed pointed cone K_Y and $M \subset X$. Let F be a multifunction $F\colon M \longrightarrow 2^Y$.

The image and graph of F are respectively defined by

$$F(M) = \bigcup_{x\in M} F(x), \qquad graph(F) = \{(x, y) \mid x \in M, y \in F(x)\}.$$

And the epigraph of F:

$$epi(F) = \{(x, y) \in X \times Y \mid x \in M,\ y \in F(x) + K_Y\}.$$

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Given $\bar{x} \in M$ we will consider the contingent cone to M at $\bar{x}$ defined by (see [12]):

$$T(M, \bar{x}) = \{v \in X \mid \exists (t_n) \to 0^+, (u_n) \to v \text{ with } \bar{x} + t_n u_n \in M \text{ for all } n \in \mathbb{N}\}.$$

Definition 1. Given $\mu \in Y$ we call the limit set of F at $\bar{x} \in M$ on the direction $v \in T(M, \bar{x})$ to the set

$$Y_F^{\mu}(\bar{x}, v) = \left\{ z \in Y \mid z = \lim_{\substack{t_n \to 0^+ \\ u_n \to v}} \frac{f(\bar{x} + t_n u_n) - f(\bar{x})}{t_n}; \mu = f(\bar{x}), f \in CS(F) \right\},$$

where $CS(F)$ denotes the set of continuous selections of F.

Definition 2 (*[2]*)**.** Let $(\bar{x}, \bar{y}) \in graph(F)$. A set-valued map $D_C F(\bar{x}, \bar{y}): X \to 2^Y$ whose graph equals the contingent cone to the graph of F at $(\bar{x}, \bar{y})$, i.e.,

$$graph(D_C F(\bar{x}, \bar{y})) = T(graph(F), (\bar{x}, \bar{y})),$$

is called contingent derivative of F at $(\bar{x}, \bar{y})$.

Recall that the contingent cone to the graph of F is denoted by $T(graph(F), (\bar{x}, \bar{y}))$ and it consists of all tangent vectors $(h, k) = \lim_{n\to\infty} \mu_n(x_n - \bar{x}, y_n - \bar{y})$, with $(\bar{x}, \bar{y}) = \lim_{n\to\infty}(x_n, y_n)$, $(x_n, y_n) \in graph(F)$ and $\mu_n > 0$ for all $n \in N$. Or equivalently, there exist a sequence of real numbers $(t_n) \to 0^+$ and a sequence of vectors $(h_n, k_n) \to (h, k)$ such that $(\bar{x} + t_n h_n, \bar{y} + t_n k_n) \in graph(F)$ for all $n \in \mathbb{N}$.

Remark 3. Observe that if F is single-valued, $F = f$, f continuous on M, then $Y_f^{f(\bar{x})}(\bar{x}, v) = D_C f(\bar{x}, f(\bar{x}))(v)$ for all $v \in T(M, \bar{x})$ (see [13]). In consequence

$$Y_F^{\mu}(\bar{x}, v) = \bigcup_{\substack{f \in CS(F) \\ \mu = f(\bar{x})}} D_C f(\bar{x}, \mu)(v) \quad \text{for all } v \in T(M, \bar{x}).$$

In certain conditions the limit set of F coincides with the contingent derivative of F. One requirement for the contingent derivative is to be lower semicontinuous:

Definition 4 (*[12]*)**.** A set-valued map $F: M \longrightarrow 2^Y$ is called lower semicontinuous at $x \in M$ if for any $y \in F(x)$ and for any sequence of elements $(x_n) \subset M$ converging to x, there exists a sequence of elements $(y_n) \subset F(x_n)$ converging to y. It is called lower semicontinuous if it is lower semicontinuous at each point of M.

Proposition 5. *Let $F: M \longrightarrow 2^{\mathbb{R}^n}$ be a continuous set-valued map with convex and closed images. Let $(\bar{x}, \mu) \in graph(F)$. Assume that $dom(D_C F(\bar{x}, \mu)) = X$ and $D_C F(\bar{x}, \mu)(v)$ is lower semicontinuous with convex images. Then*

$$Y_F^{\mu}(\bar{x}, v) = D_C F(\bar{x}, \mu)(v) \quad \textit{for all } v \in X.$$

