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Determination of an unknown diffusion coefficient in a semilinear parabolic problem

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ABSTRACT

A semilinear parabolic problem of second order with an unknown diffusion coefficient in a subregion is considered. The missing data are compensated by a total flux condition through a given surface. The solvability of this problem is proved. A numerical algorithm based on Rothe's method is designed and the convergence of approximations towards the solution is shown. The results of numerical experiments are discussed.

1. Introduction

Recovery of a possible discontinuous diffusion coefficient from boundary measurements of solutions can be found in many applications, such as heat conduction and hydrology. The complete inverse problem is ill posed, so a numerical solution is quite difficult. Spontaneous potential (SP) well-logging is an important technique to detect parameters of the formation in petroleum exploitation. The SP log is a measurement of the natural potential difference or self potential between an electrode in the borehole and a reference electrode at the surface. No artificial currents are applied. This method has been mathematically studied, such as in [1–4]. The resistivity can depend on temperature and humidity in some geological formations. This makes the problem of the resistivity identification time-dependent. The aim of this paper is to study the recovery of a diffusion coefficient in a subregion from nonlocal boundary conditions for a transient problem. We assume that the unknown coefficient can change in time, but its shape in space is known. It should be noted that nonlocal boundary conditions have already been used for identification of some missing parameters at the boundary; cf. [5,6].

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz continuous boundary Γ. Ω is split into two non-overlapping parts Ω_0 and $\Omega \setminus \overline{\Omega}_0$. We consider a transient diffusion process in Ω. The diffusion coefficient K takes the form $K = k(t,x)\kappa(t,x)$ for a known κ and $k(t,x) = 1$ for $x \in \Omega \setminus \overline{\Omega}_0$ and $k(t,x) = k(t)$ for $x \in \Omega_0$. Γ is split into three non-overlapping parts, namely Γ_N (Neumann part), Γ_D (Dirichlet part) and Γ_0, where besides a Dirichlet boundary condition (BC) also the total flux through this part is prescribed, i.e.,

$$\begin{cases} \displaystyle\int_{\Gamma_0} -K\nabla u \cdot \boldsymbol{\nu} = h(t) & \text{in } (0,T); \\ \qquad\quad u = U(t) & \text{on } (0,T) \times \Gamma_0. \end{cases} \tag{1}$$

We assume that $\overline{\Gamma}_D \cap \overline{\Gamma}_0 = \emptyset$, $\text{meas}(\Gamma_0) > 0$.

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The goal of this work is to study the following parabolic initial boundary value problem (IBVP) (1)–(2): Find a couple (K, u) such that ($T > 0$ fixed)

$$\begin{aligned} \partial_t u - \nabla \cdot (K \nabla u) &= f(u) && \text{in } Q_T := (0,T) \times \Omega; \\ u &= g^D && \text{in } (0,T) \times \Gamma_D; \\ -K\nabla u \cdot \boldsymbol{\nu} &= g^N && \text{in } (0,T) \times \Gamma_N; \\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{2}$$

We use the variational framework. Without loss of generality we assume that $g^D = 0$ and $g^N = 0$. This will increase readability of the text. The suitable choice of a test space is

$$V = \{\varphi \in H^1(\Omega);\ \varphi|_{\Gamma_D} = 0,\ \varphi|_{\Gamma_0} = \text{const}\},$$

which is clearly a Hilbert space with the norm $\|u\|_V^2 = \|u\|^2 + \|\nabla u\|^2$, where $\|\cdot\|$ represents the norm in $L_2(\Omega)$.

To prove the existence of a weak solution to problem (1)–(2), we apply Rothe's method (cf. [7]). We use an equidistant time-partitioning with a step $\tau = T/n$, for any $n \in \mathbb{N}$, and introduce the notation $t_i = i\tau$ and for any function z

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}.$$

We suggest the following recursive approximation scheme for $i = 1, \dots, n$; $K_i = k_i \kappa_i$, with the unknown $(k_i, u_i) \in \mathbb{R}_+ \times V$ on each time-step

$$\begin{aligned} \delta u_i - \nabla \cdot (K_i \nabla u_i) &= f(u_{i-1}) && \text{in } \Omega; \\ u_i &= 0 && \text{on } \Gamma_D; \\ -K_i \nabla u_i \cdot \boldsymbol{\nu} &= 0 && \text{on } \Gamma_N; \\ \int_{\Gamma_0} -K_i \nabla u_i \cdot \boldsymbol{\nu} &= h_i && \\ u_i &= U_i && \text{on } \Gamma_0. \end{aligned} \tag{3}$$

First, we have to show the existence of (K_i, u_i) for any $i = 1, \dots, n$. Then we derive the stability estimates and finally we pass to the limit for $n \to \infty$ to get the existence of a solution to (1)–(2).

