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Inertial extragradient algorithms for strongly pseudomonotone variational inequalities

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ABSTRACT

The purpose of this paper is to study and analyze two different kinds of inertial type iterative methods for solving variational inequality problems involving strongly pseudomonotone and Lipschitz continuous operators in Hilbert spaces. The projection method is used to design the algorithms which can be computed more easily. The construction of solution approximations and the proof of convergence of the algorithms are performed without prior knowledge of the modulus of strong pseudomonotonicity and the Lipschitz constant of cost operator. Instead of that, the algorithms use variable stepsize sequences which are diminishing and non-summable. The numerical behaviors of the proposed algorithms on a test problem are illustrated and compared with several previously known algorithms.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $A : H \to H$ be an operator. The variational inequality problem (VIP) [1–5] for A on C is to find $p \in C$ such that

$$\langle Ap, x - p \rangle \geq 0 \ \forall x \in C. \tag{VIP}$$

It is well known that the VIP is a central problem of nonlinear analysis. It is a useful mathematical model which unifies many important concepts in applied mathematics, such as necessary optimality conditions, network equilibrium problems, complementarity problems and systems of nonlinear equations, for instance [2,5–9]. Two notable and general directions for solving VIPs can be the regularized method and projection method. It is also emphasized that the first direction is often applied for the class of monotone operators. Regularized subproblem in this method is strongly monotone and its unique solution can be found more easily than solutions of the original problem. Regularized solutions can converge finitely or asymptotically to some solution of the original solution. For more general monotone VIPs, for example, pseudomonotone VIPs, which have been widely studied in recent years [10–12], the strong monotonicity of regularized subproblems can be destroyed. Thus, regularized methods cannot be applied in these cases.

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In this paper, we focus on the second direction: the projection methods which are well known for easier numerical computations. The early known projection method for VIPs is the gradient projection, after which, many other projection methods were developed including the extragradient method [13], the subgradient extragradient method [14–17], the projected reflected gradient methods [18–20] and others [21–27]. The aforementioned methods have been studied for VIPs which are monotone, strongly monotone or inverse strongly monotone. Moreover, a common point of these methods is that, in constructing solution approximations and establishing their convergence, the fixed or variable stepsizes often depend on factorials of cost operators, for instance, the strong monotone or Lipschitz constants of cost operators. This can make restrictions in applications because, in some cases, these constants can be unknown or difficult to approximate.

Recently, based on known projection methods, the authors in [28–33] introduced new methods for strongly pseudomonotone and Lipschitzian VIPs where the stepsizes are variable and independent from the strongly pseudomonotone and Lipschitz constants of cost operators. It is worth mentioning that the class of strongly pseudomonotone VIPs properly contains the class of strongly monotone VIPs.

Now, let us mention an inertial-type algorithm. Based on the heavy ball methods of the two-order time dynamical system, Polyak [34] firstly proposed an inertial extrapolation as an acceleration process to solve the smooth convex minimization problem. The inertial algorithm is a two-step iterative method, and the next iterate is defined by making use of the previous two iterates and it can be regarded as a procedure of speeding up the convergence properties, see [34–38]. Recently, a lot of researchers constructed fast iterative algorithms by using inertial extrapolation, including inertial forward–backward splitting methods [39–42], inertial Douglas–Rachford splitting method [43], inertial ADMM [44,45], inertial forward–backward–forward method [46], inertial proximal-extragradient method [47], inertial contraction method [48], inertial subgradient extragradient method [49], and inertial Mann method [50].

Motivated by the presented results, in this paper, we propose two algorithms for solving strongly pseudomonotone and Lipschitzian VIPs. First, we propose an inertial subgradient extragradient method (ISEGM). This method works and bases on the subgradient extragradient method [14] and the inertial method [35]. Second, we combine the inertial method and the Tseng's extragradient [51] to introduce an inertial Tseng's extragradient method (ITEGM). In these methods, a projection onto a feasible set and two values of cost operator need to be computed per each iteration. Finally, we consider a test problem and illustrate the numerical behaviors of the algorithms in [30,52,53] and compare them with the two algorithms presented in this paper.

This paper is organized as follows: In Section 2, we recall some definitions and preliminary results for further use. Section 3 deals with proposing the algorithms and analyzing their convergence. Finally, in Section 4 we perform several numerical experiments to illustrate the computational performance of the proposed algorithm over several previously known algorithms.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written as $x_n \to x$ as $n \to \infty$.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that $\|x - P_C x\| \le \|x - y\| \quad \forall y \in C$. P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive.

Lemma 2.1 (*[20,54]*)**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x \iff \langle x - z, z - y\rangle \ge 0 \quad \forall y \in C$.*

Lemma 2.2 (*[20,54]*)**.** *Let C be a closed and convex subset in a real Hilbert space H, $x \in H$. Then*
(i) $\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle \quad \forall y \in C$;
(ii) $\|P_C x - y\|^2 \le \|x - y\|^2 - \|x - P_C x\|^2 \quad \forall y \in C$.

For properties of the metric projection, the interested reader could be referred to Section 3 in [54]. We present some concepts of monotonicity of an operator.

Definition 2.1. An operator $A : H \to H$ is said to be:
(i) *strongly monotone* on C if there exists $\gamma > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \gamma\|x - y\|^2 \quad \forall x, y \in C;$$

(ii) *monotone* on C if

$$\langle Ax - Ay, x - y\rangle \ge 0 \quad \forall x, y \in C;$$

(iii) *strongly pseudomonotone* on C if there exists $\gamma > 0$ such that

$$\langle Ax, y - x\rangle \ge 0 \Longrightarrow \langle Ay, x - y\rangle \le -\gamma\|x - y\|^2 \quad \forall x, y \in C.$$

(iv) *pseudomonotone* on C if

$$\langle Ax, y - x\rangle \ge 0 \Longrightarrow \langle Ay, x - y\rangle \le 0 \quad \forall x, y \in C.$$

(v) *L - Lipschitz continuous* on C if there exists $L > 0$ such that

$$\|Ax - Ay\| \le L\|x - y\| \quad \forall x, y \in C.$$

From the aforementioned definitions, it is easy to see that (i)$\Longrightarrow$(ii)$\Longrightarrow$(iv) and (i)$\Longrightarrow$(iii)$\Longrightarrow$(iv). The converse implications are not true in general. We need the following technical lemmas.

Lemma 2.3 (*[55]*). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers. If $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n b_n < \infty$ then* $\liminf_{n\to\infty} b_n = 0$.

Lemma 2.4 (*[35]*). *Let $\{\varphi_n\}$, $\{\delta_n\}$ and $\{\alpha_n\}$ be sequences in $[0, +\infty)$ such that*

$$\varphi_{n+1} \le \varphi_n + \alpha_n(\varphi_n - \varphi_{n-1}) + \delta_n \; \forall n \ge 1, \sum_{n=1}^{+\infty} \delta_n < +\infty, \tag{1}$$

and there exists a real number α with $0 \le \alpha_n \le \alpha < 1$ for all $n \in \mathbb{N}$. Then the following hold:
(i) $\sum_{n=1}^{+\infty}[\varphi_n - \varphi_{n-1}]_+ < +\infty$, where $[t]_+ := \max\{t, 0\}$;
(ii) there exists $\varphi^* \in [0, +\infty)$ such that $\lim_{n\to+\infty} \varphi_n = \varphi^*$.

3. Main results

In this paper, we assume that $A : H \to H$ is a strongly pseudomonotone with the constant γ and L-Lipschitz mapping on H. However, the information of L is not necessary to be known and we also suppose that VIP has a solution, and so, by strong pseudomonotonicity of A, it has a unique solution, denoted by p.

First, we introduce an inertial subgradient extragradient algorithm for solving strongly variational inequality which combines the subgradient extragradient method and the inertial method.

Theorem 3.1. *Let $\{x_n\}$ be a sequence in H defined by*

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \tau_n A w_n), \\ T_n = \{x \in H | \langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \le 0\}, \\ x_{n+1} = P_{T_n}(w_n - \tau_n A y_n), \end{cases} \tag{2}$$

where the sequence $\{\alpha_n\}$ is non-decreasing and $0 \le \alpha_n \le \alpha \le \frac{1}{10}$. Let $\{\tau_n\}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \tau_n = 0 \text{ and } \sum_{n=1}^{\infty} \tau_n = \infty. \tag{3}$$

Then the sequence $\{x_n\}$ converges strongly to the unique solution p of VIP.