Proof. "$\subset$": Observe that if $dom(D_C F(\bar{x}, \mu)) = X$ it follows that $T(M, \bar{x}) = X$. Let $v \in X$, $w \in Y_F^{\mu}(\bar{x}, v)$ then there exists $f \in CS(F)$ with $\mu = f(\bar{x})$, $(t_n) \to 0^+$, $(u_n) \to v$ such that

$$w = \lim_{\substack{t_n \to 0^+ \\ u_n \to v}} \frac{f(\bar{x} + t_n u_n) - f(\bar{x})}{t_n}.$$

Let $k_n = \frac{f(\bar{x}+t_n u_n) - f(\bar{x})}{t_n}$. Then $(\bar{x} + t_n u_n, f(\bar{x}) + t_n k_n) \in graph(F)$ with $(u_n, k_n) \to (v, w)$. In consequence $(v, w) \in T(graph(F), (\bar{x}, \mu))$ and $w \in D_C F(\bar{x}, \mu)(v)$.

"$\supset$": Under these hypotheses on F, for each $v \in X$ and $w \in D_C F(\bar{x}, \mu)(v)$ there exists a continuous selection f of F such that $\mu = f(\bar{x})$ and $w \in Y_f(\bar{x}, v)$ (see [13, Theorem 17]). It follows that $D_C F(\bar{x}, \mu)(v) \subset \bigcup_{\substack{f \in CS(F) \\ \mu = f(\bar{x})}} Y_f(\bar{x}, v)$. In virtue of Remark 3 we deduce

$$\bigcup_{\substack{f \in CS(F) \\ \mu = f(\bar{x})}} Y_f^{\mu}(\bar{x}, v) = \bigcup_{\substack{f \in CS(F) \\ \mu = f(\bar{x})}} D_C f(\bar{x}, \mu)(v) = Y_F^{\mu}(\bar{x}, v) \quad \text{for all } v \in X.$$

In consequence $D_C F(\bar{x}, \mu)(v) \subset Y_F^{\mu}(\bar{x}, v)$ for all $v \in X$. □

Definitions of upper K-continuity and pseudoconvexity which we recall here, are used in Section 3.

Definition 6 (*[3]*)**.** F is upper K-continuous at $\bar{x} \in M$ if for each neighborhood V of $F(\bar{x})$ in Y, there is a neighborhood U of $\bar{x}$ in X such that

$$F(x) \subset V + K_Y \quad \text{for all } x \in U \cap M.$$

Definition 7 (*[12]*)**.** A multifunction F is pseudoconvex at $(\bar{x}, \bar{y})$ if

$$epi(F) \subset (\bar{x}, \bar{y}) + T(epi(F), (\bar{x}, \bar{y})),$$

where $T(epi(F), (\bar{x}, \bar{y}))$ is the contingent cone to the epigraph of F at $(\bar{x}, \bar{y})$.

2. Approximate solutions in set-valued optimization with vector criterion

From now on we will suppose that the cone K_Y has nonempty interior ($\text{int}K_Y \neq \emptyset$).

We recall the notion of ε-efficient element introduced in [11], for a multifunction $G: V \longrightarrow 2^{\mathbb{R}^n}$, where V is a complete metric space and $K \subset \mathbb{R}^n$ is a pointed closed convex cone with $\text{int}K \neq \emptyset$:

Definition 8 (*[11]*)**.** Given $\varepsilon > 0$, we say that $y_\varepsilon \in G(V)$ is an ε-efficient element with respect to K if there exists an element $\beta_\varepsilon \in \mathbb{R}^n$ such that $\|\beta_\varepsilon\| < \varepsilon$ and

$$G(V) \cap \{y_\varepsilon - \beta_\varepsilon - (K \setminus \{0\})\} = \emptyset.$$

From the above notion of ε-efficient element we define the concept of weak ε-efficient element as follows:

Definition 9. Given $\varepsilon > 0$, we say that $y_\varepsilon \in F(M)$ is a weak ε-efficient element of F with respect to K_Y if there exists an element $\alpha_\varepsilon \in Y$ such that $\|\alpha_\varepsilon\| < \varepsilon$ and

$$F(M) \cap \{y_\varepsilon - \alpha_\varepsilon - \text{int}K_Y\} = \emptyset.$$

In the future we will call a weak ε-efficient element of F a weak ε-efficient element of F with respect to K_Y.

