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Convergence and non-negativity preserving of the solution of balanced method for the delay CIR model with jump

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In this work, we propose the balanced implicit method (BIM) to approximate the solution of the delay Cox–Ingersoll–Ross (CIR) model with jump which often gives rise to model an asset price and stochastic volatility dependent on past data. We show that this method preserves non-negativity property of the solution of this model with appropriate control functions. We prove the strong convergence and investigate the pth moment boundedness of the solution of BIM. Finally, we illustrate those results in the last section.

1. Introduction

Consider $(\Omega, \mathcal{F}, P)$ as a complete probability space with right continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$ while $\mathcal{F}_0$ contains all P-null sets. We consider the delay CIR model with jump introduced by Jiang, Shen and Wu [1]

$$\begin{cases} dS(t) = \lambda(\mu - S(t))dt + \sigma S(t-\tau)^{\gamma}\sqrt{S(t)}dW(t) + \delta S(t^-)d\tilde{N}(t), \quad t \geq 0, \\ S(t) = \xi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1}$$

where γ, λ, μ and σ are positive constants, $W(t)$ is a standard Brownian motion and $\tilde{N}(t) = N(t) - \beta t$ is a compensated Poisson process, in which $N(t)$ is a Poisson process with intensity β and also $S(t^-) := \lim_{s\to t^-} S(s)$. The positive initial value ξ is an $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^+)$-valued random variable satisfying

$$E[\sup_{-\tau\leq t\leq 0} |\xi(t)|^p] < +\infty, \tag{2}$$

for any $p > 0$. In particular, the CIR model (1) without jump and delay, $\tau = 0$, $\delta = 0$, was introduced by Cox, Ingersoll and Ross [2], as a model for stochastic volatility, interest rate and other financial quantities. Also, the CIR model (1) without jump, $\delta = 0$, was introduced by Wu, et al. [3] with regard to the fact that stock prices depend on past behaviors. (See also [4,5].) Unfortunately, SDDEs with jumps have no explicit solution. Thus, constructing an appropriate numerical method

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to approximate and study the properties of the true solutions of these models is essential. Furthermore, in recent years, researchers are interested in numerical methods satisfying the same properties of the solutions such as positivity.

Strong convergence for stochastic differential equations (SDEs) with jumps is studied in the literature [6–11] and strong convergence for the mean-reverting square root process with jump is discussed in [12]. There are also some works concerned with positivity of numerical methods of SDEs; for example, see [13–19].

For the CIR model (1), with $\tau = 0, \delta = 0$, Dereich et al. [20] investigated the drift non-negativity preserving of implicit Euler method and in 2013, Higham et al. [21] introduced a new implicit Milstein scheme which preserves non-negativity of solution. Also, Halidias and Stamatiou in [22,23] constructed numerical schemes which preserve positivity of Heston 3/2 model and the CEV process, respectively. Lately, Yang and Wang [24] showed that the backward Euler scheme preserves positivity for the CIR model with jump.

In this manuscript, we choose a balanced implicit scheme in order to obtain the positivity of our approximation process. Non-negativity preserving of BIM for SDEs without jumps is well studied (see; e.g. [25,26]), and of SDEs with jumps is discussed in [27,24], under an appropriate choice of control functions. Also, Tan et al. [28] showed that the BIM preserves positivity for the stochastic age-dependent population equation.

Strong convergence for SDDEs with jumps is studied in the literature [29–34] and in [35] for SDDEs without jumps. Wu et al. [3], showed the existence of non-negative solution of the delay CIR model without jump and Jiang, et al. [1] proved it for the delay mean-reverting square root process with jump (1). Also, they showed the Euler Maruyama method converges strongly to the solution and proved the boundedness of the pth moments of the solution to the model and the method. Fatemioon et al. [36] investigated strong convergence of BIM for the CIR model and showed that the scheme preserves positivity.

To the best of our knowledge, there is no positivity preserving result of numerical method for SDDEs with jumps. The aim of this paper is to preserve positivity of BIM for SDDEs (the delay CIR models) with jumps (1). To do this, we cannot examine traditional control functions used in BIM for SDEs to reach the positivity of BIM for these SDDEs, for instance, see [27,14]. We define a new appropriate control function and prove that the positive solution of the BIM converges to the solution of the model (1) in the strong sense. Also we show the boundedness of p-moments of the method for any $p > 0$.

The paper is organized as follows. In Section 2, we propose the BIM for the SDDE with jump (1) and choose the appropriate control functions that imply the non-negativity property of the method. Also, we introduce the continuous case of the method to prove convergence results in the next section. In Section 3, we prove the convergence of the BIM applied to the model (1). Some numerical experiments in last section illustrate the obtained theoretical results of this paper.

2. Introduction of BIM and its properties

In this section, we describe the balanced method to approximate the solution of the delay CIR model with jump (1). Then, we state the non-negativity preserving concept of solution of numerical methods for this model, based on definitions in [26]. Also, we investigate the properties of p-moments for the balanced method in continuous time, which we need in the next section.

2.1. BIM and non-negativity preserving of method in discrete case

Set a uniform mesh on $[0, T]$, $t_n = nh$, $n = 0, \ldots, N$, $N \in \mathbb{N}$ for a step size $h \in (0, 1)$ as $h = \frac{\tau}{m}$, for a positive integer m. We introduce the BIM for SDDE with jump (1) by $s_n = \xi_n = \xi(t_n)$ for $n = -m, -m+1, \ldots, 0$ and for $n \geq 0$,

$$s_{n+1} = s_n + \lambda(\mu - s_n)h + \sigma s_{n-m}^{\gamma}\sqrt{s_n}\Delta W_n + \delta s_n \Delta\tilde{N}_n + C_n(s_n - s_{n+1}), \tag{3}$$

where $C_n = C_0(s_n, s_{n-m})h + C_1(s_n, s_{n-m})\,|\Delta W_n| + C_2(s_n, s_{n-m})\left|\Delta\tilde{N}_n\right|$, such that for control functions $C_0(s_n, s_{n-m})$, $C_1(s_n, s_{n-m})$ and $C_2(s_n, s_{n-m})$, the expression $(1 + C_0(s_n, s_{n-m})h + C_1(s_n, s_{n-m})\,|\Delta W_n| + C_2(s_n, s_{n-m})\left|\Delta\tilde{N}_n\right|)^{-1}$ always exists and is uniformly bounded.