Proof. We show that

$$\|x_{n+1} - p\|^2 \le \|w_n - p\|^2 - (1 - \tau_n L)\|y_n - w_n\|^2 - (1 - \tau_n L)\|x_{n+1} - y_n\|^2 - 2\tau_n\gamma\|y_n - p\|^2. \tag{4}$$

Indeed, let $u_n = w_n - \tau_n A y_n$. We have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|P_{T^n}(u_n) - p\|^2 = \langle P_{T_n}(u_n) - u_n + u_n - p, P_{T_n}(u_n) - u_n + u_n - p\rangle \\ &= \|u_n - p\|^2 + \|P_{T_n}(u_n) - u_n\|^2 + 2\langle P_{T_n}(u_n) - u_n, u_n - p\rangle. \end{aligned} \tag{5}$$

Since $p \in VI(C, A) \subset C \subset T_n$ we have

$$\begin{aligned} 2\|P_{T_n}(u_n) - u_n\|^2 &+ 2\langle P_{T_n}(u_n) - u_n, u_n - p\rangle \\ &= 2\langle P_{T_n}(u_n) - u_n, P_{T_n}(u_n) - u_n\rangle + 2\langle P_{T_n}(u_n) - u_n, u_n - p\rangle \\ &= 2\langle P_{T_n}(u_n) - u_n, P_{T_n}(u_n) - p\rangle \\ &= 2\langle x_{n+1} - (w_n - \tau_n A y_n), x_{n+1} - p\rangle \le 0. \end{aligned}$$

This implies that

$$\|P_{T_n}(u_n) - u_n\|^2 + 2\langle P_{T_n}(u_n) - u_n, u_n - p\rangle \le -\|P_{T_n}(u_n) - u_n\|^2. \tag{6}$$

From (5) and (6) we obtain

$$\|x_{n+1} - p\|^2 \le \|u_n - p\|^2 - \|P_{T_n}(u_n) - u_n\|^2$$

$$
\begin{aligned}
&= \|w_n - \tau_n A y_n - p\|^2 - \|x_{n+1} - w_n + \tau_n A y_n\|^2 \\
&= \|w_n - p\|^2 - \|x_{n+1} - w_n\|^2 + 2\tau_n \langle A y_n, p - x_{n+1}\rangle. \qquad (7)
\end{aligned}
$$

Since p is the solution of VIP, we have $\langle Ap, x-p\rangle \geq 0$ for all $x \in C$. By the strong pseudomonotonicity of A on C we have $\langle Ax, x-p\rangle \geq \gamma\|x-p\|^2$ for all $x \in C$. Taking $x := y_n \in C$ we get

$$\langle Ay_n, p - y_n\rangle \leq -\gamma\|y_n - p\|^2.$$

Thus,

$$\langle Ay_n, p - x_{n+1}\rangle = \langle Ay_n, p - y_n\rangle + \langle Ay_n, y_n - x_{n+1}\rangle \leq -\gamma\|y_n - p\|^2 + \langle Ay_n, y_n - x_{n+1}\rangle. \qquad (8)$$

From (7) and (8) we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \leq& \|w_n - p\|^2 - \|x_{n+1} - w_n\|^2 + 2\tau_n\langle Ay_n, y_n - x_{n+1}\rangle - 2\tau_n\gamma\|y_n - p\|^2 \\
=& \|w_n - p\|^2 - \|x_{n+1} - y_n\|^2 - \|y_n - w_n\|^2 - 2\langle x_{n+1} - y_n, y_n - w_n\rangle \\
&+ 2\tau_n\langle Ay_n, y_n - x_{n+1}\rangle - 2\tau_n\gamma\|y_n - p\|^2 \\
=& \|w_n - p\|^2 - \|x_{n+1} - y_n\|^2 - \|y_n - w_n\|^2 + 2\langle w_n - \tau_n A y_n - y_n, x_{n+1} - y_n\rangle \\
&- 2\tau_n\gamma\|y_n - p\|^2. \qquad (9)
\end{aligned}
$$

Since $y_n = P_{T_n}(w_n - \tau_n A w_n)$ and $x_{n+1} \in T_n$ we have

$$
\begin{aligned}
2\langle w_n - \tau_n A y_n - y_n, x_{n+1} - y_n\rangle \\
&= 2\langle w_n - \tau_n A w_n - y_n, x_{n+1} - y_n\rangle + 2\tau_n\langle Aw_n - Ay_n, x_{n+1} - y_n\rangle \\
&\leq 2\tau_n\langle Aw_n - Ay_n, x_{n+1} - y_n\rangle \\
&\leq 2\tau_n L\|w_n - y_n\|\|x_{n+1} - y_n\| \\
&\leq \tau_n L\|w_n - y_n\|^2 + \tau_n L\|x_{n+1} - y_n\|^2. \qquad (10)
\end{aligned}
$$

From (9) and (10) we get

$$\|x_{n+1} - p\|^2 \leq \|w_n - p\|^2 - (1 - \tau_n L)\|y_n - w_n\|^2 - (1 - \tau_n L)\|x_{n+1} - y_n\|^2 - 2\tau_n\gamma\|y_n - p\|^2.$$

By (4) we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \leq& \|w_n - p\|^2 - (1 - \tau_n L)\|y_n - w_n\|^2 - (1 - \tau_n L)\|x_{n+1} - y_n\|^2 \\
\leq& \|w_n - p\|^2 - (1 - \tau_n L)[\|y_n - w_n\|^2 + \|x_{n+1} - y_n\|^2] \\
\leq& \|w_n - p\|^2 - \frac{1 - \tau L}{2}[\|y_n - w_n\| + \|x_{n+1} - y_n\|]^2 \\
\leq& \|w_n - p\|^2 - \frac{1 - \tau_n L}{2}\|x_{n+1} - w_n\|^2. \qquad (11)
\end{aligned}
$$

It follows from $\lim_{n\to\infty}\tau_n = 0$ that there exists $n_0 \in \mathbb{N}$ such that $\tau_n \leq \frac{1}{2L}$ for all $n \geq n_0$, thus $\frac{1-\tau_n L}{2} \geq \frac{1}{4}$. It follows from (11) that for all $n \geq n_0$ we have

$$\|x_{n+1} - p\|^2 \leq \|w_n - p\|^2 - \frac{1}{4}\|x_{n+1} - w_n\|^2. \qquad (12)$$

By the definition of w_n, we have

$$
\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \alpha_n(x_n - x_{n-1}) - p\|^2 \\
&= \|(1 + \alpha_n)(x_n - p) - \alpha_n(x_{n-1} - p)\|^2 \\
&= (1 + \alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + \alpha_n(1 + \alpha_n)\|x_n - x_{n-1}\|^2. \qquad (13)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\|x_{n+1} - w_n\|^2 &= \|x_{n+1} - x_n - \alpha_n(x_n - x_{n-1})\|^2 \\
&= \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \\
&\geq \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\|x_{n+1} - x_n\|\|x_n - x_{n-1}\| \\
&\geq (1 - \alpha_n)\|x_{n+1} - x_n\|^2 + \left(\alpha_n^2 - \alpha_n\right)\|x_n - x_{n-1}\|^2. \qquad (14)
\end{aligned}
$$