The values $C, \varepsilon, C_\varepsilon$ are generic and positive constants independent of the discretization parameter τ. The value ε is small and $C_\varepsilon = C\left(\varepsilon^{-1}\right)$.

2. A single time-step

We present two different ways for solving (3). In the first one we assume that k_i is given and we look for a solution of

$$\begin{aligned} \delta u_i - \nabla \cdot (K_i \nabla u_i) &= f(u_{i-1}) && \text{in } \Omega; \\ u_i &= 0 && \text{on } \Gamma_D; \\ -K_i \nabla u_i \cdot \boldsymbol{\nu} &= 0 && \text{on } \Gamma_N; \\ \int_{\Gamma_0} -K_i \nabla u_i \cdot \boldsymbol{\nu} &= h_i. && \end{aligned}$$

We prove that the trace of u_i on Γ_0 continuously depends on k_i. We seek for such k_i for which $u_i|_{\Gamma_0} = U_i$.

In the second method we solve

$$\begin{aligned} \delta u_i - \nabla \cdot (K_i \nabla u_i) &= f(u_{i-1}) && \text{in } \Omega; \\ u_i &= 0 && \text{on } \Gamma_D; \\ -K_i \nabla u_i \cdot \boldsymbol{\nu} &= 0 && \text{on } \Gamma_N; \\ u_i &= U_i && \text{on } \Gamma_0 \end{aligned} \tag{4}$$

for a given k_i. We prove that the total flux $\int_{\Gamma_0} -K_i \nabla u_i \cdot \boldsymbol{\nu}$ through Γ_0 continuously depends on k_i. We seek for such k_i that gives $\int_{\Gamma_0} -K_i \nabla u_i \cdot \boldsymbol{\nu} = h_i$.

We adopt the following assumptions on the data

$$0 < C_0 \le k \le C_1; \tag{5}$$

$$0 < D_0 \le \kappa \le D_1; \tag{6}$$

$$U, h, \kappa \in C([0,T]); \tag{7}$$

$$|f(x) - f(y)| \le C|x - y|, \quad \forall x, y; \tag{8}$$

$$u_0 \in L_2(\Omega). \tag{9}$$

2.1. Auxiliary problem (10)

Consider the following problem

$$\frac{1}{\tau}(u,\varphi) + (K\nabla u, \nabla\varphi) + h\varphi|_{\Gamma_0} = (f,\varphi) \quad \varphi \in V. \tag{10}$$

For any given $k > 0$ (recall that $K = k\kappa$) this admits a unique weak solution $u_k \in H^1(\Omega)$, which follows from the theory of linear elliptic equations (cf. [8]).

Uniform bound for u_k. We set $\varphi = u_k$ into (10). Applying the Nečas inequality (see [9])

$$\|z\|^2_{\Gamma} \le \varepsilon \|\nabla z\|^2 + C_\varepsilon \|z\|^2, \quad \forall z \in H^1(\Omega),\ 0 < \varepsilon < \varepsilon_0 \tag{11}$$

and using the uniform bounds (5) and (6) one can easily get

$$\left(\frac{1}{\tau} - C_\varepsilon\right)\|u_k\|^2 + (C_0 D_0 - \varepsilon)\|\nabla u_k\|^2 \le C\left(h^2 + \|f\|^2\right).$$

Fixing a sufficiently small positive ε we see that for $\tau < \tau_0$

$$\|u_k\|^2 + \|\nabla u_k\|^2 \le C\left(h^2 + \|f\|^2\right) \quad \text{for } C_0 \le k \le C_1.$$

u_k ***depends continuously on*** k. Subtract (10) for $k = \beta$ from (10) for $k = \alpha$ and set $\varphi = u_\alpha - u_\beta$ to get

$$\frac{1}{\tau}\left\|u_\alpha - u_\beta\right\|^2 + \left(\alpha\kappa\nabla(u_\alpha - u_\beta), \nabla(u_\alpha - u_\beta)\right) = \left((\beta-\alpha)\kappa\nabla u_\beta, \nabla(u_\alpha - u_\beta)\right).$$

An obvious calculation implies that

$$\left\|u_\alpha - u_\beta\right\|^2 + \left\|\nabla(u_\alpha - u_\beta)\right\|^2 \le C(\alpha-\beta)^2.$$

Using the trace theorem we deduce that for $\mathcal{T}(k) := u_k|_{\Gamma_0}$ we have

$$|\mathcal{T}(\alpha) - \mathcal{T}(\beta)| \le C\left\|u_\alpha - u_\beta\right\|_{L_2(\Gamma)} \le C\sqrt{\left\|\nabla(u_\alpha - u_\beta)\right\|^2 + \left\|u_\alpha - u_\beta\right\|^2} \le C|\alpha - \beta|.$$

2.2. Auxiliary problem (12)

Consider

$$\frac{1}{\tau}(u,\varphi) + (K\nabla u, \nabla\varphi) = (f,\varphi) \quad \varphi \in \{\psi \in H^1(\Omega);\ \psi|_{\Gamma_0\cup\Gamma_D} = 0\}. \tag{12}$$

For any given $k > 0$ this admits a unique weak solution $u_k \in H^1(\Omega)$—cf. [8].