The control functions for the BIM (3) that ensure preserving non-negativity of the solution of delay CIR model with jump (1) are

$$C_0(s_n, s_{n-m}) = C_0 \geq \lambda, \tag{4}$$

$$C_1(s_n, s_{n-m}) = \begin{cases} \sigma s_{n-m}^{\gamma}\epsilon^{-\frac{1}{2}}, & s_n < \epsilon, \\ \sigma \dfrac{s_{n-m}^{\gamma}}{\sqrt{s_n}}, & s_n \geq \epsilon, \end{cases} \tag{5}$$

$$C_2(s_n, s_{n-m}) = C_2 \geq \delta, \tag{6}$$

where $C_0,\ C_2$ are positive constants.

Definition 2.1. Let s_n be a numerical solution which is computed by a numerical method for solving SDDE with jump (1). The numerical solution s_n is said to have eternal life time if

$$P(s_n \geq 0 | \xi_n \geq 0) = 1, \quad \text{for all } n \geq -m. \tag{7}$$

If (7) does not hold, then the numerical solution is said to have finite life time.

Definition 2.2. Let s_n be a numerical solution which is computed by a numerical method for solving SDDE with jump (1). The numerical solution s_n is said to have ϵ-life time if

$$P(s_{n+1} \geq 0 | s_n \geq \epsilon,\ s_{n-m} \geq 0) = 1, \quad \text{for some } \epsilon > 0. \tag{8}$$

Theorem 2.3. *The solution of the BIM (3) with control functions (4), (5) and (6) has ϵ-life time.*

Proof. Assume that $s_n \geq \epsilon,\ s_{n-m} \geq 0$. According to the BIM (3) with control functions (4), (5) and (6), we have

$$\begin{aligned} s_{n+1} &= s_n + \frac{\lambda(\mu - s_n)h + \sigma s_{n-m}^{\gamma}\sqrt{s_n}\Delta W_n + \delta s_n \Delta\tilde{N}_n}{1 + C_0\, h + \sigma \frac{s_{n-m}^{\gamma}}{\sqrt{s_n}} |\Delta W_n| + C_2 \left|\Delta\tilde{N}_n\right|} \\ &= s_n \left(\frac{(1 + C_0\, h - \lambda\, h)\sqrt{s_n} + \sigma s_{n-m}^{\gamma}(\Delta W_n + |\Delta W_n|) + (C_2 \left|\Delta\tilde{N}_n\right| + \delta\Delta\tilde{N}_n)\sqrt{s_n}}{(1 + C_0\, h)\sqrt{s_n} + \sigma s_{n-m}^{\gamma} |\Delta W_n| + C_2\sqrt{s_n} \left|\Delta\tilde{N}_n\right|} \right) \\ &\quad + \frac{\lambda\mu h}{1 + C_0\, h + \sigma \frac{s_{n-m}^{\gamma}}{\sqrt{s_n}} |\Delta W_n| + C_2 \left|\Delta\tilde{N}_n\right|}. \end{aligned}$$

So, it is clear that $s_{n+1} \geq 0$. $\square$

2.2. BIM and boundedness of the p-moments in continuous case

It is more convenient to use the time-continuous approximation of the BIM (3) as

$$s(t) = \begin{cases} \xi(t), & -\tau \leq t \leq 0, \\ \xi(0) + \lambda \displaystyle\int_0^t \frac{(\mu - \hat{s}(r))}{1 + C_r(\hat{s}(r), \hat{s}(r-\tau))} dr & \\ \quad + \sigma \displaystyle\int_0^t \frac{\hat{s}^{\gamma}(r-\tau)\sqrt{\hat{s}(r)}}{1 + C_r(\hat{s}(r), \hat{s}(r-\tau))} dW(r) + \delta \int_0^t \frac{\hat{s}(r)}{1 + C_r(\hat{s}(r), \hat{s}(r-\tau))} d\tilde{N}(r), & t \geq 0, \end{cases} \tag{9}$$

where

$$\hat{s}(t) = \begin{cases} \xi(t), & -\tau \leq t \leq 0, \\ \displaystyle\sum_{n=0}^{[\frac{T}{h}]} s_n 1_{[nh,(n+1)h)}(t), & t \geq 0, \end{cases} \tag{10}$$

with $[\frac{T}{h}]$ as the integer part of $\frac{T}{h}$, $C_r(\hat{s}(r), \hat{s}(r-\tau)) = C_0(\hat{s}(r), \hat{s}(r-\tau))h + C_1(\hat{s}(r), \hat{s}(r-\tau)) |\Delta W(r)| + C_2(\hat{s}(r), \hat{s}(r-\tau)) \left|\Delta\tilde{N}(r)\right|$, in which C_0, C_1, C_2 are defined in (4), (5) and (6) and $\Delta W(r) = W(t_{k+1}) - W(t_k)$ and $\Delta\tilde{N}(r) = \tilde{N}(t_{k+1}) - \tilde{N}(t_k)$ for $r \in [t_k, t_{k+1})$. For simplicity of notation, we set $C_r := C_r(\hat{s}(r), \hat{s}(r-\tau))$.

It is easy to observe that $s(nh) = s_n$, so an error bound for $s(t)$ will automatically imply an error bound for s_n. Also, it is easy to obtain the following natural relationship

$$\sup_{0\leq t\leq T} \left|\hat{s}(t)\right| \leq \sup_{0\leq t\leq T} |s(t)|. \tag{11}$$

Now, we study the pth moment properties of the balanced method.

Theorem 2.4. *There exists a constant K_1, which is independent of h, such that*

$$E(\sup_{-\tau\leq t\leq T} |s(t)|^p) \leq K_1, \tag{12}$$

holds for $p > 2$, and

$$E|s(t)|^p \leq E[|s(t)|^3]^{\frac{p}{3}} \leq K_1^{\frac{p}{3}}, \tag{13}$$

holds for $0 < p \leq 2$.