Combining (12) with (13) and (14) we obtain

$$\|x_{n+1} - p\|^2 \leq (1 + \alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + \alpha_n(1 + \alpha_n)\|x_n - x_{n-1}\|^2 \qquad (15)$$

$$
\begin{aligned}
&- \frac{1}{4}(1-\alpha_n)\|x_{n+1}-x_n\|^2 - \frac{1}{4}\left(\alpha_n^2-\alpha_n\right)\|x_n-x_{n-1}\|^2 \\
=&(1+\alpha_n)\|x_n-p\|^2 - \alpha_n\|x_{n-1}-p\|^2 - \frac{1}{4}(1-\alpha_n)\|x_{n+1}-x_n\|^2 \\
&+ [\alpha_n(1+\alpha_n) - \frac{1}{4}\left(\alpha_n^2-\alpha_n\right)]\|x_n-x_{n-1}\|^2 \\
=&(1+\alpha_n)\|x_n-p\|^2 - \alpha_n\|x_{n-1}-p\|^2 - \frac{1}{4}(1-\alpha_n)\|x_{n+1}-x_n\|^2 \\
&+ [\frac{3}{4}\alpha_n^2 + \frac{5}{4}\alpha_n]\|x_n-x_{n-1}\|^2 \\
=&(1+\alpha_n)\|x_n-p\|^2 - \alpha_n\|x_{n-1}-p\|^2 - \gamma_n\|x_{n+1}-x_n\|^2 + \mu_n\|x_n-x_{n-1}\|^2,
\end{aligned} \tag{16}
$$

where

$$\gamma_n := \frac{1}{4}(1-\alpha_n) \geq 0$$

and

$$\mu_n := \frac{3}{4}\alpha_n^2 + \frac{5}{4}\alpha_n \geq 0.$$

Put $\Gamma_n := \|x_n-p\|^2 - \alpha_n\|x_{n-1}-p\|^2 + \mu_n\|x_n-x_{n-1}\|^2$. It follows from (16) and the sequence $\{\alpha_n\}$ is non-decreasing that

$$
\begin{aligned}
\Gamma_{n+1} - \Gamma_n =&\|x_{n+1}-p\|^2 - (1+\alpha_{n+1})\|x_n-p\|^2 + \alpha_n\|x_{n-1}-p\|^2 \\
&+ \mu_{n+1}\|x_{n+1}-x_n\|^2 - \mu_n\|x_n-x_{n-1}\|^2 \\
\leq&\|x_{n+1}-p\|^2 - (1+\alpha_n)\|x_n-p\|^2 + \alpha_n\|x_{n-1}-p\|^2 \\
&+ \mu_{n+1}\|x_{n+1}-x_n\|^2 - \mu_n\|x_n-x_{n-1}\|^2 \\
\leq& - \gamma_n\|x_{n+1}-x_n\|^2 + \mu_{n+1}\|x_{n+1}-x_n\|^2 \\
=& - (\gamma_n - \mu_{n+1})\|x_{n+1}-x_n\|^2.
\end{aligned} \tag{17}
$$

For all $n \geq n_0$ we have

$$
\begin{aligned}
\gamma_n - \mu_{n+1} =&\frac{1}{4}(1-\alpha_n) - \frac{3}{4}\alpha_{n+1}^2 - \frac{5}{4}\alpha_{n+1} \\
\geq&\frac{1}{4}(1-\alpha_{n+1}) - \frac{3}{4}\alpha_{n+1}^2 - \frac{5}{4}\alpha_{n+1} \\
=&\frac{1}{4}(1-\alpha) - \frac{3}{4}\alpha^2 - \frac{5}{4}\alpha \\
=&\frac{1}{4} - \frac{6}{4}\alpha - \frac{3}{4}\alpha^2.
\end{aligned} \tag{18}
$$

It follows from (17) and (18) that

$$\Gamma_{n+1} - \Gamma_n \leq -\delta\|x_{n+1}-x_n\|^2 \text{ for all } n \geq n_0, \tag{19}$$

where $\delta := \frac{1}{4} - \frac{6}{4}\alpha - \frac{3}{4}\alpha^2 > 0$. This implies that

$$\Gamma_{n+1} - \Gamma_n \leq 0 \text{ for all } n \geq n_0.$$

Thus, the sequence $\{\Gamma_n\}_{n=n_0}^{\infty}$ is nonincreasing. On the other hand, we have

$$
\begin{aligned}
\Gamma_n &= \|x_n-p\|^2 - \alpha_n\|x_{n-1}-p\|^2 + \mu_n\|x_n-x_{n-1}\|^2 \\
&\geq \|x_n-p\|^2 - \alpha_n\|x_{n-1}-p\|^2.
\end{aligned}
$$

This implies that for all $n \geq n_0$ we get

$$
\begin{aligned}
\|x_n-p\|^2 &\leq \alpha_n\|x_{n-1}-p\|^2 + \Gamma_n \\
&\leq \alpha\|x_{n-1}-p\|^2 + \Gamma_{n_0} \\
&\leq \ldots \leq \alpha^n\|x_{n_0}-p\|^2 + \Gamma_{n_0}(\alpha^{n-1} + \cdots + 1) \\
&\leq \alpha^{n-n_0}\|x_{n_0}-p\|^2 + \frac{\Gamma_{n_0}}{1-\alpha}.
\end{aligned} \tag{20}
$$

We also have

$$\Gamma_{n+1} = \|x_{n+1}-p\|^2 - \alpha_{n+1}\|x_n-p\|^2 + \mu_{n+1}\|x_{n+1}-x_n\|^2$$

$$\geq -\alpha_{n+1}\|x_n - p\|^2. \tag{21}$$

It follows from (20) and (21) that for all $n \geq n_0$ we get

$$\begin{aligned} -\Gamma_{n+1} &\leq \alpha_{n+1}\|x_n - p\|^2 \leq \alpha\|x_n - p\|^2 \leq \alpha^{n-n_0+1}\|x_{n_0} - p\|^2 + \frac{\alpha\Gamma_{n_0}}{1-\alpha} \\ &\leq \|x_{n_0} - p\|^2 + \frac{\alpha\Gamma_{n_0}}{1-\alpha}. \end{aligned} \tag{22}$$

It follows from (19) and (22) that

$$\delta\sum_{n=n_0}^{k}\|x_{n+1} - x_n\|^2 \leq \Gamma_{n_0} - \Gamma_{k+1} \leq \|x_{n_0} - p\|^2 + \frac{\Gamma_{n_0}}{1-\alpha}.$$

Letting $k \to \infty$ we have

$$\sum_{n=1}^{\infty}\|x_{n+1} - x_n\|^2 < +\infty. \tag{23}$$

Therefore, we obtain

$$\|x_{n+1} - x_n\| \to 0. \tag{24}$$

We have

$$\|x_{n+1} - w_n\|^2 = \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle.$$

By (24) we obtain

$$\|x_{n+1} - w_n\| \to 0. \tag{25}$$

By (15), (23) and Lemma 2.4 we have

$$\lim_{n\to\infty}\|x_n - p\|^2 = l \tag{26}$$

By (13) we obtain

$$\lim_{n\to\infty}\|w_n - p\|^2 = l. \tag{27}$$

Since (24) and (25) we get

$$0 \leq \|x_n - w_n\| = \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \to 0. \tag{28}$$

It follows from (4), (26) and (27) that

$$(1 - \tau_n L)\|y_n - w_n\|^2 \leq \|w_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0. \tag{29}$$

Combining (3) and (29) we obtain

$$\lim_{n\to\infty}\|y_n - w_n\| = 0. \tag{30}$$

Since (28) and (30) we have

$$\lim_{n\to\infty}\|x_n - y_n\| \leq \lim_{n\to\infty}\|x_n - w_n\| + \lim_{n\to\infty}\|w_n - y_n\| = 0. \tag{31}$$

Now, we show that the sequence $\{x_n\}$ converges strongly to p. Indeed, it follows from (4) that

$$\begin{aligned} 2\tau_n\gamma\|y_n - p\|^2 &\leq -\|x_{n+1} - p\|^2 + \|w_n - p\|^2 \\ &\leq -\|x_{n+1} - p\|^2 + (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 \\ &\quad + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 \\ &\leq (\|x_n - p\|^2 - \|x_{n+1} - p\|^2) + (\alpha_n\|x_n - p\|^2 - \alpha_{n-1}\|x_{n-1} - p\|^2) \\ &\quad + 2\alpha\|x_n - x_{n-1}\|^2. \end{aligned}$$

This implies that

$$\sum_{n=1}^{k} 2\tau_n\gamma\|y_n - p\|^2 \leq \|x_1 - p\|^2 - \|x_{k+1} - p\|^2 + \alpha_k\|x_k - p\|^2 - \alpha_0\|x_0 - p\|^2$$

$$
\begin{aligned}
&+ 2\alpha \sum_{n=1}^{k} \|x_n - x_{n-1}\|^2 \\
\le &\|x_1 - p\|^2 + \alpha\|x_k - p\|^2 + 2\alpha \sum_{n=1}^{k} \|x_n - x_{n-1}\|^2 \\
\le &M,
\end{aligned}
$$

for some $M > 0$. This implies that

$$\sum_{n=1}^{\infty} 2\tau_n\gamma\|y_n - p\|^2 < +\infty.$$

It implies from $\sum_{n=1}^{\infty}\tau_n = \infty$ and Lemma 2.3 that

$$\liminf_{n\to\infty}\|y_n - p\| = 0. \tag{32}$$

From (32) there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\lim_{k\to\infty}\|y_{n_k} - p\| = 0.$$

Since $\lim_{n\to\infty}\|x_n - y_n\| = 0$, we obtain $\lim_{k\to\infty}\|x_{n_k} - p\| = 0$. On the other hand, we have $\lim_{n\to\infty}\|x_n - p\| \in \mathbb{R}$. Therefore $\lim_{n\to\infty}\|x_n - p\| = 0$, that is $x_n \to p$ as $n \to \infty$. This completes the proof.