Uniform bound for u_k. We set $\varphi = u_k$ into (12). One can readily get

$$\|u_k\|^2 + \|\nabla u_k\|^2 \le C\|f\|^2 \quad \text{for } C_0 \le k \le C_1.$$

u_k ***depends continuously on*** k. Subtract (12) for $k = \beta$ from (12) for $k = \alpha$ and set $\varphi = u_\alpha - u_\beta$ to get

$$\frac{1}{\tau}\left\|u_\alpha - u_\beta\right\|^2 + \left(\alpha\kappa\nabla(u_\alpha - u_\beta), \nabla(u_\alpha - u_\beta)\right) = \left((\beta-\alpha)\kappa\nabla u_\beta, \nabla(u_\alpha - u_\beta)\right),$$

which implies

$$\left\|u_\alpha - u_\beta\right\|^2 + \left\|\nabla(u_\alpha - u_\beta)\right\|^2 \le C(\alpha-\beta)^2.$$

Take any smooth function Φ such that $\Phi|_{\Gamma_D} = 0$ and $\Phi|_{\Gamma_0} = 1$. We recall that $\overline{\Gamma}_D \cap \overline{\Gamma}_0 = \emptyset$. Therefore, the existence of such a function is guaranteed by [10, Lemma 5.1]. Then

$$\Psi(k) := (-k\kappa\nabla u_k \cdot \boldsymbol{\nu}, 1)_{\Gamma_0} = -\frac{1}{\tau}(u_k, \Phi) - (k\kappa\nabla u_k, \nabla\Phi) + (f, \Phi)$$

obeys

$$|\Psi(\alpha) - \Psi(\beta)| = \left|\frac{1}{\tau}\left(u_\alpha - u_\beta, \Phi\right) + \left(\alpha\kappa\nabla(u_\alpha - u_\beta), \nabla\Phi\right) + \left((\alpha-\beta)\kappa\nabla u_\beta, \nabla\Phi\right)\right| \le C|\alpha - \beta|.$$

2.3. Solvability of (3)

A simple consequence of Sections 2.1 and 2.2 reads as follows:

Lemma 1. *Assume (5)–(9). If $U(t) \in \mathcal{T}([C_0, C_1])\ \forall t \in [0, T]$ or $h(t) \in \Psi([C_0, C_1])\ \forall t \in [0, T]$, then there exist a $\tau_0 > 0$ and a couple $(k_i, u_i) \in \mathbb{R}_+ \times V$ which solves (3) for $\tau < \tau_0$.*

3. Convergence

The variational formulation of (3) reads as

$$\begin{aligned} (\delta u_i, \varphi) + (K_i \nabla u_i, \nabla \varphi) + h_i \varphi|_{\Gamma_0} &= (f(u_{i-1}), \varphi) \quad \varphi \in V \\ u_i|_{\Gamma_0} &= U_i. \end{aligned} \tag{13}$$

According to Lemma 1 we see that (13) has a solution on each t_i. The next step is the stability analysis.

Lemma 2. *Let the assumptions of Lemma 1 be fulfilled. Then*

$$\max_{1 \le i \le n} \|u_i\|^2 + \sum_{i=1}^{n} \|u_i - u_{i-1}\|^2 + \sum_{i=1}^{n} \|\nabla u_i\|^2 \tau \le C.$$

Proof. Set $\varphi = u_i \tau$ into (13) and sum it up for $i = 1, \dots, j$ keeping $1 \le j \le n$. We obtain

$$\frac{1}{2}\left(\|u_j\|^2 - \|u_0\|^2 + \sum_{i=1}^{j} \|u_i - u_{i-1}\|^2 \right) + \sum_{i=1}^{j} (K_i \nabla u_i, \nabla u_i)\, \tau = \sum_{i=1}^{j} (f(u_{i-1}), u_i)\, \tau - \sum_{i=1}^{j} h_i U_i \tau.$$

Using the Cauchy and Young inequalities we readily get

$$\|u_j\|^2 + \sum_{i=1}^{j} \|u_i - u_{i-1}\|^2 + \sum_{i=1}^{j} \|\nabla u_i\|^2 \tau \le C \left(1 + \sum_{i=1}^{j} \|u_i\|^2 \tau + \|u_0\|^2 + \sum_{i=1}^{j} h_i^2 \tau + \sum_{i=1}^{j} U_i^2 \tau \right).$$

An application of Gronwall's lemma implies that

$$\|u_j\|^2 + \sum_{i=1}^{j} \|u_i - u_{i-1}\|^2 + \sum_{i=1}^{j} \|\nabla u_i\|^2 \tau \le C,$$

which is valid for all $1 \le j \le n$. From this we conclude the proof. □

Let us denote by V^* the dual space to V. Then:

Lemma 3. *Let the assumptions of Lemma 2 be fulfilled. Then*

$$\sum_{i=1}^{n} \|\delta u_i\|_{V^*}^2 \tau \le C.$$

Proof. The relation (13) gives

$$(\delta u_i, \varphi) = (f(u_{i-1}), \varphi) - (K_i \nabla u_i, \nabla \varphi) - h_i \varphi|_{\Gamma_0}.$$

A standard argumentation yields

$$|(\delta u_i, \varphi)| \le C\,(1 + |h_i| + \|\nabla u_i\|)\, \|\varphi\|_V,$$

which implies

$$\|\delta u_i\|_{V^*} = \sup_{\substack{\varphi \in V \\ \|\varphi\|_V \le 1}} |(\delta u_i, \varphi)| \le C\,(1 + |h_i| + \|\nabla u_i\|).$$

Taking into account Lemma 2 we conclude the proof. □

The variational formulation of (1)–(2) reads as: find (K, u) such that

$$(\partial_t u, \varphi) + (K \nabla u, \nabla \varphi) + h \varphi|_{\Gamma_0} = (f(u), \varphi) \quad \varphi \in V \tag{14a}$$

$$u|_{\Gamma_0} = U. \tag{14b}$$

Now, let us introduce the following piecewise linear in time function

$$\begin{aligned} &u_n(0) = u_0 \\ &u_n(t) = u_{i-1} + (t - t_{i-1})\delta u_i \quad \text{for } t \in (t_{i-1}, t_i] \end{aligned}$$

and a step function $\overline{u}_n$

$$\overline{u}_n(0) = u_0, \qquad \overline{u}_n(t) = u_i, \quad \text{for } t \in (t_{i-1}, t_i].$$

Similarly we define $\overline{K}_n, \overline{h}_n, \overline{U}_n$. The variational formulation (13) can be rewritten as

$$(\partial_t u_n, \varphi) + \left(\overline{K}_n \nabla \overline{u}_n, \nabla\varphi\right) + \overline{h}_n\varphi|_{\Gamma_0} = (f(\overline{u}_n(t-\tau)), \varphi) \quad \varphi \in V \tag{15a}$$

$$\overline{u}_n|_{\Gamma_0} = \overline{U}_n. \tag{15b}$$

We want to pass to the limit for $\tau \to 0$ in (15) and to arrive at (14).

Theorem 4. *Let the assumptions of Lemma 1 be fulfilled. Then there exists a weak solution to (14).*

Proof. Take any $\xi \in (0, T)$ and integrate (15) on $(0, \xi)$ to get

$$\int_0^\xi (\partial_t u_n, \varphi) + \int_0^\xi \left(\overline{K}_n \nabla \overline{u}_n, \nabla\varphi\right) + \int_0^\xi \overline{h}_n\varphi|_{\Gamma_0} = \int_0^\xi (f(\overline{u}_n(t-\tau)), \varphi) \quad \varphi \in V. \tag{16}$$

Using the results of Lemma 2 and applying [11, Theorem 2.13.1], we get the existence of a subsequence of $\overline{u}_n$ (denoted by the same symbol again) such that

$$\lim_{n\to\infty} \overline{u}_n \to u \quad \text{in } L_2(Q_T).$$

Therefore we also get

$$\overline{u}_n \to u \quad \text{a.e. in } Q_T. \tag{17}$$

Using Lemma 3 we may write for $\xi \in (t_{i-1}, t_i]$ and $\varphi \in V$ that

$$|(\overline{u}_n(\xi) - u_n(\xi), \varphi)| = \left|\int_\xi^{t_i} (\partial_t u_n, \varphi)\right| \le \sqrt{\int_0^T \|\partial_t u_n\|^2_{V^*}}\, \|\varphi\|_V\, \tau^{\frac{1}{2}}.$$

Hence we have for $\tau \to 0$ and $\varphi \in V$ that

$$\begin{array}{ccccc} \int_0^\xi (\partial_t u_n, \varphi) & = (\overline{u}_n(\xi) - u_0, \varphi) & + & (u_n(\xi) - \overline{u}_n(\xi), \varphi) \\ \downarrow & \downarrow & & \downarrow \\ \int_0^\xi (z, \varphi) & = (u(\xi) - u_0, \varphi) & + & 0. \end{array}$$

This is valid for any $\xi \in [0, T]$, thus $z = \partial_t u$ in $L_2((0, T), V^*)$. It holds

$$\left|\int_0^\xi (f(\overline{u}_n(t-\tau)) - f(\overline{u}_n(t)), \varphi)\right| \le C \int_0^\xi \|\partial_t u_n\| \, \|\varphi\| \, \tau = \mathcal{O}\left(\tau^{\frac{1}{2}}\right) \|\varphi\|.$$