Proof. Define the stopping time, for any $k > 0$,

$$\tau_k = T \wedge \inf\{t \geq 0, |s(t)| > k\}.$$

We set $\inf \emptyset = \infty$, where $\emptyset$ denotes the empty set. For any $t_1 \in [0, T]$, from the Hölder inequality and the Burkholder–Davis–Gundy inequality [37,38] and applying the fact that $\frac{1}{1+C_r} \leq 1$ and the relation (9), we conclude that there exist positive constants C_p and $C_{p,\beta}$ such that

$$\begin{aligned}
E(\sup_{0\leq t\leq t_1} |s(t \wedge \tau_k)|^p) &\leq 4^{p-1}\Bigg[E|\xi(0)|^p + \lambda^p T^{p-1} E\int_0^{t_1\wedge\tau_k} \left|\frac{(\mu - \hat{s}(r))}{1+C_r}\right|^p dr \\
&+ \sigma^p E\left[\sup_{0\leq t\leq t_1}\left|\int_0^{t\wedge\tau_k} \frac{\hat{s}(r-\tau)^\gamma\sqrt{\hat{s}(r)}}{1+C_r} dW(r)\right|^p\right] \\
&+ \delta^p E\left[\sup_{0\leq t\leq t_1}\left|\int_0^{t\wedge\tau_k} \frac{\hat{s}(r)}{1+C_r} d\tilde{N}(r)\right|^p\right]\Bigg] \\
&\leq 4^{p-1}\Bigg[E|\xi(0)|^p + \lambda^p T^{p-1} E\int_0^{t_1\wedge\tau_k} \left|\frac{(\mu - \hat{s}(r))}{1+C_r}\right|^p dr \\
&+ \sigma^p C_p E\left|\int_0^{t_1\wedge\tau_k} \frac{\hat{s}(r-\tau)^{2\gamma}\hat{s}(r)}{(1+C_r)^2} dr\right|^{\frac{p}{2}} \\
&+ \delta^p C_{p,\beta} E\left|\int_0^{t_1\wedge\tau_k} \frac{\hat{s}(r)^p}{(1+C_r)^p} dr\right|\Bigg] \\
&\leq 4^{p-1}\Bigg[E|\xi(0)|^p + \lambda^p T^{p-1} E\int_0^{t_1\wedge\tau_k} \left|\mu - \hat{s}(r)\right|^p dr \\
&+ \sigma^p C_p E\left|\int_0^{t_1\wedge\tau_k} \left|\hat{s}(r-\tau)^{2\gamma}\right| \left|\hat{s}(r)\right| dr\right|^{\frac{p}{2}} \\
&+ \delta^p C_{p,\beta} E\int_0^{t_1\wedge\tau_k} \left|\hat{s}(r)\right|^p dr\Bigg].
\end{aligned} \tag{14}$$

Then, following the proof of Lemma 3.1 in [1], the proof of the stated result for $p > 2$ is completed.

For the case $0 < p \leq 2$, the stated result follows directly from the Hölder inequality. □

Theorem 2.4 states that the pth moment of the numerical solution of the balanced method (3), is bounded for any $p > 0$.

Jiang et al. [1], showed that Eq. (1) is mean reversion as $t \to \infty$. Also, they proved that the Euler–Maruyama method keeps this property. In the following theorem we show that μ is also an upper bound for the mean of solution of the BIM (3), for every $h > 0$, when $n \to \infty$.

Theorem 2.5. *For the BIM (3) with control functions (4), (5) and (6), we have*

$$E(s_{n+1}) \leq (1 - \lambda h + C_0\lambda h^2)^n(E(\xi(0)) - \mu) + \mu + O(h^2 + h(1 + \epsilon^{-\frac{1}{2}})), \tag{15}$$

and hence for every $h > 0$, we have $E(s_n) \leq \mu$ as $n \to \infty$.

Proof. Taking expectation from the both sides of the BIM (3), and using $\frac{1}{1+C_n} \leq 1$, one can derive that

$$\begin{aligned}
E(s_{n+1}) &= E(s_n) + E(\frac{\lambda\mu h}{1+C_n}) - E\left(\frac{\lambda s_n h}{1+C_n}\right) + \sigma E\left(\frac{s_{n-m}^\gamma\sqrt{s_n}\Delta W_n}{1+C_n}\right) + \delta E\left(\frac{s_n\Delta\tilde{N}_n}{1+C_n}\right) \\
&\leq E(s_n) + \lambda\mu h - \lambda h E\left(\frac{s_n}{1+C_n}\right) + \sigma E\left(\frac{s_{n-m}^\gamma\sqrt{s_n}\Delta W_n}{1+C_n}\right) + \delta E\left(\frac{s_n\Delta\tilde{N}_n}{1+C_n}\right).
\end{aligned} \tag{16}$$

We then have

$$\begin{aligned}
-E(\frac{s_n}{1+C_n}) &= -E(s_n) + E(\frac{s_nC_n}{1+C_n}) \leq -E(s_n) + E(s_nC_n) \\
&= -E(s_n) + C_0\, hE(s_n) + \sigma E(\sqrt{s_n}s_{n-m}^\gamma\, |\Delta W_n|\, \mathbb{I}_{s_n\geq\epsilon}) \\
&+ \sigma\epsilon^{-\frac{1}{2}} E(s_n s_{n-m}^\gamma\, |\Delta W_n|\, \mathbb{I}_{s_n<\epsilon}) + C_2\, E\left(s_n\left|\Delta\tilde{N}_n\right|\right).
\end{aligned} \tag{17}$$

For every $\gamma > 0$, from Theorem 2.4 and the Hölder inequality, there exists a constant $U_1 > 0$ such that

$$E(\sqrt{s_n}s_{n-m}^\gamma) \leq E(s_n^3)^{\frac{1}{6}} E(s_{n-m}^{\frac{6\gamma}{5}})^{\frac{5}{6}} \leq U_1, \tag{18}$$

$$E(s_n s_{n-m}^{\gamma}) \le E(s_n^6)^{\frac{1}{6}} E(s_{n-m}^{\frac{6\gamma}{5}})^{\frac{5}{6}} \le U_1, \tag{19}$$

$$E\left(s_n \left|\Delta\tilde{N}_n\right|\right) \le E(s_n^2)^{\frac{1}{2}} E\left(\left|\Delta\tilde{N}_n\right|^2\right)^{\frac{1}{2}} \le U_1\sqrt{\beta h}. \tag{20}$$

We know that $E(|\Delta W_n|) = \sqrt{\frac{2h}{\pi}}$, also s_n and s_{n-m} are $\mathcal{F}_{t_n}$-measurable, so substituting inequalities (18), (19) and (20) in (17), we obtain

$$\begin{aligned} -E(\frac{s_n}{1+C_n}) &\le -E(s_n) + C_0\, hE(s_n) + \sigma\sqrt{\frac{2h}{\pi}} E(\sqrt{s_n} s_{n-m}^{\gamma} 1_{s_n>\epsilon}) \\ &\quad + \sigma\sqrt{\frac{2h}{\pi}}\epsilon^{-\frac{1}{2}} E(s_n s_{n-m}^{\gamma} 1_{s_n<\epsilon}) + C_2 U_1\sqrt{\beta h} \\ &\le -E(s_n) + C_0\, hE(s_n) + U_1\,\sigma\sqrt{\frac{2h}{\pi}}(1+\epsilon^{-\frac{1}{2}}) + C_2 U_1 \sqrt{\beta h}. \end{aligned} \tag{21}$$