Now applying Theorem 3.1 with $\alpha_n = 0$, we obtain the following result.

Corollary 3.1. *Let $\{x_n\}$ be a sequence in H defined by*

$$
\begin{cases}
x_0 \in H, \\
y_n = P_C(x_n - \tau_n Ax_n), \\
T_n = \{x \in H | \langle x_n - \tau_n Aw_n - y_n, x - y_n\rangle \le 0\}, \\
x_{n+1} = P_{T_n}(x_n - \tau_n Ay_n).
\end{cases}
$$

Let $\{\tau_n\}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty}\tau_n = 0 \text{ and } \sum_{n=1}^{\infty}\tau_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to the unique solution p of VIP.

Second, we introduce an inertial Tseng's extragradient algorithm for solving strongly variational inequality which combines the Tseng's extragradient method and the inertial method.

Theorem 3.2. *Let $\{x_n\}$ be a sequence in H defined by*

$$
\begin{cases}
x_0, x_1 \in H, \\
w_n = x_n + \alpha_n(x_n - x_{n-1}), \\
y_n = P_C(w_n - \tau_n Aw_n), \\
x_{n+1} = y_n - \tau_n(Ay_n - Aw_n)
\end{cases}
$$

where the sequence $\{\alpha_n\}$ is non-decreasing and $0 \le \alpha_n \le \alpha \le \sqrt{5} - 2$. Let $\{\tau_n\}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty}\tau_n = 0 \text{ and } \sum_{n=1}^{\infty}\tau_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to the unique solution p of VIP.

Proof. First, we prove that

$$\|x_{n+1} - p\|^2 \le \|w_n - p\|^2 - (1 - \tau_n^2 L^2)\|y_n - w_n\|^2 - 2\tau_n\gamma\|y_n - p\|^2. \tag{33}$$

Indeed, we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 = &\|y_n - \tau_n(Ay_n - Aw_n) - p\|^2 \\
= &\|y_n - p\|^2 + \tau_n^2\|Ay_n - Aw_n\|^2 - 2\tau_n\langle y_n - p, Ay_n - Aw_n\rangle \\
= &\|w_n - p\|^2 + \|w_n - y_n\|^2 + 2\langle y_n - w_n, w_n - p\rangle + \tau_n^2\|Ay_n - Aw_n\|^2
\end{aligned}
$$

$$
\begin{aligned}
&- 2\tau_n\langle y_n - p, Ay_n - Aw_n\rangle \\
=&\|w_n - p\|^2 + \|w_n - y_n\|^2 - 2\langle y_n - w_n, y_n - w_n\rangle + 2\langle y_n - w_n, y_n - p\rangle \\
&+ \tau_n^2\|Ay_n - Aw_n\|^2 - 2\tau_n\langle y_n - p, Ay_n - Aw_n\rangle \\
=&\|w_n - p\|^2 - \|w_n - y_n\|^2 + 2\langle y_n - w_n, y_n - p\rangle + \tau_n^2\|Ay_n - Aw_n\|^2 \\
&- 2\tau_n\langle y_n - p, Ay_n - Aw_n\rangle. \qquad (34)
\end{aligned}
$$

Noting that since $y_n = P_C(w_n - \tau_n Aw_n)$ this implies that

$$\langle y_n - w_n + \tau_n Aw_n, y_n - p\rangle \leq 0,$$

or equivalently

$$\langle y_n - w_n, y_n - p\rangle \leq -\tau_n\langle Aw_n, y_n - p\rangle. \qquad (35)$$

From (34) and (35), we get

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \leq& \|w_n - p\|^2 - \|w_n - y_n\|^2 - 2\tau_n\langle Aw_n, y_n - p\rangle + \tau_n^2\|Ay_n - Aw_n\|^2 \\
&- 2\tau_n\langle y_n - p, Ay_n - Aw_n\rangle \\
=& \|w_n - p\|^2 - \|w_n - y_n\|^2 + \tau_n^2\|Ay_n - Aw_n\|^2 - 2\tau_n\langle y_n - p, Ay_n\rangle. \qquad (36)
\end{aligned}
$$

Since p is the solution of VIP, we have $\langle Ap, x - p\rangle \geq 0$ for all $x \in C$. By the strong pseudomonotonicity of A on C we have $\langle Ax, x - p\rangle \geq \gamma\|x - p\|^2$ for all $x \in C$. Taking $x := y_n \in C$ we get

$$\langle Ay_n, p - y_n\rangle \leq -\gamma\|y_n - p\|^2. \qquad (37)$$

It follows from (36) and (37) that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \leq& \|w_n - p\|^2 - \|w_n - y_n\|^2 + \tau_n^2 L^2\|y_n - w_n\|^2 - 2\tau_n\gamma\|y_n - p\|^2 \\
\leq& \|w_n - p\|^2 - (1 - \tau_n^2 L^2)\|y_n - w_n\|^2 - 2\tau_n\gamma\|y_n - p\|^2.
\end{aligned}
$$

By (33) we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \leq& \|w_n - p\|^2 - (1 - \tau_n^2 L^2)\|y_n - w_n\|^2 - 2\tau_n\gamma\|y_n - p\|^2 \qquad (38) \\
\leq& \|w_n - p\|^2 - (1 - \tau_n^2 L^2)\|y_n - w_n\|^2. \qquad (39)
\end{aligned}
$$

By the definition of x_{n+1} we have

$$
\begin{aligned}
\|x_{n+1} - y_n\| &= \|y_n - \tau_n(Ay_n - Aw_n) - y_n\| \\
&\leq \tau_n\|Ay_n - Aw_n\| \\
&\leq \tau_n L\|y_n - w_n\|.
\end{aligned}
$$

Therefore

$$\|x_{n+1} - w_n\| \leq \|x_{n+1} - y_n\| + \|y_n - w_n\| \leq (1 + \tau_n L)\|y_n - w_n\|.$$

This implies

$$\|y_n - w_n\| \geq \frac{1}{1 + \tau_n L}\|x_{n+1} - w_n\|. \qquad (40)$$

From (39) and (40) we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|w_n - p\|^2 - \frac{1 - \tau_n^2 L^2}{(1 + \tau_n L)^2}\|x_{n+1} - w_n\|^2 \\
&= \|w_n - p\|^2 - \frac{1 - \tau_n L}{1 + \tau_n L}\|x_{n+1} - w_n\|^2. \qquad (41)
\end{aligned}
$$

It follows from $\lim_{n\to\infty}\tau_n = 0$ that there exists $n_0 \in \mathbb{N}$ such that $\tau_n \leq \frac{1}{3L}$ for all $n \geq n_0$, this implies that $1 - \tau_n L \geq \frac{2}{3}$ and $1 + \tau_n L \leq \frac{4}{3}$. Therefore

$$\frac{1 - \tau_n L}{1 + \tau_n L} \geq \frac{1}{2}. \qquad (42)$$

It implies from (41) and (42) that for all $n \geq n_0$ we have

$$\|x_{n+1} - p\|^2 \leq \|w_n - p\|^2 - \frac{1}{2}\|x_{n+1} - w_n\|^2. \qquad (43)$$

By the definition of w_n, we have

$$\begin{aligned}\|w_n - p\|^2 &= \|x_n + \alpha_n(x_n - x_{n-1}) - p\|^2 \\ &= \|(1+\alpha_n)(x_n - p) - \alpha_n(x_{n-1} - p)\|^2 \\ &= (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{44}$$

