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The existence and uniqueness of positive monotone solutions for a class of nonlinear Schr$\ddot{o}$dinger equations on infinite domains

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Abstract: In this paper, by constructing a new weighted norm method and analysis technique, we establish the conditions for the existence and uniqueness of positive monotone solutions to a class of nonlinear Schr$\ddot{o}$dinger equations in planar exterior domains. Some examples are also given to demonstrate the application of our main results.

MSC: 34B18, 35A05, 35J65.

Keywords: Nonlinear Schrödinger equation, positive solutions, fixed point, weighted norm

1 Introduction

We consider the existence and uniqueness of positive monotone solutions to a class of nonlinear Schr$\ddot{o}$dinger equations in planar exterior domains

$$\Delta y + g(|x|)b(y) = 0, \tag{1.1}$$

where $|x| \in E_D,\ g(|x|) \in C^\lambda_{loc}(E_D, R),\ \lambda \in (0,1),\ b(y) \in C^\lambda_{loc}(R, R)$ (locally Hölder continuous), and we denote $E_D = \{x \in R^2 : |x| > D\},\ S_D = \{x \in R^2 : |x| = D\}$, for $D > 0$.

Nonlinear differential equations arise in a variety of different areas of applied mathematics and physics (see, e.g. [10]-[20]). Recently, boundary value problems for differential equations on the infinite domain have received much attention; and

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for details we refer the reader to Dub*é* and Mingarelli [2], Wahl*é*n [19], Yao and L*ü* [21], Yao [22], [23] and the references therein. Many results relevant to the study of equation (1.1) have been achieved in recent years.

By making use of the classical variational principle, Buffoni et al [1] established the condition for the existence and conditional energentic stability of the three-dimensional fully localised solitary gravity-capillary water waves governed by

$$\begin{cases} \phi_{xx} + \phi_{yy} + \phi_{zz} = 0, & 0 < y < 1 + \eta, \\ \phi_y = 0, & y = 0, \\ \eta_t = \phi_y - \eta_x \phi_x - \eta_z \phi_z, & y = 1 + \eta, \\ \phi_t = -\dfrac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) - \eta & \\ \qquad\qquad + \beta \left[\dfrac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_z^2}}\right]_x + \beta \left[\dfrac{\eta_z}{\sqrt{1 + \eta_x^2 + \eta_z^2}}\right]_z, & y = 1 + \eta \end{cases}$$

where the function η relies on the two horizontal spatial directions z and x; $\beta = \sigma/gh^2$ in which h is the depth of the water in its undisturbed state and g is the acceleration due to gravity; and $\sigma > 0$ is the coefficient of surface tension.

Groves and Wahlén [7] established the existence and conditional energetic stability of solitary water waves with weak surface tension. Groves and Wahlén[8] presented an existence theory for small-amplitude Stokes and solitary gravity water waves solutions to the classical water-wave problem with an arbitrary distribution of vorticity. Wahlén [4] studied the governing equations for a Hamiltonian formulation of water waves with constant vorticity, and the results obtained generalized the well-known formulation established by Zakharov. Groves, Sun and Wahlén [9] showed the existence of three-dimensional periodically modulated solitary waves which are periodic in the transverse direction and have a solitary-wave profile in the direction of propagation. Lair [11] gave a necessary and sufficient condition for the existence of large solutions to a class of sublinear elliptic systems. It is well known that many elliptic equations arise from physical phenomena.

Takahashi [18] studied the following elliptic systems

$$\begin{cases} -\Delta u + u = 0, \quad \text{on} \quad \Omega, \\ u > 0, \quad \text{on} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = u^p \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^2$, $p > 1$ is any positive number, ν is the outer unit normal vector to $\partial\Omega$. The author considered the asymptotic behavior of least energy solutions to the problem (1.2) and showed that the least energy solutions remain bounded uniformly in p.

Li and Zhang et al [16] showed the existence of entire positive radial solutions to the following elliptic system

$$\begin{cases} \Delta u = p(|x|)f(v), & x \in R^N \ (N \geq 3), \\ \Delta v = q(|x|)g(u), & x \in R^N, \end{cases} \tag{1.3}$$

where $p,\ q,\ f,\ g : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous, in which $\mathbb{R} := [0, +\infty)$.