From the Hölder inequality and $(\frac{1}{1+C_n})^2 \le 1$ and similar to the inequality (18), there exists a constant U_2 such that

$$E\left(\frac{s_{n-m}^{\gamma}\sqrt{s_n}}{1+C_n}\Delta W_n\right) \le E(s_{n-m}^{2\gamma} s_n)^{\frac{1}{2}} E\left((\frac{\Delta W_n}{1+C_n})^2\right)^{\frac{1}{2}} \le E(s_{n-m}^{2\gamma} s_n)^{\frac{1}{2}} E(\Delta W_n^2)^{\frac{1}{2}} \le U_2\, h^{\frac{1}{2}}, \tag{22}$$

$$E\left(\frac{s_n\Delta\tilde{N}_n}{1+C_n}\right) \le E(s_n^2)^{\frac{1}{2}} E\left((\frac{\Delta\tilde{N}_n}{1+C_n})^2\right)^{\frac{1}{2}} \le E(s_n^2)^{\frac{1}{2}} E(\Delta\tilde{N}_n^2)^{\frac{1}{2}} \le U_2\sqrt{\beta h}. \tag{23}$$

Now, inequalities (16), (21), (22) and (23) result

$$\begin{aligned} E(s_{n+1}) &\le E(s_n)(1-\lambda\, h + C_0\lambda h^2) + \lambda\mu h \\ &\quad + \lambda\, h\left(U_1\,\sigma\sqrt{\frac{2h}{\pi}}(1+\epsilon^{-\frac{1}{2}}) + C_2 U_1\sqrt{\beta h}\right) + \sigma\, U_2\, h^{\frac{1}{2}} + \delta\, U_2\sqrt{\beta h}. \end{aligned} \tag{24}$$

This establishes the inequality (15). □

3. Convergence analysis

In this section, we prove the convergence of the BIM by using suitable stopping times and uniformly boundedness of the moments of $S(t)$ and $s(t)$.

For any integer j, define the stopping times

$$u_j := \inf\{t \ge 0 : |S(t)| \ge j \text{ or } S(t) < \frac{1}{j}\}, \quad \nu_j := \inf\{t \ge 0 : |s(t)| \ge j \text{ or } s(t) < \frac{1}{j}\}, \quad \rho_j := u_j \wedge \nu_j,$$

and $\nu := t \wedge \rho_j$, for every $0 \le t \le T$.

Lemma 3.1. *For $h \in (0,1)$, there exist positive constants D_1 and D_2 such that*

$$E\left(\int_0^{\nu} \frac{C_r}{1+C_r} dr\right) \le (D_1 + D_2\epsilon^{-\frac{1}{2}})\, h^{\frac{1}{2}}. \tag{25}$$

Proof. We need the following version of (2), i.e., there exists a positive constant D_3, such that $E(\xi(r\wedge\rho_j - \tau)^{2\gamma}) \le D_3$, for $0 \le r\wedge\rho_j < \tau$.

According to the defined control functions in (4), (5) and (6), for $h \in (0,1)$, and $\frac{1}{1+C_r} \le 1$, we have

$$\begin{aligned} E\left(\int_0^{\nu} \frac{C_r}{1+C_r} dr\right) &\le E\int_0^{\nu} C_r dr \le \int_0^t E\left(C_{r\wedge\rho_j}\right) dr \le \int_0^T E(C_{r\wedge\rho_j}) dr \\ &= \int_0^T E\left(C_0\, h + \sigma\frac{\hat{s}(r\wedge\rho_j - \tau)^{\gamma}}{\sqrt{\hat{s}(r\wedge\rho_j)}}\left|\Delta W(r\wedge\rho_j)\right| 1_{\hat{s}(r\wedge\rho_j)\ge\epsilon}\right. \\ &\quad \left. + \sigma\frac{\hat{s}(r\wedge\rho_j-\tau)^{\gamma}}{\sqrt{\epsilon}}\left|\Delta W(r\wedge\rho_j)\right| 1_{\hat{s}(r\wedge\rho_j)<\epsilon} + C_2\left|\Delta\tilde{N}(r\wedge\rho_j)\right|\right) dr \end{aligned}$$

$$\leq \int_0^T E\Big(C_0\, h + \sigma j^{\gamma+\frac{1}{2}}\ \left|\Delta W(r\wedge\rho_j)\right| 1_{\{\hat{s}(r\wedge\rho_j)\geq\epsilon\wedge(r\wedge\rho_j\geq\tau)\}}$$
$$+\ \sigma j^{\frac{1}{2}}\xi(r\wedge\rho_j-\tau)^{\gamma}\ \left|\Delta W(r\wedge\rho_j)\right| 1_{\{\hat{s}(r\wedge\rho_j)\geq\epsilon\wedge(0\leq r\wedge\rho_j<\tau)\}}$$
$$+\ \sigma j^{\gamma}\,\epsilon^{-\frac{1}{2}}\ \left|\Delta W(r\wedge\rho_j)\right| 1_{\{\hat{s}(r\wedge\rho_j)<\epsilon\wedge(r\wedge\rho_j\geq\tau)\}})$$
$$+\ \sigma\epsilon^{-\frac{1}{2}}\xi(r\wedge\rho_j-\tau)^{\gamma}\left|\Delta W(r\wedge\rho_j)\right| 1_{\{\hat{s}(r\wedge\rho_j)<\epsilon\wedge(0\leq r\wedge\rho_j<\tau)\}}$$
$$+C_2\left|\Delta\tilde{N}(r\wedge\rho_j)\right|\Big)\,dr. \tag{26}$$

The Cauchy Schwarz inequality implies

$$E(\xi(r\wedge\rho_j-\tau)^{\gamma}\left|\Delta W(r\wedge\rho_j)\right|)\leq E(\xi(r\wedge\rho_j-\tau)^{2\gamma})^{\frac{1}{2}}\,E(\left|\Delta W(r\wedge\rho_j)\right|^2)^{\frac{1}{2}}\leq D_3^{\frac{1}{2}}\,h^{\frac{1}{2}}. \tag{27}$$