Proposition 10. *Let $y_\varepsilon \in F(x_\varepsilon)$ be a weak ε-efficient element of F. Let $\alpha_\varepsilon \in (-K_Y)$. Then*

$$Y_F^{y_\varepsilon}(x_\varepsilon, v) \cap (-\alpha_\varepsilon - \text{int}K_Y) = \emptyset \quad \textit{for all } v \in T(M, x_\varepsilon).$$

Proof. Let $z \in Y_F^{y_\varepsilon}(x_\varepsilon, v)$ with $v \in T(M, x_\varepsilon)$. Then there exist $(t_n) \to 0^+$, $(u_n) \to v$, $f \in CS(F)$ such that

$$z = \lim_{\substack{t_n \to 0^+ \\ u_n \to v}} \frac{f(x_\varepsilon + t_n u_n) - f(x_\varepsilon)}{t_n} \quad \text{with } f(x_\varepsilon) = y_\varepsilon.$$

Since y_ε is a weak ε-efficient element of F then

$$(F(x_\varepsilon + t_n u_n) - y_\varepsilon) \cap (-\alpha_\varepsilon - \text{int}K_Y) = \emptyset \quad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

Let us prove that $z \notin (-\alpha_\varepsilon - \text{int}K_Y)$. In other case, since $-\alpha_\varepsilon - \text{int}K_Y$ is open there exists $N \in \mathbb{N}$ such that

$$\frac{f(x_\varepsilon + t_n u_n) - f(x_\varepsilon)}{t_n} \in (-\alpha_\varepsilon - \text{int}K_Y) \quad \text{for all } n > N,$$

hence

$$f(x_\varepsilon + t_n u_n) - f(x_\varepsilon) \in (-t_n\alpha_\varepsilon - t_n\text{int}K_Y) \subset (-\alpha_\varepsilon - \text{int}K_Y) \quad \text{for all } n > N,$$

therefore

$$(F(x_\varepsilon + t_n u_n) - y_\varepsilon) \cap (-\alpha_\varepsilon - \text{int}K_Y) \neq \emptyset \quad \text{for all } n > N$$

in contradiction with (2.1). Then $z \notin (-\alpha_\varepsilon - \text{int}K_Y)$ and as a consequence

$$Y_F^{y_\varepsilon}(x_\varepsilon, v) \cap (-\alpha_\varepsilon - \text{int}K_Y) = \emptyset \quad \text{for all } v \in T(M, x_\varepsilon). \quad \square$$

Next lemma proved in [6] will be useful for providing a sufficient condition of existence of weak ε-efficient element.

Lemma 11. *Let M be convex. Let F be pseudoconvex at $(x_\varepsilon, y_\varepsilon) \in graph(F)$. Let us suppose that for each pair of sequences $(t_n) \to 0^+$, $(h_n, k_n) \to (h, k)$, such that $(x_\varepsilon + t_n h_n, y_\varepsilon + t_n k_n) \in epi(F)$, there exists a continuous selection f of F such that $y_\varepsilon = f(x_\varepsilon)$ and $(x_\varepsilon + t_n h_n, y_\varepsilon + t_n k_n) \in epi(f)$. Then*

$$F(x) - y_\varepsilon \subset Y_F^{y_\varepsilon}(x_\varepsilon, x - x_\varepsilon) + K_Y$$

for all $x \in M$.

Proposition 12. *Let M be convex. Let F be pseudoconvex at $(x_\varepsilon, y_\varepsilon) \in graph(F)$. Let $\alpha_\varepsilon \in K_Y$ with $\|\alpha_\varepsilon\| < \varepsilon$. Assume that F satisfies the hypothesis of Lemma 11. If*

$$Y_F^{y_\varepsilon}(x_\varepsilon, v) \cap (-\alpha_\varepsilon - \mathrm{int}K_Y) = \emptyset \quad \text{for all } v \in T(M, x_\varepsilon),$$

then y_ε is a weak ε-efficient element of F.

Proof. Suppose that y_ε is not a weak ε-efficient element of F, then for all $\alpha_\varepsilon \in Y$ with $\|\alpha_\varepsilon\| < \varepsilon$ it satisfies:

$$F(M) \cap (y_\varepsilon - \alpha_\varepsilon - \mathrm{int}K_Y) \neq \emptyset.$$

By Lemma 11

$$F(x) - y_\varepsilon \subset Y_F^{y_\varepsilon}(x_\varepsilon, x - x_\varepsilon) + K_Y \quad \text{for all } x \in M,$$

then there exists $x \in M$ such that

$$(Y_F^{y_\varepsilon}(x_\varepsilon, x - x_\varepsilon) + K_Y) \cap (-\alpha_\varepsilon - \mathrm{int}K_Y) \neq \emptyset,$$

and as a consequence

$$Y_F^{y_\varepsilon}(x_\varepsilon, x - x_\varepsilon) \cap (-\alpha_\varepsilon - \mathrm{int}K_Y) \neq \emptyset. \quad \square$$

3. Approximate solutions in set-valued optimization with set criterion

The optimality concept that we will use in this section is determined by a relation defined by Kuroiwa et al. in [14] as follows:

Definition 13 (*[14]*). Let A, B be two subsets of Y, $A \neq \emptyset$, $B \neq \emptyset$. $A \leq^l B$ if for all $b \in B$ there exists $a \in A$ such that $a \leq b$. "$\leq^l$" is called the lower relation.