Feng [5] investigated the existence, uniqueness and exact asymptotic behavior of solutions of semilinear elliptic problems with boundary blow-up of the form

$$\begin{cases} -\Delta u = \lambda g(u) - b(x)f(u), & \text{in} \ \ \Omega, \\ u = +\infty \ \ \text{on} \ \ \partial\Omega, \end{cases} \tag{1.4}$$

where $\lambda \in \mathbb{N}, \Omega$ is a smooth bounded domain of $\mathbb{R}$, and $b(x) \in C^{\alpha}(\overline{\Omega}, \mathbb{R}^+)$ for some $\alpha \in (0, 1)$.

Inspired and motivated by the works mentioned above, in this work we consider the existence of positive solutions to the nonlinear Schrödinger equations (1.1) in planar exterior domains. We shall first prove that problem (1.1) has a monotone positive solution, and then construct a weighted norm. Finally, by employing the Banach fixed point theorem, some sufficient conditions guaranteeing the existence of a unique positive monotone solution are established for the nonlinear Schrödinger equations (1.1) in planar exterior domains. We should address here that our new results extend and complement some known results.

The rest of the article is organized as follows. In section 2, the main results and proofs are presented. Then some examples are given in section 3 to demonstrate the application of our main results, followed by some discussion in section 4.

2 Main results

For all $L > 0$, denote $B_L = \{y \in C([m_1, +\infty)) : 0 \leq y(t) \leq L, \ \ t \geq m_1 > 0\}$.

Throughout the paper we make the following assumptions:

(A_1) $g \in C([m_1,\ +\infty), [0,\ +\infty))$, $b \in C([0,\ +\infty), [0,\ +\infty))$, and $b(y)$ satisfies the locally Lipschitz condition;
(A_2) $0 < \int_{m_1}^{+\infty} t \ln\left(\frac{t}{m_1}\right) g(t)dt < +\infty$;
(A_3) there exists a constant c such that $0 \leq \limsup\limits_{y \to 0+} \frac{b(y)}{y} < \frac{1}{c}$.

(A_4)

$$\int_{m_1}^{r} \frac{1}{t} \int_{m_1}^{t} sg(s)b(y(s))dsdt < +\infty.$$

Denote

$$M_1 = \int_{m_1}^{r} \frac{1}{t} \int_{m_1}^{t} sg(s)b(y(s))dsdt. \tag{2.1}$$

Lemma 2.1. Equation (1.1) can be changed into the following equation

$$ry''(r) + y'(r) + rg(r)b(y(r)) = 0 \tag{2.2}$$

on $[m_1, +\infty)$.

Proof. Let $y = y(r)$, where $r = \sqrt{s^2 + t^2}$. Then

$$\frac{\partial y}{\partial s} = \frac{dy}{dr} \cdot \frac{\partial r}{\partial s} = \frac{s}{r} \cdot \frac{dy}{dr} = \frac{s}{r} \cdot y'(r), \qquad \frac{\partial y}{\partial t} = \frac{t}{r} \cdot y'(r),$$
$$\frac{\partial^2 y}{\partial s^2} = (\frac{\partial^2 y}{\partial r^2} \cdot \frac{\partial r}{\partial s})\frac{s}{r} + \frac{\partial y}{\partial r}(\frac{1}{r} - \frac{s}{r^2} \cdot \frac{\partial r}{\partial s}) = \frac{d^2 y}{dr^2} \cdot \frac{s^2}{r^2} + \frac{dy}{dr}(\frac{1}{r} - \frac{s^2}{r^3}),$$

By the same way, we have

$$\frac{\partial^2 y}{\partial t^2} = \frac{d^2 y}{dr^2} \cdot \frac{t^2}{r^2} + \frac{dy}{dr}(\frac{1}{r} - \frac{t^2}{r^3}).$$

Therefore

$$\Delta y = \frac{\partial^2 y}{\partial s^2} + \frac{\partial^2 y}{\partial t^2} = \frac{d^2 y}{dr^2} \cdot \frac{s^2 + t^2}{r^2} + \frac{dy}{dr}(\frac{2}{r} - \frac{s^2 + t^2}{r^3}) = \frac{d^2 y}{dr^2} + \frac{dy}{dr} \cdot \frac{1}{r}.$$

Then

$$\Delta y + g(|r|)b(y(r)) = 0 \Longleftrightarrow ry''(r) + y'(r) + rg(r)b(y(r)) = 0, \qquad r \in R^2.$$

Thus finding the radial symmetric solution of equation (1.1) is equivalent to solving equation (2.2). □

Lemma 2.2. Suppose that $(A_1) - (A_4)$ hold. Let $y(r)$ be a positive radial symmetric solution of equation (2.2), then $y(r)$ is a bounded solution on $[m_1, +\infty)$ and equation (1.1) has a positive bounded radial symmetric solution.