Then, using (27) in (26), we obtain

$$E\int_0^{\nu}\frac{C_r}{1+C_r}dr\leq T(C_0\,h+\sigma(j^{\gamma+\frac{1}{2}}+D_3^{\frac{1}{2}}\,j^{\frac{1}{2}})\,h^{\frac{1}{2}}+\sigma(D_3^{\frac{1}{2}}+j^{\gamma})\,\epsilon^{-\frac{1}{2}}\,h^{\frac{1}{2}}+C_2\sqrt{\beta}h^{\frac{1}{2}})$$
$$=:(D_1+D_2\epsilon^{-\frac{1}{2}})\,h^{\frac{1}{2}}.$$

□

Let $M_1:=(D_1+D_2\epsilon^{-\frac{1}{2}})$. We note that since $\frac{C_r}{1+C_r}\leq 1$, we also derive

$$E\int_0^{\nu}\left(\frac{C_r}{1+C_r}\right)^2dr\leq E\int_0^{\nu}\frac{C_r}{1+C_r}dr\leq M_1\,h^{\frac{1}{2}}. \tag{28}$$

Lemma 3.2. *Let $S(t)$ be the solution of Eq. (1). Then*

$$\lim_{h\to 0}(\sup_{0\leq t\leq T}E\left|S(t\wedge\rho_j)-s(t\wedge\rho_j)\right|)=0. \tag{29}$$

Proof. Let $a_0=1$ and $a_n=exp(-\frac{n(n+1)}{2})$ for $n\geq 1$, so that $\int_{a_n}^{a_{n-1}}\frac{du}{u}=n$. For each $n\geq 1$, there exists a continuous function $\psi_n(u)$ with support in (a_n,a_{n-1}), such that

$$0\leq\psi_n(u)\leq\frac{2}{nu}\quad\text{for}\ \ a_n<u<a_{n-1}$$

and $\int_{a_n}^{a_{n-1}}\psi_n(u)=1$. Define

$$\phi_n(x)=\int_0^{|x|}dy\int_0^{y}\psi_n(u)du. \tag{30}$$

Then $\phi_n\in C^2(\mathbb{R},\mathbb{R})$, $\phi_n(0)=0$, and for all $x\in\mathbb{R}$; $\left|\phi_n'(x)\right|\leq 1$, also for every $a_n<|x|<a_{n-1}$, $\left|\phi_n''(x)\right|\leq\frac{2}{n|x|}$ and otherwise $\left|\phi_n''(x)\right|=0$. One can easily observe that

$$|x|-a_{n-1}\leq\phi_n(x)\leq|x|\,. \tag{31}$$

Let $e(\nu):=S(\nu)-s(\nu)$. From (31), we have

$$E(|e(\nu)|)\leq a_{n-1}+E(\phi_n(e(\nu))). \tag{32}$$

Applying the Itô's formula, using $\frac{1}{1+C_r}\leq 1$ and definition of ϕ_n, we derive

$$E(\phi_n(e(\nu)))=\lambda\,\mu E\int_0^{\nu}\phi_n'(e(r))(\frac{C_r}{1+C_r})dr-(\lambda+\delta\beta)E\int_0^{\nu}\phi_n'(e(r))(S(r)-\frac{\hat{s}(r)}{1+C_r})dr$$
$$+\ \frac{\sigma^2}{2}E\int_0^{\nu}\phi_n''(e(r))\left[S^{\gamma}(r-\tau)\sqrt{S(r)}-\frac{\hat{s}^{\gamma}(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r}\right]^2dr$$
$$+\ \beta E\int_0^{\nu}[\phi_n((1+\delta)e(r))-\phi_n(e(r))]dr$$
$$\leq\lambda\,\mu E\int_0^{\nu}\frac{C_r}{1+C_r}dr+(\lambda+\delta\beta)E\int_0^{\nu}\left|S(r)-\frac{\hat{s}(r)}{1+C_r}\right|dr$$
$$+\ \frac{\sigma^2}{2}E\int_0^{\nu}\phi_n''(e(r))\left[S^{\gamma}(r-\tau)\sqrt{S(r)}-\frac{S^{\gamma}(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r}\right.$$

$$
\begin{aligned}
&+ \frac{S^{\gamma}(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r} - \frac{\hat{s}^{\gamma}(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r}\Bigg]^2 dr \\
&+ \delta\beta E\left(\sup_{x\in\mathbb{R}}\left|\phi_n'(x)\right| \int_0^{\nu} |e(r)|\,dr\right) \\
\le{}& \lambda\,\mu E\int_0^{\nu}\frac{C_r}{1+C_r}dr + (\lambda+\delta\beta)E\int_0^{\nu}\left(\left|S(r)-\frac{S(r)}{1+C_r}\right| + \left|\frac{S(r)}{1+C_r}-\frac{\hat{s}(r)}{1+C_r}\right|\right)dr \\
&+ \sigma^2 E\int_0^{\nu}\phi_n''(e(r))\left[S^{\gamma}(r-\tau)\sqrt{S(r)} - \frac{S^{\gamma}(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r}\right]^2 dr \\
&+ \sigma^2 E\int_0^{\nu}\phi_n''(e(r))\left[\frac{S^{\gamma}(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r} - \frac{\hat{s}^{\gamma}(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r}\right]^2 dr \\
&+ \delta\beta E\int_0^{\nu}|e(r)|\,dr \\
\le{}& (\lambda\mu+\lambda j+\delta\beta j)E\int_0^{\nu}\frac{C_r}{1+C_r}dr \\
&+ (\lambda+2\delta\beta)E\int_0^{\nu}|e(r)| + (\lambda+\delta\beta)E\int_0^{\nu}\left|s(r)-\hat{s}(r)\right|dr \\
&+ \sigma^2 j^{2\gamma}E\int_0^{\nu}\left|\phi_n''(e(r))\right|\left[\sqrt{S(r)} - \frac{\sqrt{\hat{s}(r)}}{1+C_r}\right]^2 dr \\
&+ \sigma^2 jE\int_0^{\nu}\left|\phi_n''(e(r))\right|\left[S^{\gamma}(r-\tau)-\hat{s}^{\gamma}(r-\tau)\right]^2 dr.
\end{aligned} \tag{33}
$$

Now, in order to bound the right-hand side of (33), for simplicity, we assign each term to J_1, J_2, J_3, J_4, J_5, respectively.