It follows from (43) and (44) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 \\ &\le (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + 2\alpha\|x_n - x_{n-1}\|^2. \end{aligned} \tag{45}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1} - w_n\|^2 &= \|x_{n+1} - x_n - \alpha_n(x_n - x_{n-1})\|^2 \\ &= \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \\ &\ge \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\|x_{n+1} - x_n\|\|x_n - x_{n-1}\| \\ &\ge (1-\alpha_n)\|x_{n+1} - x_n\|^2 + \left(\alpha_n^2 - \alpha_n\right)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{46}$$

Combining (43), (44) and (46) we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 \le\ & (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 \\ & - \frac{1}{2}(1-\alpha_n)\|x_{n+1} - x_n\|^2 - \frac{1}{2}\left(\alpha_n^2 - \alpha_n\right)\|x_n - x_{n-1}\|^2 \\ =&(1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 - \frac{1}{2}(1-\alpha_n)\|x_{n+1} - x_n\|^2 \\ & + [\alpha_n(1+\alpha_n) - \frac{1}{2}\left(\alpha_n^2 - \alpha_n\right)]\|x_n - x_{n-1}\|^2 \\ =&(1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 - \frac{1}{2}(1-\alpha_n)\|x_{n+1} - x_n\|^2 \\ & + [\frac{1}{2}\alpha_n^2 + \frac{3}{2}\alpha_n]\|x_n - x_{n-1}\|^2 \\ =&(1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 - \gamma_n\|x_{n+1} - x_n\|^2 + \mu_n\|x_n - x_{n-1}\|^2 \\ \le&(1+\alpha_{n+1})\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 - \gamma_n\|x_{n+1} - x_n\|^2 + \mu_n\|x_n - x_{n-1}\|^2, \end{aligned} \tag{47}$$

where $\gamma_n := \frac{1}{2}(1-\alpha_n)$ and $\mu_n := \frac{1}{2}\alpha_n^2 + \frac{3}{2}\alpha_n \ge 0$.

Put

$$\Gamma_n := \|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + \mu_n\|x_n - x_{n-1}\|^2.$$

It follows from (47) that

$$\begin{aligned}\Gamma_{n+1} - \Gamma_n =&\|x_{n+1} - p\|^2 - (1+\alpha_{n+1})\|x_n - p\|^2 + \alpha_n\|x_{n-1} - p\|^2 \\ & + \mu_{n+1}\|x_{n+1} - x_n\|^2 - \mu_n\|x_n - x_{n-1}\|^2 \\ \le&\|x_{n+1} - p\|^2 - (1+\alpha_n)\|x_n - p\|^2 + \alpha_n\|x_{n-1} - p\|^2 \\ & + \mu_{n+1}\|x_{n+1} - x_n\|^2 - \mu_n\|x_n - x_{n-1}\|^2 \\ \le& - (\gamma_n - \mu_{n+1})\|x_{n+1} - x_n\|^2. \end{aligned} \tag{48}$$

For all $n \ge n_0$ we have

$$\begin{aligned}\gamma_n - \mu_{n+1} &= \frac{1}{2}(1-\alpha_n) - \frac{1}{2}\alpha_{n+1}^2 - \frac{3}{2}\alpha_{n+1} \\ &\ge \frac{1}{2}(1-\alpha_{n+1}) - \frac{1}{2}\alpha_{n+1}^2 - \frac{3}{2}\alpha_{n+1} \\ &= \frac{1}{2} - 2\alpha_{n+1} - \frac{1}{2}\alpha_{n+1}^2 \\ &\ge \frac{1 - 4\alpha - \alpha^2}{2}. \end{aligned} \tag{49}$$

Combining (48) and (49) we get

$$\Gamma_{n+1} - \Gamma_n \le -\delta\|x_{n+1} - x_n\|^2, \tag{50}$$

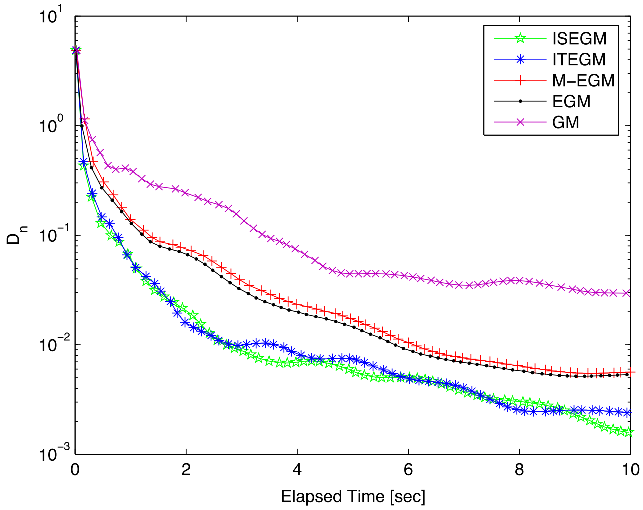

Fig. 1. Example 1 for $\tau_n = \frac{1}{\sqrt{n+1}}$ and $m = 50$. Numbers of iterations (resp.) are 644, 626, 311, 318, 605.

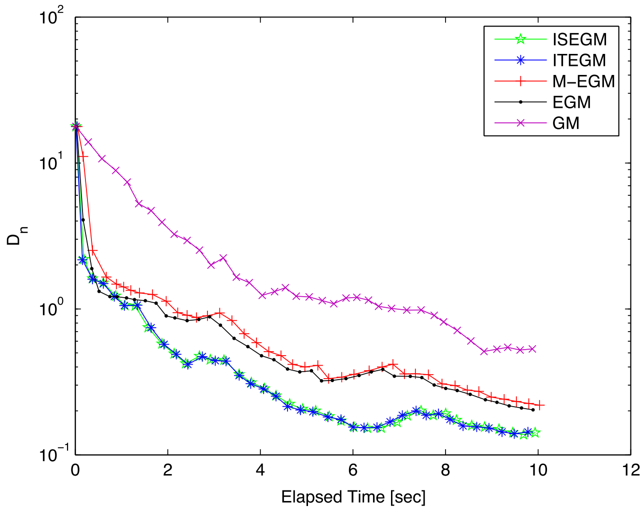

Fig. 2. Example 1 for $\tau_n = \frac{1}{\sqrt{n+1}}$ and $m = 100$. Numbers of iterations (resp.) are 373, 378, 200, 201, 375.

where $\delta := \frac{1-4\alpha-\alpha^2}{2} > 0$. This implies that

$$\Gamma_{n+1} - \Gamma_n \le 0 \text{ for all } n \ge n_0.$$

Thus, the sequence $\{\Gamma_n\}_{n=n_0}^{\infty}$ is nonincreasing. On the other hand, we have

$$\begin{aligned}\Gamma_n &= \|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + \mu_n\|x_n - x_{n-1}\|^2 \\ &\ge \|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2.\end{aligned}$$

This implies that for all $n \ge n_0$ we get

$$\begin{aligned}\|x_n - p\|^2 &\le \alpha_n\|x_{n-1} - p\|^2 + \Gamma_n \\ &\le \alpha\|x_{n-1} - p\|^2 + \Gamma_{n_0} \\ &\le \cdots \le \alpha^n\|x_{n_0} - p\|^2 + \Gamma_{n_0}(\alpha^{n-1} + \cdots + 1) \\ &\le \alpha^{n-n_0}\|x_{n_0} - p\|^2 + \frac{\Gamma_{n_0}}{1-\alpha}.\end{aligned} \tag{51}$$

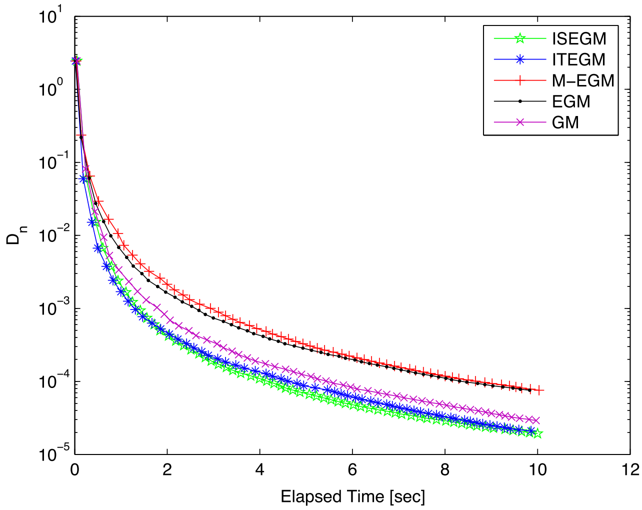

Fig. 3. Example 1 for $\tau_n = \frac{1}{n+1}$ and $m = 50$. Numbers of iterations (resp.) are 592, 588, 295, 288, 551.