Proof. Let $z = y'$, and denote $y(m_1) = a_1$. From (2.2), we have

$$z'(r) + \frac{1}{r}z(r) + g(r)b\left(a_1 + \int_{m_1}^{r} z(s)ds\right) = 0. \tag{2.3}$$

Using the formula of variation of constants, we get a special solution of (2.3) as follows:

$$z(r) = \frac{1}{r} \int_{m_1}^{r} g(t)b\left(a_1 + \int_{m_1}^{t} z(s)ds\right) dt. \tag{2.4}$$

Integrating (2.4) from m_1 to r and noticing $z = y'$, we get

$$y(r) = a_1 - \int_{m_1}^{r} \frac{1}{t} \int_{m_1}^{t} sg(s)b(y(r))dsdt \tag{2.5}$$

By making use of (2.1), we see that $y(r)$ in (2.5) is bounded on $[m_1, +\infty)$. Therefore equation (2.2) has a bounded solution on $[m_1, +\infty)$. Thus from Lemma 2.1, equation (1.1) has a bounded solution on E_D. □

Lemma 2.3. The equation (2.2) can be changed into the following equation

$$h''(s) + e^{2s}g(e^s)b(h(s)) = 0. \tag{2.6}$$

Proof. Let $r = e^s$, then $y(r) = y(e^s) = h(s)$,

$$h'(s) = \frac{dy}{ds} = \frac{dy}{dr} \cdot \frac{dr}{ds} = y'(r) \cdot e^s, \quad h''(s) = \frac{d^2y}{ds^2} = y''(r) \cdot e^s \cdot e^s + y'(r)e^s.$$

Therefore

$$\begin{aligned} h''(s) + e^{2s}g(e^s)b(h(s)) = 0 &\Longleftrightarrow \frac{d^2y}{ds^2} + e^{2s}g(e^s)b(y(e^s)) = 0 \\ &\Longleftrightarrow y''(r)e^{2s} + y'(r)e^s + e^{2s}g(e^s)b(y(e^s)) = 0 \\ &\Longleftrightarrow y''(r) + \frac{1}{r}y'(r) + g(r)b(y(r)) = 0. \end{aligned}$$

This completes the proof. □

Remark 2.1. Denote $a = \int_{m_1}^{+\infty} t \ln\left(\frac{t}{m_1}\right) g(t)dt$. Then it follows from (A_2) that $a > 0$ is a constant.

Theorem 2.1. *Suppose that $(A_1) - (A_3)$ hold. Then equation (1.1) has a radial positive solution.*

Proof. The proof of Theorem 2.1. is divided into four steps:

(I) Firstly we show that there exists $L > 0$ sufficiently small such that for all $y \in B_L$, we have

$$\int_{m_1}^{+\infty} tln\left(\frac{t}{m_1}\right) g(t)b(y(t))dt \le L. \tag{2.7}$$

In fact, from (A_3), there exists $L > 0$ (L may be chosen sufficiently small) such that for $0 \le y \le \frac{c}{a} \cdot L$, we have $\frac{b(y)}{y} < \frac{1}{c}$, that is $b(y) < \frac{1}{c}y$. Thus, for $y \in B_L$, by virtue of (A_2), we get that

$$\begin{aligned} \int_{m_1}^{+\infty} t\ln\left(\frac{t}{m_1}\right) g(t)b(y(t))dt &\le \int_{m_1}^{+\infty} t\ln\left(\frac{t}{m_1}\right) g(t)\frac{1}{c}y(t)dt \\ &\le \frac{c}{a} \cdot \frac{L}{c} \int_{m_1}^{+\infty} t\ln\left(\frac{t}{m_1}\right) g(t)dt = L. \end{aligned}$$

Hence, $\int_{m_1}^{+\infty} t\ln\left(\frac{t}{m_1}\right) g(t)b(y(t))dt \leq L$.