To bound the term J_3, by definition (9), for $r\in[0,\nu]$, we have

$$
s(r)-\hat{s}(r) = \frac{\lambda(\mu - s_{[\frac{r}{h}]})(r-[\frac{r}{h}]h)}{1+C_{[\frac{r}{h}]}} + \frac{\sigma s^{\gamma}_{([\frac{r}{h}]-N)}\sqrt{s_{[\frac{r}{h}]}}(W(r)-W([\frac{r}{h}]h))}{1+C_{[\frac{r}{h}]}} + \frac{\delta s_{[\frac{r}{h}]}(\tilde{N}(r)-\tilde{N}([\frac{r}{h}]h))}{1+C_{[\frac{r}{h}]}}. \tag{34}
$$

So, for every $h\in(0,1)$,

$$
\begin{aligned}
E\int_0^{\nu}\left|s(r)-\hat{s}(r)\right|dr &\le \lambda(\mu+j)hT + \sigma j^{\gamma+\frac{1}{2}}E\int_0^{\nu}\left|W(r)-W\left([\frac{r}{h}]h\right)\right|dr \\
&\quad + \delta jE\int_0^{\nu}\left|\tilde{N}(r)-\tilde{N}([\frac{r}{h}]h)\right|dr \\
&\le \lambda(\mu+j)hT + \sigma j^{\gamma+\frac{1}{2}}\int_0^{T}E\left|W(r\wedge\rho_j)-W\left([\frac{r\wedge\rho_j}{h}]h\right)\right|dr \\
&\quad + \delta jE\int_0^{T}\left|\tilde{N}(r\wedge\rho_j)-\tilde{N}\left([\frac{r\wedge\rho_j}{h}]h\right)\right|dr \\
&\le \lambda(\mu+j)hT + \sigma Tj^{\gamma+\frac{1}{2}}h^{\frac{1}{2}} + \delta jT\sqrt{\beta}h^{\frac{1}{2}} =: Dh^{\frac{1}{2}}.
\end{aligned} \tag{35}
$$

To bound J_4, from definition of ϕ_n and using Lemma 3.1 and inequality (35), we obtain

$$
\begin{aligned}
E\int_0^{\nu}\left|\phi_n''(e(r))\right|\left[\sqrt{S(r)}-\frac{\sqrt{\hat{s}(r)}}{1+C_r}\right]^2 dr &= E\int_0^{\nu}\left|\phi_n''(e(r))\right|\left[\sqrt{S(r)}-\sqrt{\hat{s}(r)}+\sqrt{\hat{s}(r)}-\frac{\sqrt{\hat{s}(r)}}{1+C_r}\right]^2 dr \\
&\le 2E\int_0^{\nu}\left|\phi_n''(e(r))\right|\left[\sqrt{S(r)}-\sqrt{\hat{s}(r)}\right]^2 dr + 2E\int_0^{\nu}\left|\phi_n''(e(r))\right|\left[\sqrt{\hat{s}(r)}-\frac{\sqrt{\hat{s}(r)}}{1+C_r}\right]^2 dr \\
&\le 2E\int_0^{\nu}\left|\phi_n''(e(r))\right|\left|S(r)-\hat{s}(r)\right|dr + 2jE\int_0^{\nu}\left|\phi_n''(e(r))\right|\left(\frac{C_r}{1+C_r}\right)^2 dr \\
&\le 2E\int_0^{\nu}\left|\phi_n''(e(r))\right|\left|S(r)-\hat{s}(r)\right|dr + \frac{4j}{na_n}E\int_0^{\nu}\left(\frac{C_r}{1+C_r}\right)^2 dr \\
&\le 2E\int_0^{\nu}\frac{2}{n}dr + \frac{4}{na_n}E\int_0^{\nu}\left|s(r)-\hat{s}(r)\right|dr + \frac{4j}{na_n}M_1h^{\frac{1}{2}}
\end{aligned}
$$

$$\leq \frac{4T}{n} + \frac{4}{na_n}(jM_1 + D)h^{\frac{1}{2}}. \tag{36}$$

Now considering J_5 and definition of ϕ_n,

$$\begin{aligned} E\int_0^{\nu} &\left|\phi_n''(e(r))\right| \left[S^{\gamma}(r-\tau) - \hat{s}^{\gamma}(r-\tau)\right]^2 dr \\ &\leq C_j E\int_0^{\nu} \left|\phi_n''(e(r))\right| \left[S(r-\tau) - \hat{s}(r-\tau)\right]^{2\gamma} dr \\ &\leq \frac{2C_j\bar{C}}{na_n} E\int_0^{\nu} \left[s(r-\tau) - \hat{s}(r-\tau)\right]^{2\gamma} dr + \frac{2C_j\bar{C}}{na_n} E\int_0^{\nu} |e(r-\tau)|^{2\gamma} dr, \end{aligned} \tag{37}$$

where

$$\bar{C} = \begin{cases} 1, & 0 < \gamma \leq \frac{1}{2}, \\ 2^{2\gamma-1}, & \gamma > \frac{1}{2}, \end{cases}$$

and

$$C_j = \begin{cases} \bar{C}_j^2, & \gamma > 1, \\ 1, & \text{otherwise}. \end{cases}$$

The last inequality is true due to the following fact. For $x, y > 0$ and $\theta \in (0, 1]$,

$$\left|x^{\theta} - y^{\theta}\right| \leq |x - y|^{\theta},$$

and for $|x| \leq j$, $|y| < j$ and $\theta > 1$,

$$\left|x^{\theta} - y^{\theta}\right| \leq \bar{C}_j |x - y|^{\theta}.$$

Here $\bar{C}_j$, is a constant depending on j. Substituting (25), (35), (36) and (37), in (33), we derive

$$\begin{aligned} E(\phi_n(e(\nu))) \leq \sigma^2 j &\left(\frac{2C_j\bar{C}}{na_n} E\int_0^{\nu} \left[s(r-\tau) - \hat{s}(r-\tau)\right]^{2\gamma} dr + \frac{2C_j\bar{C}}{na_n} E\int_0^{\nu} |e(r-\tau)|^{2\gamma} dr\right) \\ &+ \sigma^2 j^{2\gamma}\left(\frac{4T}{n} + \frac{4j}{na_n} M_1 h^{\frac{1}{2}}\right) + (\lambda + 2\delta\beta) E\int_0^{\nu} |e(r)|\, dr \\ &+ (\lambda\mu + \lambda j + \delta\beta j) M_1 h^{\frac{1}{2}} + (\lambda + \delta\beta) D\, h^{\frac{1}{2}} + D\sigma^2 j^{2\gamma} \frac{4}{na_n} h^{\frac{1}{2}}. \end{aligned}$$