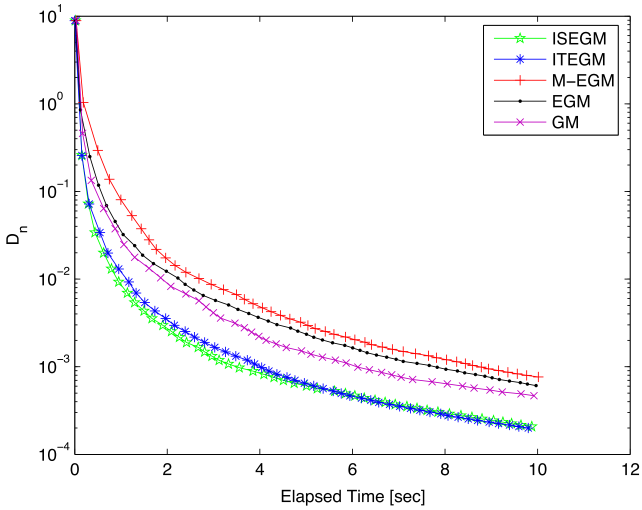

Fig. 4. Example 1 for $\tau_n = \frac{1}{n+1}$ and $m = 100$. Numbers of iterations (resp.) are 478, 488, 235, 246, 404.

We also have

$$\begin{aligned}\Gamma_{n+1} &= \|x_{n+1} - p\|^2 - \alpha_{n+1}\|x_n - p\|^2 + \mu_{n+1}\|x_{n+1} - x_n\|^2 \\ &\geq -\alpha_{n+1}\|x_n - p\|^2.\end{aligned} \tag{52}$$

It follows from (51) and (52) that for all $n \geq n_0$ we get

$$\begin{aligned}-\Gamma_{n+1} &\leq \alpha_{n+1}\|x_n - p\|^2 \leq \alpha\|x_n - p\|^2 \leq \alpha^{n-n_0+1}\|x_{n_0} - p\|^2 + \frac{\alpha\Gamma_{n_0}}{1-\alpha} \\ &\leq \|x_{n_0} - p\|^2 + \frac{\alpha\Gamma_{n_0}}{1-\alpha}.\end{aligned} \tag{53}$$

It follows from (50) and (53) that

$$\delta \sum_{n=n_0}^{k} \|x_{n+1} - x_n\|^2 \leq \Gamma_{n_0} - \Gamma_{k+1} \leq \|x_{n_0} - p\|^2 + \frac{\Gamma_{n_0}}{1-\alpha}.$$

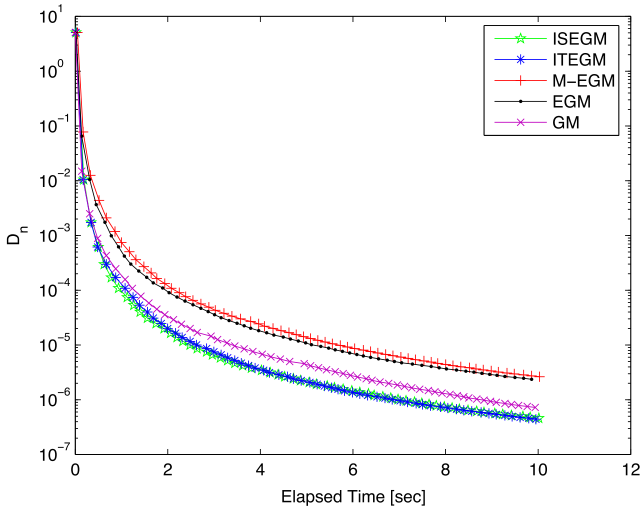

Fig. 5. Example 1 for $\tau_n = \frac{1}{(n+1)\log(n+1)}$ and $m = 50$. Numbers of iterations (resp.) are 590, 604, 305, 299, 575.

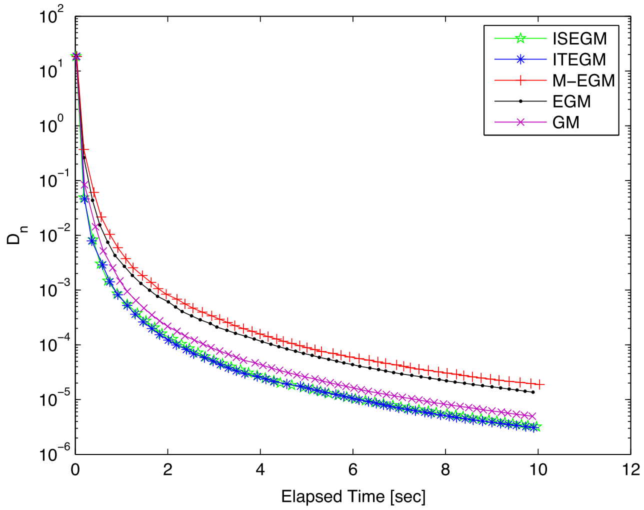

Fig. 6. Example 1 for $\tau_n = \frac{1}{(n+1)\log(n+1)}$ and $m = 100$. Numbers of iterations (resp.) are 534, 535, 270, 273, 558.

This implies

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < +\infty. \tag{54}$$

Therefore, we obtain

$$\|x_{n+1} - x_n\| \to 0. \tag{55}$$

We have

$$\|x_{n+1} - w_n\|^2 = \|x_{n+1} - x_n\|^2 + \alpha_n^2 \|x_n - x_{n-1}\|^2 - 2\alpha_n \langle x_{n+1} - x_n, x_n - x_{n-1}\rangle.$$

This implies that

$$\|x_{n+1} - w_n\| \to 0. \tag{56}$$

By (45), (54) and Lemma 2.4 we have

$$\lim_{n\to\infty} \|x_n - p\|^2 = l \tag{57}$$

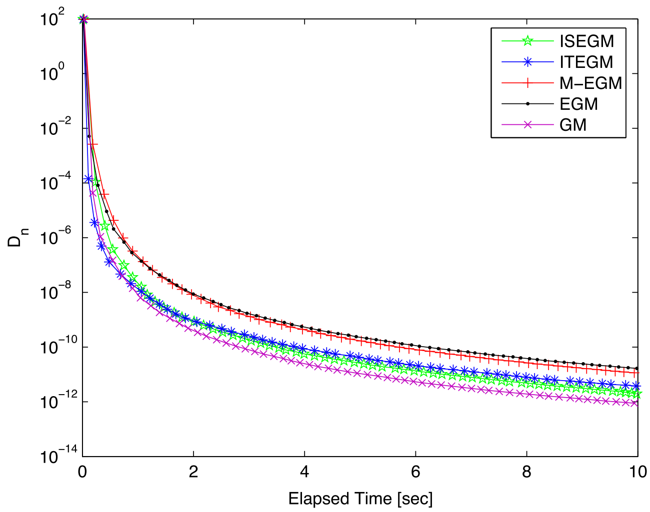

Fig. 7. Example 1 for $\tau_n = \frac{1}{(n+1)\log^5(n+1)}$ and $m = 50$. Numbers of iterations (resp.) are 603, 543, 283, 311, 585.

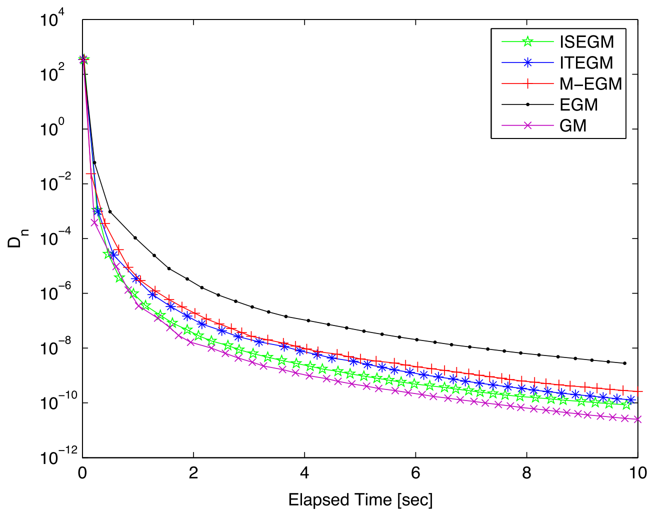

Fig. 8. Example 1 for $\tau_n = \frac{1}{(n+1)\log^5(n+1)}$ and $m = 100$. Numbers of iterations (resp.) are 409, 356, 221, 158, 421.