(II) Choose $L > 0$ sufficiently small such that (2.7) holds. Define integral operator A on B_L:

$$(Ax)(t) = L - \int_t^{+\infty}(s-t)g(s)b(x(s))ds, \quad x \in B_L, \quad t \geq m_1. \tag{2.8}$$

Obviously, $(Ax)(t) \geq 0, \quad t \geq m_1$. By simple computation we know that the following conclusions hold:

(1) If $x \in B_L$, then $(Ax)''(t) = -g(t)b(x), \quad t \geq m_1, \quad Ax \in B_L \cap C^2(0, +\infty)$.

(2) If $x \in B_L$ is a solution of equation (2.6), then $Ax = x$. Thus, we only need to study the operator equation $y(t) = Ay(t), \quad t \geq m_1$.

Hence, $\forall x \in B_L, \quad t \geq m_1$, from (I) we get that

$$0 \leq \int_t^{+\infty}(s-t)g(s)b(y(s))ds \leq \int_0^{+\infty} sg(s)b(y(s))ds \leq L.$$

Thus, $0 \leq Ax \leq L$. So $Ax \in B_L$. Consequently $AB_L \subseteq B_L$.

It follows from (2.3) that the integral $\int_t^{+\infty}(s-t)g(s)b(y(s))ds$ is non-increasing. Thus, if there exists a solution $y(t)$ of the equation (2.6), then $y(t)$ must be non-decreasing, and $y(t) \to L$ for $t \to +\infty$. By virtue of (2.7) we get that $y(t)$ is nonnegative and $y(t) > 0$ for $t > 0$.

(III) From (A_1), there exists a positive constant $k > 0$ such that $|b(x) - b(y)| \leq k|x - y|$ for $x, y \in [0, L]$. Define the function

$$f(t) := exp\left\{\alpha \int_t^{\infty} ksg(s)ds\right\}, \quad t \geq m_1, \ \alpha > 2.$$

We then introduce the norm $\|\cdot\|_g$ on $BC(R^+)$, the space of bounded continuous functions $y : R^+ \longrightarrow R^+$, defined by

$$\|y\|_g = \sup_{t \geq 0}\left\{\frac{|y(t)|}{f(t)}\right\}, \quad y \in BC(R^+).$$

Obviously the weighted norm $\|\cdot\|_g$ is equivalent to the used supremum norm on $BC(R^+)$. Since B_L is a closed subset of $BC(R^+)$ with the supremum norm $\|\cdot\|_{B_L}$, B_L is also a closed subset of $BC(R^+)$ with the supremum norm $\|\cdot\|_g$. Consequently, it follows that a closed subset of a complete metric space is a complete metric space. Hence we know that B_L is a complete normed space under the corresponding norm generated by $\|\cdot\|_g$.

(IV) We now prove that equation (2.6) has a bounded monotone positive solution. We only need to prove that A is a contraction mapping on $(B_L, \|\cdot\|_g)$. For all $x,\, y \in B$, we have

$$\begin{aligned}
\frac{\mid (Ax)(t) - (Ay)(t) \mid}{f(t)} &\leq \frac{1}{f(t)} \int_t^{+\infty} (s-t) g(s) \mid b(x(s)) - b(y(s)) \mid ds \\
&\leq \frac{1}{f(t)} \int_t^{+\infty} ksg(s) \mid x(s) - y(s) \mid ds \\
&\leq \frac{1}{f(t)} \|x-y\|_g \int_t^{+\infty} ksg(s) f(s) ds \\
&= \frac{1}{f(t)} \|x-y\|_g \int_t^{+\infty} \left(-\frac{1}{\alpha}\right) f'(s) ds \\
&= \frac{1}{\alpha} \frac{f(t)-1}{f(t)} \|x-y\|_g \leq \frac{1}{\alpha} \|x-y\|_g .
\end{aligned}$$

Thus, $\|Ax - Ay\|_g \leq \frac{1}{\alpha}\|x-y\|_g, \quad 0 < \frac{1}{\alpha} < \frac{1}{2}$. Hence, A is a contraction mapping. Therefore, by virtue of the Banach fixed point theorem we get that equation (2.6) has a unique bounded monotone positive solution. From Lemma 2.1. we then get that equation (1.1) has a radial positive solution. □

Theorem 2.2. *Suppose that (A_1) and (A_2) are satisfied. In addition, assume that $b(y)$ satisfies*

(A_5) $\;0 \leq \limsup\limits_{y\to+\infty} \frac{b(y)}{y} < \frac{1}{c}$, *where c is a constant.*

Then equation (1.1) has a radial positive solution.