From (32), we then obtain

$$\begin{aligned} E\,|e(\nu)| &\leq a_{n-1} + \sigma^2 j^{2\gamma}\left(\frac{4T}{n} + \frac{4j}{na_n} M_1 h^{\frac{1}{2}}\right) + (\lambda\mu + \lambda j + \delta\beta j) M_1 h^{\frac{1}{2}} \\ &\quad + (\lambda + \delta\beta) D h^{\frac{1}{2}} + D\sigma^2 j^{2\gamma} \frac{4}{na_n} h^{\frac{1}{2}} + (\lambda + 2\delta\beta) E\int_0^{\nu} |e(r)|\, dr \\ &\quad + \sigma^2 j \left(\frac{2C_j\bar{C}}{na_n} E\int_0^{\nu} \left[s(r-\tau) - \hat{s}(r-\tau)\right]^{2\gamma} dr + \frac{2C_j\bar{C}}{na_n} E\int_0^{\nu} |e(r-\tau)|^{2\gamma} dr\right) \\ &=: a_{n-1} + \frac{\alpha_1}{n} + \left(\alpha_2 + \frac{\alpha_3}{na_n} + \alpha_4 \epsilon^{-\frac{1}{2}}\right) h^{\frac{1}{2}} + \frac{\alpha_5}{na_n} E\int_0^{\nu} \left[s(r-\tau) - \hat{s}(r-\tau)\right]^{2\gamma} dr \\ &\quad + \frac{\alpha_6}{na_n} E\int_0^{\nu} |e(r-\tau)|^{2\gamma} dr + (\lambda + 2\delta\beta) E\int_0^{\nu} |e(r)|\, dr, \end{aligned} \tag{38}$$

where α_i, for $i = 1, \ldots, 6$ are independent of n,. Following the proof of Lemma 4.2 in [1] (right after Eq. (22)), when $\gamma \geq \frac{1}{2}$, for any given $\epsilon_0 > 0$, we choose $n_{[T/\tau]+1} \geq 1$ such that

$$\left(a_{n_{[T/\tau]+1}-1} + \frac{(\alpha_1)_{[T/\tau]+1}}{n_{[T/\tau]+1}}\right) e^{(\lambda+2\beta\delta)T} < \frac{\epsilon_0}{2},$$

and then choose $\epsilon > 0$ and $h \in (0, 1)$ such that

$$\left((\alpha_2)_{[T/\tau]+1} + \frac{(\alpha_3)_{[T/\tau]+1}}{n_{[T/\tau]+1} a_{n_{[T/\tau]+1}}} + (\alpha_4)_{[T/\tau]+1} \epsilon^{-\frac{1}{2}}\right) h^{\frac{1}{2}} e^{(\lambda+2\beta\delta)T} < \frac{\epsilon_0}{2},$$

and when $\gamma \in (0, \frac{1}{2})$ we can choose ϵ, h such that

$$\left((\alpha_2)_{[T/\tau]+1} + \frac{(\alpha_3)_{[T/\tau]+1}}{n_{[T/\tau]+1}a_{n_{[T/\tau]+1}}} + (\alpha_4)_{[T/\tau]+1}\epsilon^{-\frac{1}{2}}\right)h^{\frac{(2\gamma)^{[T/\tau]}}{2}}e^{(\lambda+2\beta\delta)T} < \frac{\epsilon_0}{2}.$$

This completes the proof. □

Lemma 3.3. *For the stopping times u_j, ν_j and ρ_j, we have*

$$\lim_{h\to 0} E\left(\sup_{0\le t\le T}\left|S(t\wedge\rho_j) - s(t\wedge\rho_j)\right|^2\right) = 0. \tag{39}$$

Proof. From (9) and the Hölder inequality, we get

$$\begin{aligned}(S(\nu)-s(\nu))^2 \le\ & 4T\lambda^2\mu^2\int_0^\nu\left(1-\frac{1}{1+C_r}\right)^2 dr + 4T\lambda^2\int_0^\nu\left(S(r)-\frac{\hat{s}(r)}{1+C_r}\right)^2 dr\\ &+ 4\sigma^2\left[\int_0^\nu\left(S(r-\tau)^\gamma\sqrt{S(r)} - \frac{\hat{s}^\gamma(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r}\right)dW(r)\right]^2\\ &+ 4\delta^2\left[\int_0^\nu\left(S(r^-) - \frac{\hat{s}(r^-)}{1+C_r}\right)d\tilde{N}(r)\right]^2.\end{aligned} \tag{40}$$

Let $\nu_1 = t_1\wedge\rho_j$. By the Doob martingale inequality [39], for every $t_1\in[0,T]$, we have

$$\begin{aligned}&E\left(\sup_{0\le t\le t_1}\left|S(t\wedge\rho_j)-s(t\wedge\rho_j)\right|^2\right)\\ &\le 4T\lambda^2\mu^2 E\int_0^{\nu_1}\left(1-\frac{1}{1+C_r}\right)^2 dr + (4T\lambda^2+16\beta\delta^2)E\int_0^{\nu_1}\left(S(r)-\frac{\hat{s}(r)}{1+C_r}\right)^2 dr\\ &+ 16\sigma^2 E\int_0^{\nu_1}\left[S(r-\tau)^\gamma\sqrt{S(r)} - \frac{\hat{s}^\gamma(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r}\right]^2 dr.\end{aligned} \tag{41}$$

On the other hand, from (28), $\frac{1}{1+C_r}\le 1$ and (35), we have

$$\begin{aligned}E\int_0^{\nu_1}\left(S(r)-\frac{\hat{s}(r)}{1+C_r}\right)^2 dr &\le 2E\int_0^{\nu_1}\left(S(r)-\frac{S(r)}{1+C_r}\right)^2 dr + 2E\int_0^{\nu_1}\left(\frac{S(r)}{1+C_r}-\frac{\hat{s}(r)}{1+C_r}\right)^2 dr\\ &\le 2j^2E\int_0^{\nu_1}\left(\frac{C_r}{1+C_r}\right)^2 dr + 4E\int_0^{\nu_1}(S(r)-s(r))^2 dr + 4E\int_0^{\nu_1}(s(r)-\hat{s}(r))^2 dr\\ &\le 2j^2M_1h^{\frac{1}{2}} + 8jE\int_0^{\nu_1}\left|s(r)-\hat{s}(r)\right|dr + 4E\int_0^{\nu_1}(S(r)-s(r))^2 dr\\ &\le 2j^2M_1h^{\frac{1}{2}} + 8jDh^{\frac{1}{2}} + 4E\int_0^{\nu_1}(S(r)-s(r))^2 dr.\end{aligned} \tag{42}$$