Applying (17) we get

$$\lim_{\tau\to 0} \int_0^\xi (f(\overline{u}_n(t)), \varphi) = \int_0^\xi (f(u(t)), \varphi).$$

Lemma 2 and the reflexivity of $L_2((0, T), V)$ give (for a subsequence)

$$\overline{u}_n \rightharpoonup u \quad \text{in } L_2((0, T), V).$$

This, together with (11), implies

$$\int_0^T \|\overline{u}_n - u\|^2_\Gamma \le \varepsilon \int_0^T \|\nabla(\overline{u}_n - u)\|^2 + C_\varepsilon \int_0^T \|\overline{u}_n - u\|^2 \le \varepsilon + C_\varepsilon \int_0^T \|\overline{u}_n - u\|^2.$$

Passing to the limit for $\tau \to 0$ and applying (17) we obtain

$$\lim_{\tau\to 0} \int_0^T \|\overline{u}_n - u\|^2_\Gamma \le \varepsilon,$$

which is valid for any small $\varepsilon > 0$. Hence

$$\lim_{\tau\to 0} \int_0^T \|\overline{u}_n - u\|^2_\Gamma = 0 \quad \text{and} \quad \overline{u}_n \to u \quad \text{a.e. in } (0, T) \times \Gamma.$$

Repeating this consideration for Ω_0 instead of Ω we deduce that

$$\lim_{\tau\to 0}\int_0^T \|\overline{u}_n - u\|^2_{\partial\Omega_0} = 0 \quad \text{and} \quad \overline{u}_n \to u \quad \text{a.e. in } (0,T)\times\partial\Omega_0. \tag{18}$$

Due to the construction we have that $C_0 \le \overline{k}_n \le C_1$. This yields that $\overline{k}_n \rightharpoonup k$ (for a subsequence) in $L_2((0,T))$. Now, applying the Green theorem and taking a sufficiently smooth φ we deduce that

$$\begin{aligned}\int_0^\xi \left(\overline{K}_n\nabla\overline{u}_n, \nabla\varphi\right) &= \int_0^\xi \overline{k}_n\,(\overline{\kappa}_n\nabla\overline{u}_n, \nabla\varphi)_{\Omega_0} + \int_0^\xi (\overline{\kappa}_n\nabla\overline{u}_n, \nabla\varphi)_{\Omega\setminus\Omega_0}\\ &= \int_0^\xi \overline{k}_n\,(\overline{u}_n, \overline{\kappa}_n\nabla\varphi\cdot\boldsymbol{\nu})_{\partial\Omega_0} - \int_0^\xi \overline{k}_n\,(\overline{u}_n, \nabla\cdot(\overline{\kappa}_n\nabla\varphi))_{\Omega_0} + \int_0^\xi (\overline{\kappa}_n\nabla\overline{u}_n, \nabla\varphi)_{\Omega\setminus\Omega_0}.\end{aligned}$$

Passing to the limit for $\tau\to 0$ we get

$$\begin{aligned}\lim_{\tau\to 0}\int_0^\xi \left(\overline{K}_n\nabla\overline{u}_n, \nabla\varphi\right) &= \int_0^\xi k\,(u, \kappa\nabla\varphi\cdot\boldsymbol{\nu})_{\partial\Omega_0} - \int_0^\xi k\,(u, \nabla\cdot(\kappa\nabla\varphi))_{\Omega_0} + \int_0^\xi (\kappa\nabla u, \nabla\varphi)_{\Omega\setminus\Omega_0}\\ &= \int_0^\xi k\,(\kappa\nabla u, \nabla\varphi)_{\Omega_0} + \int_0^\xi (\kappa\nabla u, \nabla\varphi)_{\Omega\setminus\Omega_0}\\ &= \int_0^\xi (K\nabla u, \nabla\varphi).\end{aligned}$$

Applying the density argument we conclude that

$$\lim_{\tau\to 0}\int_0^\xi \left(\overline{K}_n\nabla\overline{u}_n, \nabla\varphi\right) = \int_0^\xi (K\nabla u, \nabla\varphi) \quad \forall\varphi\in V.$$

Collecting all considerations above and passing to the limit for $\tau\to 0$ in (16) we arrive at

$$\int_0^\xi (\partial_t u, \varphi) + \int_0^\xi (K\nabla u, \nabla\varphi) + \int_0^\xi h\varphi|_{\Gamma_0} = \int_0^\xi (f(u), \varphi) \quad \varphi\in V.$$

Differentiation with respect to ξ gives (14a). Taking the limit in (15b) and using (18) we get (14b), which concludes the proof. $\square$

4. Numerical experiments

The domain we consider is $\Omega = \left(-\frac{1}{2}, 1\right)\times(-1,1)$, with $\Omega_0 = \left(-\frac{1}{2}, 0\right)\times(-1,1)$ in $\mathbb{R}^2$. Let the time interval be $[0,1]$, i.e., $T=1$. The boundary Γ is split into three non-overlapping parts, namely Γ_D (right), Γ_N (top and bottom) and Γ_0 (left part of Γ).