Proof. We only need to prove that there exists $L > 0$ (L is chosen sufficiently large) such that

$$\int_{m_1}^{+\infty} t \ln\left(\frac{t}{m_1}\right) g(t) b(y(t)) dt \leq L, \qquad \text{for} \quad y \in B_L. \tag{2.9}$$

In fact, from (A_5), we get that there exists $K > 0,\ 0 < \varepsilon < \frac{1}{c}$ such that $0 \leq \frac{b(y)}{y} \leq \varepsilon < \frac{1}{c}$ for $y > K$. Since $b(y)$ is continuous on $[0,\, K]$, there exists a sufficiently large number $L > K$ such that $b(y) \leq (\frac{1}{c} - \varepsilon)\frac{c}{a} \cdot L$ for $y \in [0, K]$. Thus, for $y \in B_L$, by virtue of (A_2) we get that

$$\begin{aligned}
\int_{m_1}^{+\infty} t \ln\left(\frac{t}{m_1}\right) g(t) b(y(t)) dt &= \int_{0\leq y(t)\leq K} t \ln\left(\frac{t}{m_1}\right) g(t) b(y(t)) dt \\
&\quad + \int_{K<y(t)\leq L} t \ln\left(\frac{t}{m_1}\right) g(t) b(y(t)) dt \\
&\leq \left(\frac{1}{c} - \varepsilon\right) \frac{c}{a} \cdot L \int_{m_1}^{+\infty} t \ln\left(\frac{t}{m_1}\right) g(t) dt \\
&\quad + \varepsilon L \int_{m_1}^{+\infty} t \ln\left(\frac{t}{m_1}\right) g(t) dt \\
&\leq L.
\end{aligned}$$

Therefore, $\int_{m_1}^{+\infty} t\ln\left(\frac{t}{m_1}\right) g(t)b(y(t))dt \leq L, \quad \forall y \in B_L$.

The rest of the proof of Theorem 2.2 is the same as the proof of Theorem 2.1, and thus is omitted here. □

Remark 2.2. By virtue of constructing the weighted norm method, under the integral

$$\int_{m_1}^{+\infty} t\ln\left(\frac{t}{m_1}\right) g(t)b(y(t))dt = a < +\infty,$$

where a may be 1, we have proved that operator A is a contraction mapping and established that equation (1.1) has a radial positive solution.

Denote

$$G(r) = \int_0^r \frac{1}{t}\int_0^t sg(s)dsdt.$$

Theorem 2.3. Let

(A_6) $b \in C((0, +\infty); (0, +\infty))$, b is non-increasing on $(0, +\infty)$, and there exist positive constants m, M, α and $\alpha \geq 1$ such that

$$m := \inf_{y>0} \frac{b(y)}{y^\alpha} > 0, \qquad M := \sup_{y>0} \frac{b(y)}{y^\alpha} < +\infty.$$

Then equation (1.1) has a positive entire radial solution y if and only if the function $g(t)$ satisfies

$$\int_0^{+\infty} \frac{1}{t}\int_0^t sg(s)dsdt = +\infty. \tag{2.10}$$

Proof. For $\overline{a} \geq 1, r > 0$ sufficiently large, the problem

$$\begin{cases} y'' + \frac{1}{r}y' + g(r)b(y) = 0, \ r > 0, \\ y' \geq 0, \ \text{on } [0, \infty), \ y(0) = \overline{a} > 0 \end{cases} \tag{2.11}$$

has a solution y in $[0, +\infty)$.

From (2.11), we have

$$y(r) = \overline{a} - \int_0^r \frac{1}{t}\int_0^t sg(s)b(y(s))dsdt \tag{2.12}$$

By using (2.12), we can generate a positive non-increasing sequence $\{y_k\}_{k=0}^{+\infty}$, which is bounded above on $[0, R]$ for any fixed $R > 0$.