By (44) we obtain

$$\lim_{n\to\infty} \|w_n - p\|^2 = l. \tag{58}$$

Combining (55) and (56) we obtain

$$0 \le \|x_n - w_n\| = \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \to 0. \tag{59}$$

It follows from (38) that

$$(1 - \tau_n^2 L^2)\|y_n - w_n\|^2 \le \|w_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0. \tag{60}$$

Since (3.2) and (60) we get

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{61}$$

It implies from (59) and (61) that

$$\lim_{n\to\infty} \|x_n - y_n\| \le \lim_{n\to\infty} \|x_n - w_n\| + \lim_{n\to\infty} \|w_n - y_n\| = 0. \tag{62}$$

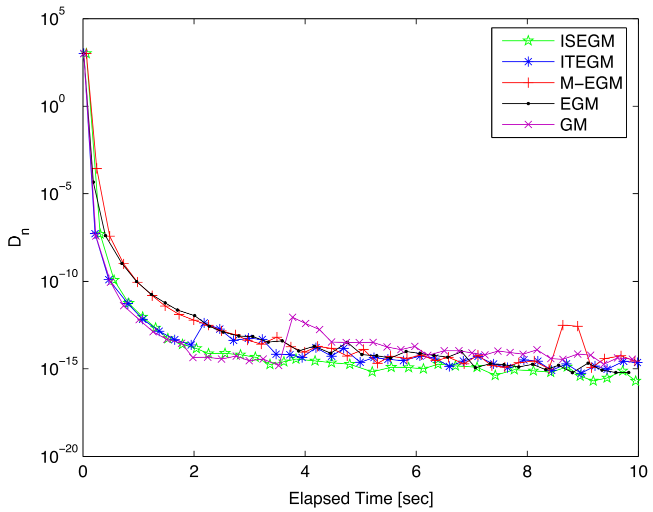

Fig. 9. Example 1 for $\tau_n = \frac{1}{(n+1)\log^{10}(n+1)}$ and $m = 50$. Numbers of iterations (resp.) are 351, 382, 197, 193, 405.

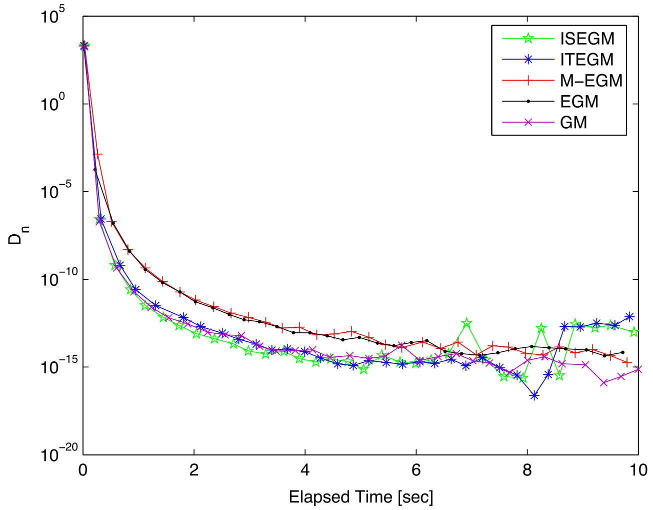

Fig. 10. Example 1 for $\tau_n = \frac{1}{(n+1)\log^{10}(n+1)}$ and $m = 100$. Numbers of iterations (resp.) are 323, 326, 163, 163, 311.

Now, we show that the sequence $\{x_n\}$ converges strongly to p. Indeed, it follows from (38) that

$$\begin{aligned}
2\tau_n\gamma\|y_n - p\|^2 \le& -\|x_{n+1} - p\|^2 + \|w_n - p\|^2\\
\le& -\|x_{n+1} - p\|^2 + (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2\\
&+ \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2\\
\le&(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) + (\alpha_n\|x_n - p\|^2 - \alpha_{n-1}\|x_{n-1} - p\|^2)\\
&+ 2\alpha\|x_n - x_{n-1}\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
\sum_{n=1}^{k} 2\tau_n\gamma\|y_n - p\|^2 \le&\|x_1 - p\|^2 - \|x_{k+1} - p\|^2 + \alpha_k\|x_k - p\|^2 - \alpha_0\|x_0 - p\|^2\\
&+ 2\alpha\sum_{n=1}^{k}\|x_n - x_{n-1}\|^2
\end{aligned}$$

$$\begin{aligned}&\leq \|x_1 - p\|^2 + \alpha\|x_k - p\|^2 + 2\alpha\sum_{n=1}^{k}\|x_n - x_{n-1}\|^2\\&\leq M,\end{aligned}$$

for some $M > 0$. This implies that

$$\sum_{n=1}^{\infty} 2\tau_n\gamma\|y_n - p\|^2 < +\infty. \tag{63}$$

It implies from $\sum_{n=1}^{\infty}\tau_n = \infty$, (63) and Lemma 2.3 that

$$\liminf_{n\to\infty}\|y_n - p\| = 0. \tag{64}$$

By (64) there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\lim_{k\to\infty}\|y_{n_k} - p\| = 0.$$

Since (62) we have $\lim_{n\to\infty}\|x_n - y_n\| = 0$, we obtain $\lim_{k\to\infty}\|x_{n_k} - p\| = 0$. On the other hand, we have $\lim_{n\to\infty}\|x_n - p\| \in \mathbb{R}$. Therefore $\lim_{n\to\infty}\|x_n - p\| = 0$, that is $x_n \to p$ as $n \to \infty$. This completes the proof.

Now applying Theorem 3.2 with $\alpha_n = 0$, we obtain the following result.

Corollary 3.2. *Let $\{x_n\}$ be a sequence in H defined by*

$$\begin{cases}x_0 \in H,\\ y_n = P_C(x_n - \tau_n A x_n),\\ x_{n+1} = y_n - \tau_n(Ay_n - Ax_n).\end{cases}$$

Let $\{\tau_n\}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty}\tau_n = 0 \text{ and } \sum_{n=1}^{\infty}\tau_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to the unique solution p of VIP.

4. Numerical illustrations

This section reports several numerical results to illustrate the convergence of the inertial subgradient extragradient method (2) (shortly, ISEGM), the inertial Tseng's extragradient method (3.2) (ITEGM) in comparison with three algorithms having the same features including the gradient method (GM) considered in [30, Theorem 5.1], the extragradient method (EGM) presented in [52, Corollary 3.3] and the modified extragradient method (M-EGM) proposed in [53, Corollary 3.1]. All the projections over C are computed effectively by the function *quadprog* in Matlab 7.0 Optimization Toolbox. All the projections over half-spaces are inherently explicit. All the programs are performed on a PC Desktop Intel(R) Core(TM) i5-3210M CPU @ 2.50 GHz 2.50 GHz, RAM 2.00 GB. The test problems are of the following form.