We use the second solution method described in Section 2 and define the exact diffusion coefficient as follows

$$K(t,x,y) = \tilde{k}(t)\mathbb{1}_{\{x<0\}} + \frac{1}{2}.$$

This is equivalent to setting

$$k(t,x,y) = \begin{cases}\tilde{k}(t) + 0.5 & \text{if } (t,x,y)\in\Omega_0;\\ 1 & \text{if } (t,x,y)\in\Omega\setminus\overline{\Omega_0};\end{cases} \qquad \kappa(t,x,y) = \begin{cases}1 & \text{if } (t,x,y)\in\Omega_0;\\ 0.5 & \text{if } (t,x,y)\in\Omega\setminus\overline{\Omega_0}\end{cases}$$

in the previous notation $K = k\kappa$.

First, we prescribe the exact solution (K,u) as follows

$$K(t,x,y) = (1+\sin(10t))\,\mathbb{1}_{\{x<0\}} + \frac{1}{2}; \qquad u(t,x,y) = (1+t)\sin\left(\frac{\pi}{2}(1-x)\right). \tag{19}$$

Remark that we choose a trigonometric discontinuous diffusion coefficient with $\tilde{k}(t) = (1+\sin(10t))$. Some simple calculations with the use of the exact solution give the exact data for the numerical experiment

$$g^D = g^N = 0; \qquad U(t) = \frac{1+t}{\sqrt{2}}; \qquad u_0(x) = \sin\left(\frac{\pi}{2}(1-x)\right). \tag{20}$$

We want to approximate the exact solution (19) given the exact data (20). Therefore, we focus on the determination of $\tilde{k}(t)$. For the recovery of $\tilde{k}(t)$ we need the value of $h(t)$ at each time $t\in[0,1]$, which is given by

$$h(t) = \frac{\pi}{\sqrt{2}}\,(1+t)\,(1.5+\sin(10t)).$$

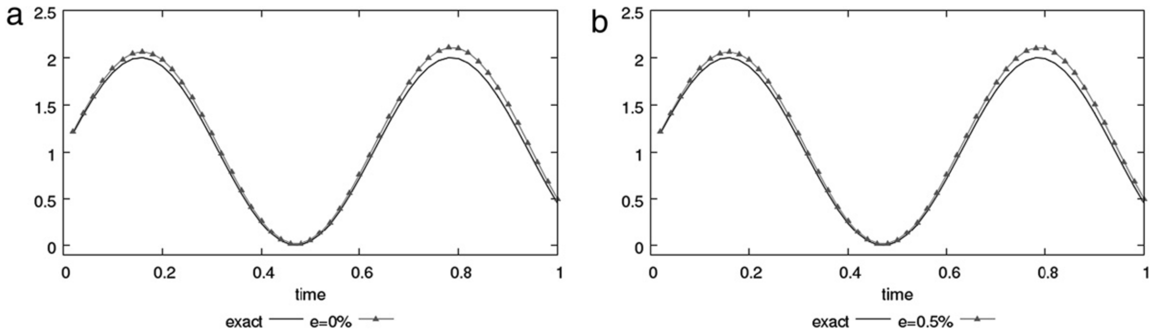

Fig. 1. Numerical value of $\tilde{k}_i$ using the P1-FEM with noise $e = 0\%$ (a) and $e = 0.5\%$ (b); $i = 1, \ldots, 50$.

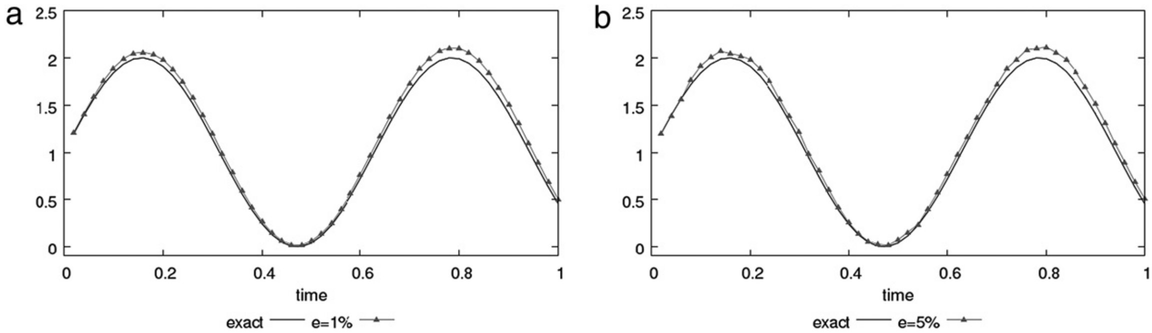

Fig. 2. Numerical value of $\tilde{k}_i$ using the P1-FEM with noise $e = 1\%$ (a) and $e = 5\%$ (b); $i = 1, \ldots, 50$.