To do this, let $\{y_k\}_{k=0}^{+\infty}$ be a sequence of positive continuous functions defined on $[0, +\infty)$ by

$$y_0(r) = \overline{a}, \ y_k(r) = \overline{a} - \int_0^r \frac{1}{t}\int_0^t sg(s)b(y_{k-1}(s))dsdt, \quad \forall r \geq 0. \tag{2.13}$$

Obviously, for all $r \geq 0$, we have

$$y_k(r) \leq \overline{a}, \quad y_0(r) \geq y_1(r), \cdots$$

The monotonicity of b yields

$$\begin{aligned} y_1(r) &= \overline{a} - \int_0^r \frac{1}{t} \int_0^t sg(s)b(y_0(s))dsdt \\ &\geq \overline{a} - \int_0^r \frac{1}{t} \int_0^t sg(s)b(y_1(s))dsdt = y_2(r). \end{aligned}$$

Repeating such arguments we deduce that

$$y_k(r) \geq y_{k+1}(r), \quad r \geq 0, \ k \geq 1.$$

Thus $\{y_k\}_{k=0}^{\infty}$ is a non-increasing sequence on $[0, +\infty)$.

On the other hand, from equation (2.13) and condition (A_6), we have

$$\begin{aligned} y_k(r) &= \overline{a} - \int_0^r \frac{1}{t} \int_0^t sg(s)b(y_{k-1}(s))dsdt \\ &= \overline{a} - \int_0^r \frac{1}{t} \int_0^t sg(s)\frac{b(y_{k-1}(s))}{y_{k-1}^{\alpha}(s)} y_{k-1}^{\alpha}(s)dsdt \\ &\leq \overline{a} - m\int_0^r \frac{1}{t} \int_0^t sg(s)y_{k-1}^{\alpha}(s)dsdt \\ &\leq \overline{a} - m\int_0^r \frac{1}{t} \int_0^t sg(s)y_{k-1}^{\alpha}(r)dsdt \\ &\leq \overline{a} - mG(r)y_k^{\alpha}(r) \leq \overline{a} - mG(r)y_k(r) \end{aligned}$$

which implies that the sequence $\{y_k\}$ is also bounded on bounded intervals and hence converges for all $0 \leq r < +\infty$. Let $y = \lim_{k \to +\infty} y_k$. It is clear that y is an entire solution of equation (2.2), and therefore an entire solution of equation (1.1).

We now assume that (2.14) holds. Then by making use of condition (A_6), we get immediately from an easy estimate

$$\begin{aligned} y(r) &= \overline{a} - \int_0^r \frac{1}{t} \int_0^t sg(s)b(y(s))dsdt \\ &= \overline{a} - M\int_0^r \frac{1}{t} \int_0^t sg(s)y^{\alpha}(s)dsdt \\ &\geq \overline{a} - \overline{a}^{\alpha}M\int_0^r \frac{1}{t} \int_0^t sg(s)dsdt \end{aligned}$$

which implies that $y(r) \longrightarrow +\infty$ as $r \longrightarrow +\infty$. Therefore y is a positive entire radial solution of equation (1.1). $\square$

Theorem 2.4. Let

(A_7) $b \in C((0, +\infty); (0, +\infty))$, b is non-increasing on $(0, +\infty)$, and there exist positive constants m, M, α and $\alpha > 1$ such that

$$m := \inf_{y>0} \frac{b(y)}{y^\alpha} > 0, \qquad M := \sup_{y>0} \frac{b(y)}{y^\alpha} < +\infty.$$

Then equation (1.1) has a positive entire radial solution y if and only if the function satisfies

$$\int_0^{+\infty} \frac{1}{t} \int_0^t sg(s)dsdt = +\infty. \tag{2.14}$$

3 Examples

Example 3.1. Consider equation

$$\Delta y + \frac{1}{\ln t} e^{-2t} \sin y = 0, \quad t \geq 2. \tag{3.1}$$

Proof. Let $g(t) = \frac{1}{\ln t} e^{-2t}$, $t \geq 2$, $b(y) = \sin 2y$, $y \in [0, +\infty)$. Obviously condition (A_1) is satisfied. Since

$$\int_2^{+\infty} t \ln \frac{t}{2} \cdot \frac{1}{\ln t} e^{-2t} dt < +\infty,$$

condition (A_2) is satisfied. Since $\lim\limits_{y \to +\infty} \frac{\sin y}{y} = 0$, condition (A_4) is satisfied. Consequently, it follows from Theorem 2.2 that equation (3.1) has a positive radial solution. □

Example 3.2. The following equation

$$\Delta y + \frac{y^5}{t^3 \ln t} = 0, \quad t \geq 1, \tag{3.2}$$

has a positive radial solution.