The above definition is equivalent to $B \subset A + K_Y$ and generalizes the order induced by K_Y in Y, in the sense: $a \leq b$ if $b \in a + K_Y$.

Definition 14 (*[14]*). Given a family $\mathcal{S}$ of subsets of Y, $A \in \mathcal{S}$ is a lower minimal (or l-minimal) if for each $B \in \mathcal{S}$ such that $B \leq^l A$ it satisfies $A \leq^l B$.

In a natural way the notion of lower weak relation and lower weak minimal element are defined as follows:

Definition 15. Let A, B be two subsets of Y, $A \neq \emptyset$, $B \neq \emptyset$. $A \preceq^l B$ if $B \subset A + \mathrm{int}K_Y$. "$\preceq^l$" is called the lower weak relation.

Definition 16. Given a family $\mathcal{S}$ of subsets of Y, $A \in \mathcal{S}$ is a lower weak minimal (or l-$w$minimal) of $\mathcal{S}$, if for each $B \in \mathcal{S}$ such that $B \preceq^l A$ it satisfies $A \preceq^l B$.

The set of l-minimals of $\mathcal{S}$ is denoted by l- min $\mathcal{S}$ and the set of l-$w$minimals of $\mathcal{S}$ is denoted by l-w min $\mathcal{S}$.
The relation $\leq^l$ determines in $\mathcal{S}$ the equivalence relation:

$$A \sim B \Leftrightarrow A \leq^l B \quad \text{and} \quad B \leq^l A$$

whose classes we will represent by $[A]^l$. Analogously the relation $\preceq^l$ determines in $\mathcal{S}$ an equivalence relation whose classes we will represent by $[A]^{lw}$.
We will define the concepts of ε-lower minimal and ε-lower weak minimal elements as follows:

Definition 17. (a) Let $\varepsilon > 0$. $A_\varepsilon \in \mathcal{S}$ is an ε-lower minimal of $\mathcal{S}$ if there exists $\alpha_\varepsilon \in Y$ with $\|\alpha_\varepsilon\| < \varepsilon$ such that for each $B \in \mathcal{S}$ with $B \leq^l A_\varepsilon - \alpha_\varepsilon$ it satisfies $A_\varepsilon - \alpha_\varepsilon \leq^l B$.

(b) Let $\varepsilon > 0$. $A_\varepsilon \in \mathcal{S}$ is an ε-lower weak minimal of $\mathcal{S}$ if there exists $\alpha_\varepsilon \in Y$ with $\|\alpha_\varepsilon\| < \varepsilon$ such that for each $B \in \mathcal{S}$ with $B \preceq^l A_\varepsilon - \alpha_\varepsilon$ it satisfies $A_\varepsilon - \alpha_\varepsilon \preceq^l B$.

We consider a multifunction F with the hypotheses of Section 1. Let $\mathcal{F} = \{F(x) \mid x \in M\}$. If $\mathcal{S} = \mathcal{F}$ we say ε-lower (weak) minimal of F instead of $\mathcal{F}$.
If $F(x_\varepsilon)$ is an ε-lower (weak) minimal of F, then x_ε is called an ε-lower (weak) minimum of F.

Proposition 18. *Let $(x_\varepsilon, y_\varepsilon) \in graph(F)$. $F(x_\varepsilon)$ is an ε-lower weak minimal of F if and only if there exists $\alpha_\varepsilon \in Y$ with $\|\alpha_\varepsilon\| < \varepsilon$ such that for each $x \in M$ one of the conditions below is satisfied:*

1. $F(x) \in [F(x_\varepsilon) - \alpha_\varepsilon]^{lw}$.
2. *There exists $\mu \in F(x_\varepsilon)$ such that $(F(x) - \mu + \alpha_\varepsilon) \cap (-\mathrm{int}K_Y) = \emptyset$.*