Example 1. Let $A : \Re^m \to \Re^m$ ($m = 50,\ 100$) be an operator in the form $A(x) = Mx + q$ [21,56] with $q \in \Re^m$ and

$$M = NN^T + S + D,$$

where N is a $m \times m$ matrix, S is a $m \times m$ skew-symmetric matrix, and D is a $m \times m$ diagonal matrix with its diagonal entries being positive. The feasible set C is given by $C = \{x \in \Re^m : -5 \leq x_i \leq 5,\ i = 1, \ldots, m\}$. The problem is to find a point $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0,\ \forall y \in C.$$

It is clear to see that A is strongly pseudomonotone and Lipschitz continuous. For experiments, all entries of q, N, S are generated randomly in $(-2, 2)$ and of D are in $(0, 2)$. The starting point is $x_0 = (1, 1, \ldots, 1)^T \in \Re^m$. We choose the inertial parameter $\alpha_n = 0.1$ for two proposed algorithms ISEGM and ITEGM and five sequences of stepsizes for all the algorithms as

$$\tau_n = \frac{1}{\sqrt{n+1}},\ \frac{1}{n+1},\ \frac{1}{(n+1)\log(n+1)},\ \frac{1}{(n+1)\log^5(n+1)},\ \frac{1}{(n+1)\log^{10}(n+1)}.$$

Note that if $x = P_C(x - \lambda Ax)$ for some $\lambda > 0$ then x is a solution of problem (VIP). So, we will use the following sequences, for each $n \geq 0$,

$$D_n = \|y_n - w_n\|^2 \text{ for ISEGM and ITEGM},$$

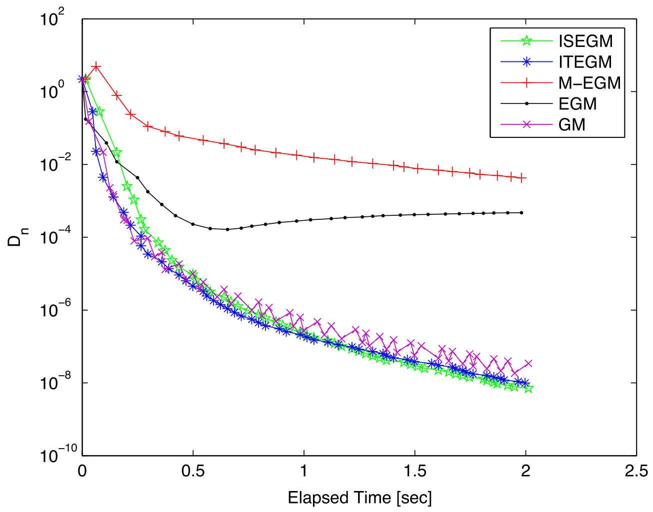

Fig. 11. Example 2 for $\tau_n = \frac{1}{n+1}$. Numbers of iterations (resp.) are 175, 183, 82, 94, 191.

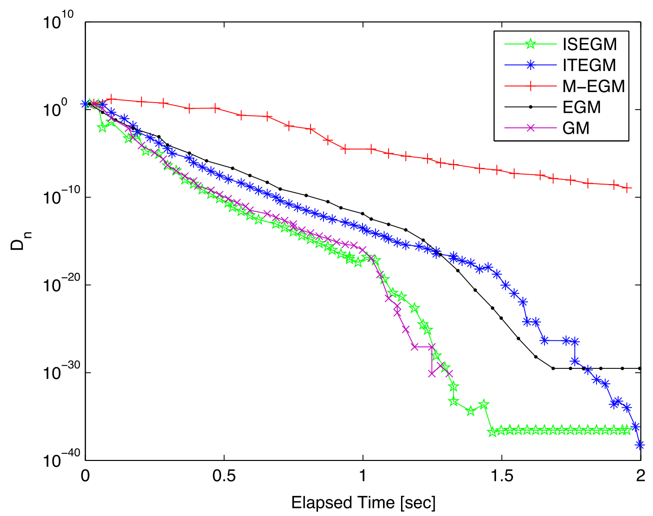

Fig. 12. Example 2 for $\tau_n = \frac{1}{\sqrt{n+1}}$. Numbers of iterations (resp.) are 205, 191, 91, 101, 205.

$$D_n = \|y_n - x_n\|^2 \text{ for EGM},$$
$$D_n = \|x_{n+1} - x_n\|^2 \text{ for GM},$$
$$D_n = \max\left\{\|x_{n+1} - y_n\|^2,\ \|y_n - x_n\|^2\right\} \text{ for M-EGM},$$

to study the convergence of all the algorithms. The convergence of D_n to 0 implies that the iterative sequence generated by each algorithm converges to the solution of the problem. To illustrate the convergence and computational performance of all the algorithms, we have shown the value of D_n (the y-axis) when the execution time in second elapses (the x-axis). Figs. 1–10 describe all the numerical results.

Example 2. Next, consider $H = \Re^2$, $C = [-5, 5] \times [-5, 5]$, and a nonlinear operator $A : \Re^2 \to \Re^2$ given by

$$A(x, y) = (x + y + \sin x; -x + y + \sin y).$$

The numerical results for this example are shown in Figs. 11 and 12.

Example 3. Finally, we consider our problem with regard to the nonlinear operator $A : \Re^m \to \Re^m$ ($m = 50$) of the form $A(x) = Mx + F(x) + q$ and M is a $m \times m$ symmetric semidefinite matrix and $F : \Re^m \to \Re^m$ is the proximal mapping of the

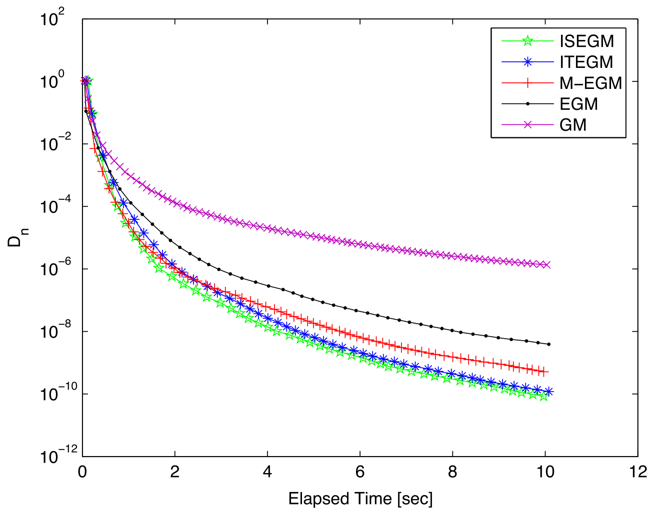

Fig. 13. Example 3 for $\tau_n = \frac{1}{n+1}$. Numbers of iterations (resp.) are 91, 90, 133, 86, 164.

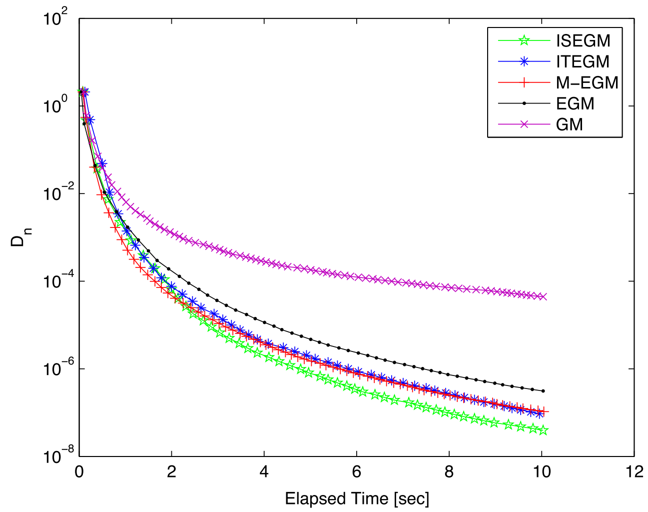

Fig. 14. Example 3 for $\tau_n = \frac{1}{\sqrt{n+1}}$. Numbers of iterations (resp.) are 94, 93, 138, 98, 160.

function $g(x) = \frac{1}{4}\|x\|^4$, i.e.,

$$F(x) = \arg\min \left\{ \frac{\|y\|^4}{4} + \frac{1}{2}\|y - x\|^2 : y \in \Re^m \right\}.$$

The feasible set C is given as in Example 1. In this case, A is strongly pseudomonotone and Lipschitz continuous. For experiment, all the entries of M and q are generated randomly in $(-2, 2)$. The numerical results are shown in Figs. 13 and 14.

In view of these figures, we would like to make the following two notable observations:
(a) The rate of convergence of all the algorithms in general depends strictly on the convergent rate of sequence of stepsizes $\{\tau_n\}$.
(b) The two ISEGM and ITEGM also have competitive advantages over other known algorithms, especially, from Figs. 1–6 and 11–14 we see that the two proposed algorithms are better than others.

5. Conclusions

The paper has proposed two algorithms, called ISEGM and ITEGM for solving strongly pseudomonotone and Lipschitz VIPs in Hilbert spaces. Under some suitable conditions imposed on parameters, we established the strong convergence of the algorithms. The result in the paper have extended and improved a previously known extragradient method for VIPs. The efficiency of the proposed algorithms has also been illustrated by several preliminary numerical experiments.

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