Proof. Obviously, equation (3.2) satisfies all the conditions of Theorem 2.1. Therefore, by Theorem 2.1, equation (3.2) has a positive radial solution. □

4 Discussions on the conditions of Theorems

We discuss the conditions in this paper. It is easy to see that the functions satisfying the conditions of the theorems are rather wide. For example, we can obtain the following corollary:

Corollary 4.1. Suppose that all $b_i(y)\,(i = 0, 1, \cdots, m)$ are nonnegative continuous functions and satisfy the locally Lipschitz condition on $[0, +\infty)$. In addition, if

$0 \le \limsup_{y\to+0} \frac{\sum_{i=1}^{m} b_i(y)}{y} < \frac{1}{c}$, where c is a constant.

Then the following equation

$$\Delta y + e^{-3t} \sum_{i=1}^{m} b_i(y) = 0, \quad t > 0 \tag{4.1}$$

has a radial positive solution.

Proof. Let $g(t) = e^{-3t}$, $b(y) = \sum_{i=1}^{m} b_i(y)$, $y \in [0, +\infty)$. Obviously, condition (A_1) is satisfied. Since

$$\int_m^{+\infty} t \ln \frac{t}{m} \cdot e^{-3t} dt < +\infty,$$

condition (A_2) is satisfied. Consequently, it follows from Theorem 2.1 that equation (4.1) has a positive radial solution. This completes the proof. □

Corollary 4.2. For any constant $\alpha > 1$, the following equation

$$\Delta y + e^{-6t} y^{\alpha} = 0, \quad t > 0. \tag{4.2}$$

has a radial positive solution.

Proof. Obviously, equation (4.2) satisfies all the conditions of Theorem 2.1. Therefore, from Theorem 2.1, equation (4.2) has a positive radial solution. This completes the proof. □

Corollary 4.3. Suppose that all $b_i(y)\,(i = 0, 1, \cdots, m)$ are nonnegative continuous functions and satisfy the locally Lipschitz condition on $[0, +\infty)$. If

$0 \le \limsup_{y\to+\infty} \frac{\sum_{i=1}^{m} b_i(y)}{y} < \frac{1}{c}$, where c is a constant,

then the following equation

$$\Delta y + e^{-7t} \sum_{i=1}^{m} b_i(y) = 0, \quad t > 0 \tag{4.3}$$

has a radial positive solution.

Proof. Let $g(t) = e^{-7t}$, $b(y) = \sum_{i=1}^{m} b_i(y)$, $y \in [0, +\infty)$. Obviously condition (A_1) is satisfied. Since

$$\int_m^{\infty} t \ln \frac{t}{m} \cdot e^{-7t} dt < +\infty,$$

condition (A_2) is satisfied. Consequently, it follows from Theorem 2.2 that equation (4.3) has a positive radial solution. This completes the proof. □

Corollary 4.4. For any constant β satisfying $0 < \beta < 1$, the following equation

$$\Delta y + e^{-5t} y^{\beta} = 0, \quad t > 0 \tag{4.4}$$

has a radial positive solution.

Proof. Let $g(t) = e^{-5t}$, $b(y) = y^{\beta}$, $y \in [0, +\infty)$. Obviously condition (A_1) is satisfied. Since

$$\int_m^{\infty} t \ln \frac{t}{m} \cdot e^{-5t} dt < +\infty,$$

condition (A_2) is satisfied. Consequently, from Theorem 2.2, equation (4.4) has a positive radial solution. This completes the proof. $\square$

Remark 4.1. The key condition in [2] requires the integral is strictly less than 1 for the existence of a positive bounded solution in E_1. Here in our work, we only require the integral to be a constant a, and the constant a may be 1. From above discussions, it is clear that our results improve and extend the results in [2] and [19].

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