We add an uncorrelated noise to this additional condition in order to simulate the errors present in real measurements. The noise is generated randomly with given magnitude $e = 0\%$, 0.5%, 1% and 5%.

For the time discretization we choose an equidistant time partitioning with time-step $\tau = 0.02$. Applying the backward Euler difference scheme into (4), we are left with a recurrent system of linear elliptic BVPs for $(K_i, u_i) \approx (K(t_i), u(t_i))$, $i = 1, 2, \ldots, 50$ and $\varphi \in \{\psi \in H^1(\Omega);\ \psi|_{\Gamma_0 \cup \Gamma_D} = 0\}$

$$\frac{1}{\tau}(u_i, \varphi) + (K_i \nabla u_i, \nabla \varphi) = (f_i, \varphi) + \frac{1}{\tau}(u_{i-1}, \varphi); \qquad u_0 = u_0; \tag{21}$$

with

$$(f_i, \varphi) = \left(\sin\left(\frac{\pi}{2}(1-x)\right), \varphi\right) + \left((1.5 + \sin(10t_i))\left(\frac{\pi}{2}\right)^2 (1+t_i)\sin\left(\frac{\pi}{2}(1-x)\right), \varphi\right)_{\Omega_0} + \left(0.5\left(\frac{\pi}{2}\right)^2 (1+t_i)\sin\left(\frac{\pi}{2}(1-x)\right), \varphi\right)_{\Omega \setminus \Omega_0}.$$

The unknown $\tilde{k}_i \approx \tilde{k}(t_i)$, $i = 1, \ldots, 50$, is determined by the nonlinear conjugate gradient method. On each time-step t_i, $i = 1, \ldots, 50$, we minimalize the functional

$$J(\tilde{k}_i) := \left(\int_{\Gamma_0} (\tilde{k}_i + 0.5)\nabla u_i \cdot \boldsymbol{\nu} - h(t_i)\right)^2.$$

Since this functional J is not convex we can only obtain convergence to a local minimum. Therefore, the initial guess has to be sufficiently close to the actual minimizer of the functional. The starting point for this algorithm on the first time-step is set as $\tilde{k}_1^{(0)} = 1$. The starting points on the following time-steps are different in the various examples. We remark that the algorithm stops after maximum 10 iterations with the prescribed error tolerance 10^{-6}.

For the space discretization we use a fixed uniform mesh consisting of 144 528 triangles. At each time-step, the resulting elliptic BVP (21) is solved numerically by the finite element method (FEM) using first order (P1-FEM) and second order (P2-FEM) Lagrange polynomials.

The results from the recovery of $\tilde{k}(t)$ using the P1-FEM and P2-FEM for the different values of the amplitude e are shown in Figs. 1, 2, 4 and 5. The exact $\tilde{k}(t_i)$ is denoted by a solid line and the approximations $\tilde{k}_i$ by linespoints; $i = 1, \ldots, 50$. The evolution of the $\tilde{k}_i$-error for the different time-steps is shown in Figs. 3(a) and 6(a). The $L_2(\Omega)$-error of the approximate solution u_i on $[0, 1]$ is depicted in Figs. 3(b) and 6(b).

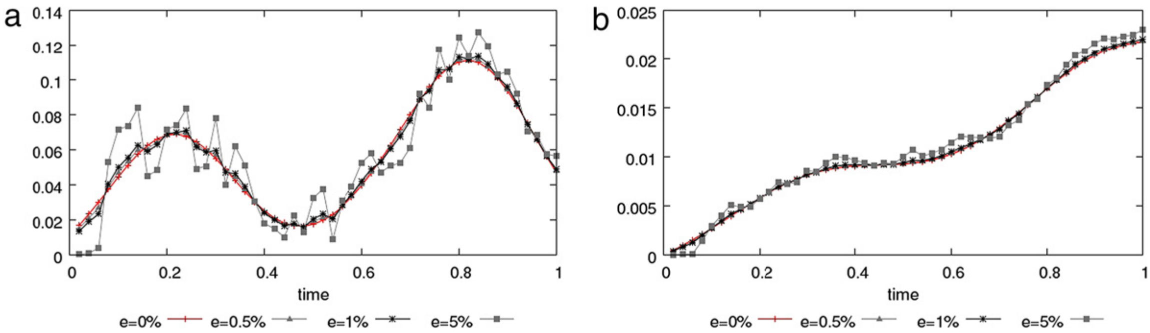

Fig. 3. The absolute $\tilde{k}_i$-error (a) and the absolute $L_2(\Omega)$-error of the approximate solution u_i (b) using the P1-FEM; $i = 1, \ldots, 50$.