Proof. Let us suppose that neither 1 nor 2 hold. Then there exists $x \in M$ such that $F(x) \notin [F(x_\varepsilon) - \alpha_\varepsilon]^{lw}$. Furthermore for all $\mu \in F(x_\varepsilon)$ there exists $y \in F(x)$ such that $u = y - \mu + \alpha_\varepsilon \in (-\text{int}K_Y)$. Thus $\mu - \alpha_\varepsilon = y - u \in F(x) + \text{int}K_Y$. Then $F(x_\varepsilon) - \alpha_\varepsilon \subset F(x) + \text{int}K_Y$ and $F(x) \preceq^l F(x_\varepsilon) - \alpha_\varepsilon$. But $F(x_\varepsilon) - \alpha_\varepsilon \npreceq^l F(x)$ because $F(x) \notin [F(x_\varepsilon) - \alpha_\varepsilon]^{lw}$. Consequently $F(x_\varepsilon)$ is not an ε-lower weak minimal of F.

Reciprocally, if $F(x_\varepsilon)$ is not an ε-lower weak minimal of F, for each $\alpha_\varepsilon \in Y$ with $\|\alpha_\varepsilon\| < \varepsilon$ there exists $x \in M$ such that $F(x) \preceq^l F(x_\varepsilon) - \alpha_\varepsilon$ and $F(x_\varepsilon) - \alpha_\varepsilon \npreceq^l F(x)$. Then $F(x) \notin [F(x_\varepsilon) - \alpha_\varepsilon]^{lw}$ and since $F(x) \preceq^l F(x_\varepsilon) - \alpha_\varepsilon$ it follows that $F(x_\varepsilon) - \alpha_\varepsilon \subset F(x) + \text{int}K_Y$. Hence there exists $\mu \in F(x_\varepsilon)$ such that $(F(x) - \mu + \alpha_\varepsilon) \cap (-\text{int}K_Y) \neq \emptyset$ in contradiction with the hypothesis. □

Definition 19. Let $F(x_\varepsilon)$ be an ε-lower weak minimal of F. $F(x_\varepsilon)$ is called strict if there exists a neighborhood U of x_ε such that $F(x) \npreceq^l F(x_\varepsilon) - \alpha_\varepsilon$ for all $x \in U \cap M$. Then x_ε is called a strict ε-lower weak minimum of F.

Remark 20. In virtue of Proposition 18, it is easy to see that, given an ε-lower weak minimal $F(x_\varepsilon)$ of F, it is strict if and only if there exist a neighborhood U of x_ε and $\alpha_\varepsilon \in Y$ with $\|\alpha_\varepsilon\| < \varepsilon$, such that for each $x \in U \cap M$ there exists $\mu \in F(x_\varepsilon)$ with $(F(x) - \mu + \alpha_\varepsilon) \cap (-\text{int}K_Y) = \emptyset$.

Next results establish conditions for the existence of ε-lower minimal and ε-lower weak minimal. First we recall some concepts defined in [6].

Definition 21 (*[6]*)**.** Let $\mathcal{S}$ be a family of subsets of Y and let $\Sigma = \cup_{A \in \mathcal{S}} A$. A cover $\left\{(x_i - K_Y)^c : x_i \in \Sigma, i \in I\right\}$ is called a $K_{\mathcal{S}}$-cover if for each $A \in \mathcal{S}$ there exists $i \in I$ such that $A \subset (x_i - K_Y)^c$.

$\mathcal{S}$ is called K-semicompact if all $K_{\mathcal{S}}$-covers have a finite $K_{\mathcal{S}}$-subcover.

The multifunction F is called $K_{\mathcal{F}}$-semicompact if $\mathcal{F}$ is K-semicompact.

Given $\varepsilon > 0$ and $\alpha_\varepsilon \in Y$ with $\|\alpha_\varepsilon\| < \varepsilon$ we consider the family

$$\mathcal{F} - \alpha_\varepsilon = \{F(x) - \alpha_\varepsilon \mid x \in M\}.$$

Proposition 22. *Let $\varepsilon > 0$ and $\alpha_\varepsilon \in Y$ with $\|\alpha_\varepsilon\| < \varepsilon$. If F is $K_{\mathcal{F}}$-semicompact then the family $\mathcal{F} - \alpha_\varepsilon$ is K-semicompact.*