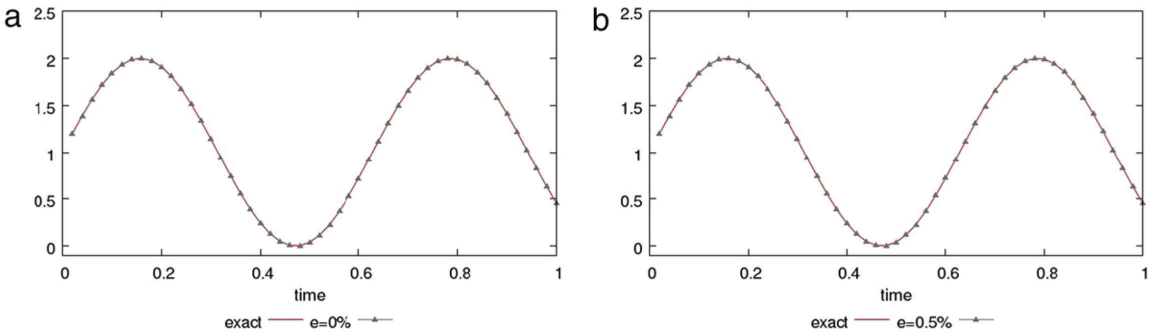

Fig. 4. Numerical value of $\tilde{k}_i$ using the P2-FEM with noise $e = 0\%$ (a) and $e = 0.5\%$ (b); $i = 1, \ldots, 50$.

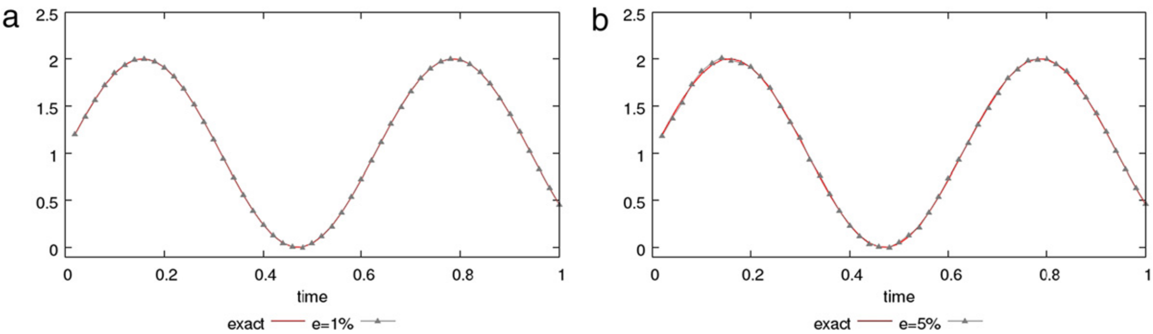

Fig. 5. Numerical value of $\tilde{k}_i$ using the P2-FEM with noise $e = 1\%$ (a) and $e = 5\%$ (b); $i = 1, \ldots, 50$.

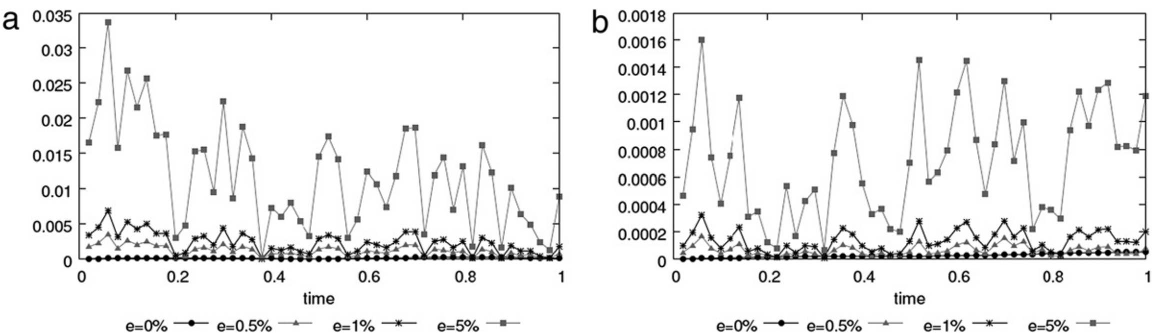

Fig. 6. The absolute $\tilde{k}_i$-error (a) and the absolute $L_2(\Omega)$-error of the approximate solution u_i (b) using the P2-FEM; $i = 1, \ldots, 50$.

The experiments show that the approximation becomes less accurate with increasing magnitude e when the number of time discretization intervals and the number of triangles in the space discretization is fixed. This result is valid for the P1-FEM as well as for the P2-FEM. We conclude, as expected, that the approximations are more accurate if we use the P2-FEM.

5. Conclusion

A semilinear parabolic problem of second order with an unknown diffusion coefficient in a subregion is considered. The existence of a weak solution for the IBVP is proved when an additional total flux condition through a given surface is prescribed. A numerical algorithm is established and its convergence is demonstrated by a numerical experiment.

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