Proof. Let $\{(y_i - K_Y)^c \mid y_i \in F(M) - \alpha_\varepsilon, i \in I\}$ be a $K_{\mathcal{F}-\alpha_\varepsilon}$-cover. Then $y_i = z_i - \alpha_\varepsilon$ with $z_i \in F(x_i)$, $x_i \in M$. Furthermore for each $F(x) - \alpha_\varepsilon \in F(M) - \alpha_\varepsilon$ there exists $i \in I$ such that $F(x) - \alpha_\varepsilon \subset (y_i - K_Y)^c = (z_i - \alpha_\varepsilon - K_Y)^c$. Hence $F(x) \subset (z_i - K_Y)^c$ and, as F is $K_{\mathcal{F}}$-semicompact, there exists a finite subcover $\{(z_j - K_Y)^c \mid j = 1, 2, \ldots, n\}$ of $\mathcal{F}$. As a consequence $\{(y_j - K_Y)^c \mid j = 1, 2, \ldots, n\}$ is a finite subcover of $\mathcal{F} - \alpha_\varepsilon$. □

The next propositions, proved in [6], will provide a condition for the existence of ε-lower minimal and ε-lower weak minimal.

Proposition 23 (*[6]*)**.** *Let F be $K_{\mathcal{F}}$-semicompact, then l-$\min F \neq \emptyset$.*

Proposition 24 (*[6]*)**.** *Let M be compact. If F is upper K-continuous, then F is $K_{\mathcal{F}}$-semicompact.*

Applying Propositions 22 and 23 we deduce the following corollary:

Corollary 25. *If F is $K_{\mathcal{F}}$-semicompact then there exists an ε-lower minimal of F.*

The next corollary is a consequence of Proposition 24 and Corollary 25.

Corollary 26. *Let M be compact. If F is upper K-continuous, then there exists an ε-lower minimal of F.*

Corollary 27. *Let M be compact. If F is upper K-continuous, then there exists an ε-lower weak minimal of F.*

Proof. We will prove that an ε-lower minimal of F is an ε-lower weak minimal of F and the result will be consequence of Corollary 26. Let $F(x_\varepsilon)$ be an ε-lower minimal of F. Then there exists $\alpha_\varepsilon \in Y$ with $\|\alpha_\varepsilon\| < \varepsilon$ such that for all $x \in M$ with $F(x) \preceq^l F(x_\varepsilon) - \alpha_\varepsilon$ it satisfies $F(x_\varepsilon) - \alpha_\varepsilon \preceq^l F(x)$. Let $x \in M$ such that $F(x) \preceq^l F(x_\varepsilon) - \alpha_\varepsilon$. Hence

$$F(x_\varepsilon) - \alpha_\varepsilon \subset F(x) + \text{int}K_Y. \tag{3.1}$$

Since $K_Y + \text{int}K_Y = \text{int}K_Y$ we have

$$F(x_\varepsilon) - \alpha_\varepsilon + K_Y \subset F(x) + \text{int}K_Y. \tag{3.2}$$

From (3.1) we deduce that

$$F(x_\varepsilon) - \alpha_\varepsilon \subset F(x) + K_Y. \tag{3.3}$$

In consequence $F(x) \leq^l F(x_\varepsilon) - \alpha_\varepsilon$. Since $F(x_\varepsilon)$ is an ε-lower minimal we deduce that $F(x_\varepsilon) - \alpha_\varepsilon \leq^l F(x)$. Then

$$F(x) \subset F(x_\varepsilon) - \alpha_\varepsilon + K_Y, \tag{3.4}$$

from that and (3.3) we obtain

$$F(x) + K_Y = F(x_\varepsilon) - \alpha_\varepsilon + K_Y,$$

add $\mathrm{int}K_Y$ and deduce

$$F(x) + \mathrm{int}K_Y = F(x_\varepsilon) - \alpha_\varepsilon + \mathrm{int}K_Y. \tag{3.5}$$

From (3.2) and (3.4) we have

$$F(x) \subset F(x_\varepsilon) - \alpha_\varepsilon + K_Y \subset F(x) + \mathrm{int}K_Y$$

and from (3.5)

$$F(x) \subset F(x_\varepsilon) - \alpha_\varepsilon + \mathrm{int}K_Y,$$

then we conclude that

$$F(x_\varepsilon) - \alpha_\varepsilon \preceq^l F(x). \quad \square$$

A necessary condition for the existence of a strict ε-lower weak minimal is enunciated with the notions of the weakly minimal element and K-$w$minimal property that we recall here.

Definition 28. Let $D \subset Y$. An element $\bar{y} \in D$ is called weakly minimal if

$$(\bar{y} - \mathrm{int}(K_Y)) \cap D = \emptyset.$$

The set of weakly minimal elements of D is denoted by K-$w \min D$.

Definition 29 (*Domination Property*)**.** A subset $A \subset Y$ has the K-$w$minimal property if for all $y \in A$ there exists a weakly minimal element $a \in A$ such that $a - y \in (-\mathrm{int}K_Y) \cup \{0\}$.

In the next theorem we enunciate a necessary condition for the existence of a strict ε-lower weak minimum. First we need a previous lemma.

Lemma 30. *Let $x_\varepsilon, x \in M$, $y_\varepsilon \in F(x_\varepsilon)$, $\alpha_\varepsilon \in Y$. If $F(x) \not\preceq^l F(x_\varepsilon) - \alpha_\varepsilon$, K-$w \min F(x_\varepsilon) = \{y_\varepsilon\}$ and $F(x_\varepsilon)$ has the K-$w$minimal property, then:*

$$(F(x) - y_\varepsilon) \cap (-\alpha_\varepsilon - \mathrm{int}K_Y) = \emptyset.$$

Proof. Let us suppose that it is false, i.e.:

$$(F(x) - y_\varepsilon) \cap (-\alpha_\varepsilon - \mathrm{int}K_Y) \neq \emptyset,$$

then there exist $z \in F(x)$, $h \in \mathrm{int}K_Y$ such that $z - y_\varepsilon = -\alpha_\varepsilon - h$. Thus $y_\varepsilon = z + h + \alpha_\varepsilon$ and $y_\varepsilon \in F(x) + \alpha_\varepsilon + \mathrm{int}K_Y$. Since K-$w \min F(x_\varepsilon) = \{y_\varepsilon\}$ and $F(x_\varepsilon)$ has the K-$w$minimal property then for all $w \in F(x_\varepsilon)$, there exists $k \in \mathrm{int}K_Y \cup \{0\}$ such that

$$w = y_\varepsilon + k \in F(x) + \alpha_\varepsilon + \mathrm{int}K_Y.$$

Therefore $F(x_\varepsilon) - \alpha_\varepsilon \subset F(x) + \mathrm{int}K_Y$ and as a consequence $F(x) \preceq^l F(x_\varepsilon) - \alpha_\varepsilon$ which contradicts the hypothesis. $\square$

Theorem 31. *Let x_ε be a strict ε-lower weak minimum of F. Let $\alpha_\varepsilon \in (-K_Y)$. If K-$w \min F(x_\varepsilon) = \{y_\varepsilon\}$ and $F(x_\varepsilon)$ has the K-$w$minimal property then*

$$Y_F^{y_\varepsilon}(x_\varepsilon, v) \cap (-\alpha_\varepsilon - \mathrm{int}K_Y) = \emptyset \quad \text{for all } v \in T(M, x_\varepsilon).$$

Proof. Since x_ε is a strict ε-lower weak minimum of F there exists a neighborhood U of x_ε such that $F(x) \not\preceq^l F(x_\varepsilon) - \alpha_\varepsilon$ for all $x \in U \cap M$. Let $v \in T(M, x_\varepsilon)$. If $z \in Y_F^{y_\varepsilon}(x_\varepsilon, v)$ then there exist $f \in CS(F)$, $(t_n) \to 0^+$, $(u_n) \to v$ such that

$$z = \lim_{\substack{t_n \to 0^+ \\ u_n \to v}} \frac{f(x_\varepsilon + t_n u_n) - f(x_\varepsilon)}{t_n}.$$

Therefore there exists $N \in \mathbb{N}$ such that for $n \geq N$ we can define $x_n = x_\varepsilon + t_n u_n \in U \cap M$ which satisfies $F(x_n) \not\preceq^l F(x_\varepsilon) - \alpha_\varepsilon$. By Lemma 30:

$$(F(x_n) - y_\varepsilon) \cap (-\alpha_\varepsilon - \mathrm{int}K_Y) = \emptyset \quad \text{for all } n \geq N.$$

In a way similar to that in Proposition 10 it is proved that $z \notin (-\alpha_\varepsilon - \mathrm{int}K_Y)$ and as a consequence

$$Y_F^{y_\varepsilon}(x_\varepsilon, v) \cap (-\alpha_\varepsilon - \mathrm{int}K_Y) = \emptyset \quad \text{for all } v \in T(M, x_\varepsilon). \quad \square$$

Observe that without the condition K-$w \min F(x_\varepsilon) = \{y_\varepsilon\}$ the conclusion is not guaranteed, as we show in the following example.

Example 32. Let $F\colon (0,1) \longrightarrow 2^{\mathbb{R}^2}$ be defined by

$$F(r) = \{(x_1, x_2) \mid x_1^2 + x_2^2 = r^2, x_2 \geq 0\} \cup \{(0, -r)\}.$$

Let $K_Y = \mathbb{R}^2_+ = \{(x_1, x_2) | x_1 \geq 0, x_2 \geq 0\}$. Given $\varepsilon > 0$, let $r_\varepsilon = 1 - \varepsilon/2$, $y_\varepsilon = (-(1-\varepsilon/2), 0)$ and $\alpha_\varepsilon = (-\varepsilon, 0)$. Then K-$w \min F(r_\varepsilon) = \{y_\varepsilon, (0, -r_\varepsilon)\} \neq \{y_\varepsilon\}$. Moreover if $\varepsilon \leq 2/3$ then r_ε is a strict ε-lower weak minimum of F, because $F(1-\varepsilon/2) - (-\varepsilon, 0) \not\subset F(r) + \mathrm{int}K_Y$ for all $r \in (0,1)$.

Let $f\colon (0,1) \longrightarrow \mathbb{R}^2$ be defined by $f(r) = (0, -r)$. It is clear that $f \in CS(F)$. Given $v \in T((0,1), r_\varepsilon)$ we have

$$\lim_{\substack{t_n \to 0^+ \\ u_n \to v}} \frac{f(r_\varepsilon + t_n u_n) - f(r_\varepsilon)}{t_n} = (0, -v) \quad \text{for all } (t_n) \to 0^+,\ (u_n) \to v.$$

It is easy to check that if $v > 0$ then $(0, -v) \in (-(-\varepsilon, 0) - \mathbb{R}^2_+)$. In consequence, given $0 < \varepsilon \leq 2/3$, $r_\varepsilon = 1 - \varepsilon/2$ is a strict ε-lower weak minimum of F, $F(r_\varepsilon)$ has the K-$w$minimal property and however

$$Y_F^{y_\varepsilon}(r_\varepsilon, v) \cap (-\alpha_\varepsilon - \mathrm{int}K_Y) \neq \emptyset.$$

Finally we obtain a sufficient condition for the existence of ε-lower weak minimum.

Theorem 33. *Let M be convex. Let F be pseudoconvex at $(x_\varepsilon, y_\varepsilon) \in graph(F)$. Let us suppose that for each pair of sequences $(t_n) \to 0^+$, $(h_n, k_n) \to (h, k)$, such that $(x_\varepsilon + t_n h_n, y_\varepsilon + t_n k_n) \in epi(F)$, there exists a continuous selection f of F such that $y_\varepsilon = f(x_\varepsilon)$ and $(x_\varepsilon + t_n h_n, y_\varepsilon + t_n k_n) \in epi(f)$. Let $\alpha_\varepsilon \in Y$ with $\|\alpha_\varepsilon\| < \varepsilon$. If every $v \in T(M, x_\varepsilon)$ satisfies $Y_F^{y_\varepsilon}(x_\varepsilon, v) \cap (-\alpha_\varepsilon - \mathrm{int}K_Y) = \emptyset$, then x_ε is an ε-lower weak minimum of F.*

Proof. Suppose that it is false, i.e. x_ε is not an ε-lower weak minimum of F. From Proposition 18, for all $\alpha_\varepsilon \in Y$ with $\|\alpha_\varepsilon\| < \varepsilon$ there exists $x \in M$ such that for each $\mu \in F(x_\varepsilon)$ there exists $\beta_\mu \in F(x)$ with

$$(\beta_\mu - \mu + \alpha_\varepsilon) \in (-\mathrm{int}K_Y). \tag{3.6}$$

From Lemma 11 we get

$$F(x) - y_\varepsilon \subset Y_F^{y_\varepsilon}(x_\varepsilon, x - x_\varepsilon) + K_Y,$$

and in particular

$$\beta_\mu - y_\varepsilon + \alpha_\varepsilon \in Y_F^{y_\varepsilon}(x_\varepsilon, x - x_\varepsilon) + K_Y + \alpha_\varepsilon.$$

From this fact and applying (3.6) with $\mu = y_\varepsilon$, we deduce that

$$(Y_F^{y_\varepsilon}(x_\varepsilon, x - x_\varepsilon) + K_Y + \alpha_\varepsilon) \cap (-\mathrm{int}K_Y) \neq \emptyset,$$

then

$$Y_F^{y_\varepsilon}(x_\varepsilon, x - x_\varepsilon) \cap (-\alpha_\varepsilon - \mathrm{int}K_Y) \neq \emptyset,$$

which is a contradiction. $\square$

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