Similarly, from (35) and the inequality $\frac{1}{1+C_r}\le 1$, we also get

$$\begin{aligned}&E\int_0^{\nu_1}\left[S(r-\tau)^\gamma\sqrt{S(r)} - \frac{\hat{s}^\gamma(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r}\right]^2 dr\\ &= E\int_0^{\nu_1}\left[S^\gamma(r-\tau)\sqrt{S(r)} - \frac{S^\gamma(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r} + \frac{S^\gamma(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r} - \frac{\hat{s}^\gamma(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r}\right]^2 dr\\ &\le 2E\int_0^{\nu_1}\left[S^\gamma(r-\tau)\sqrt{S(r)} - \frac{S^\gamma(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r}\right]^2 dr\\ &+ 2E\int_0^{\nu_1}\left[\frac{S^\gamma(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r} - \frac{\hat{s}^\gamma(r-\tau)\sqrt{\hat{s}(r)}}{1+C_r}\right]^2 dr\\ &\le 2j^{2\gamma}E\int_0^{\nu_1}\left[\sqrt{S(r)} - \frac{\sqrt{\hat{s}(r)}}{1+C_r}\right]^2 dr + 2jE\int_0^{\nu_1}(S^\gamma(r-\tau)-\hat{s}(r-\tau)^\gamma)^2 dr\end{aligned}$$

$$
\begin{aligned}
&\leq 2j^{2\gamma} E \int_0^{\nu_1} \left[\sqrt{S(r)} - \sqrt{\hat{s}(r)} + \sqrt{\hat{s}(r)} - \frac{\sqrt{\hat{s}(r)}}{1+C_r} \right]^2 dr \\
&\quad + 2jC_j E \int_0^{\nu_1} \left| S(r-\tau) - \hat{s}(r-\tau) \right|^{2\gamma} dr \\
&\leq 4j^{2\gamma} E \int_0^{\nu_1} \left| S(r) - \hat{s}(r) \right| dr + 4j^{2\gamma+1} M_1 h^{\frac{1}{2}} + 2jC_j E \int_0^{\nu_1} \left| S(r) - \hat{s}(r) \right|^{2\gamma} dr \\
&\leq 4j^{2\gamma} E \int_0^{\nu_1} |S(r) - s(r)| \, dr + 4(D + M_1 j) j^{2\gamma} h^{\frac{1}{2}} + 2jC_j E \int_0^{\nu_1} |S(r) - s(r)|^{2\gamma} dr, \qquad (43)
\end{aligned}
$$

where $M_1 = (D_1 + D_2 \epsilon^{-\frac{1}{2}})$. Then, applying (41), (42), (43) and following the proof of Lemma 4.3 in [1] (right after Eq. (40)), with the same way as the end of the proof of Lemma 3.2, the proof is completed. □

Theorem 3.4. *Let $S(t)$ be the solution of* Eq. (1)*. Then the numerical solution* (9) *converges to $S(t)$, that is,*

$$
\lim_{h \to 0} E \left(\sup_{0 \leq t \leq T} |S(t) - s(t)|^2 \right) = 0. \qquad (44)
$$

Proof. By Theorem 2.4 and Lemma 3.3 and in the same way as Theorem 4.1 in [1] the conclusion follows. □

4. Numerical examples

In this section, we illustrate some numerical examples that confirm the results in the previous sections. Also, by the convergence theory in Section 3, we show that the BIM can be used to compute some financial quantities.

Consider the delay CIR model with jump

$$
\begin{cases} dS(t) = \lambda(\mu - S(t))dt + \sigma S(t-1)^{\gamma} \sqrt{S(t)} dW(t) + \delta S(t^-) d\tilde{N}(t), & t \geq 0, \\ S(t) = 1, \quad t \in [-1, 0]. \end{cases} \qquad (45)
$$

We consider the two following examples.

Example 1. $\lambda = 5, \mu = 0.5, \sigma = 1.5, \gamma = 0.5, \delta = 1, \beta = 2.$

Example 2. $\lambda = 100, \mu = 5, \sigma = 2, \gamma = 1, \delta = 2, \beta = 4.$

Figs. 1–4 show the values $S(t)$ vs. t for Examples 1 and 2 produced by the balanced method and Euler method in [1], with ten solution paths for $\epsilon = 0.01$. From these figures it can be observed that the balanced method preserves non-negativity of the solution even for the large step size $h = 0.5$, while the Euler method does not preserve this property even for the small step size $h = 0.01$. In Figs. 5, 6, we apply the BIM to Examples 1 and 2. We estimates the rate of convergence by drawing the strong error at the endpoint $T = 1$, $e_h^{strong} := E|S(T) - s_T|^2$. We plot e_h^{strong} against h on a log–log scale. A reference line of slope $\frac{1}{2}$ is also given. Since we do not have an explicit solution of Examples 1 and 2, we take the BIM with step size$h = 2^{-14}$ as a reference solution. For showing the convergence, we compare the reference solution with the BIM evaluated with $2^{2i-1} h$, $i = 1, 2, 3, 4, 5$. We compute 5000 different solution paths. Also, we apply the Euler method for Examples 1 and 2, for comparison purpose. From these figures it can be seen that, the rate of convergence of the balanced method is better than the Euler method for both of examples. Figs. 7–10, show the values of $E(S(t))$ and $E(S(t)^2)$ vs. t for Examples 1 and 2, by the BIM with step size $h = 0.1$ and with 1000 solution paths. Figs. 7 and 9, show $\lim_{n \to \infty} E(s_n) \leq \mu$; similarly, Figs. 8 and 10, show $E(S(t)^2)$ is bounded, confirming the results of Theorems 2.4, 2.5.

Now, we use bonds and barrier options to show our results.

Example 3 (*Bonds*). In the case where the SDDE with jump (1) describes short-term interest rate dynamics, the price of a bond at the end of period is given by

$$
B(T) := E \left[exp \left(- \int_0^T S(t) dt \right) \right].
$$

By the step process $\hat{s}(t)$ in (10), a natural approximation to compute $B(T)$ is

$$
\hat{B}_h(T) := E \left[exp \left(- \int_0^T \hat{s}(t) dt \right) \right].
$$

We have

$$
\lim_{h \to 0} |B(T) - \hat{B}_h(T)| = 0.
$$

The proof is the similar to that of Theorem 4.1 in [40].

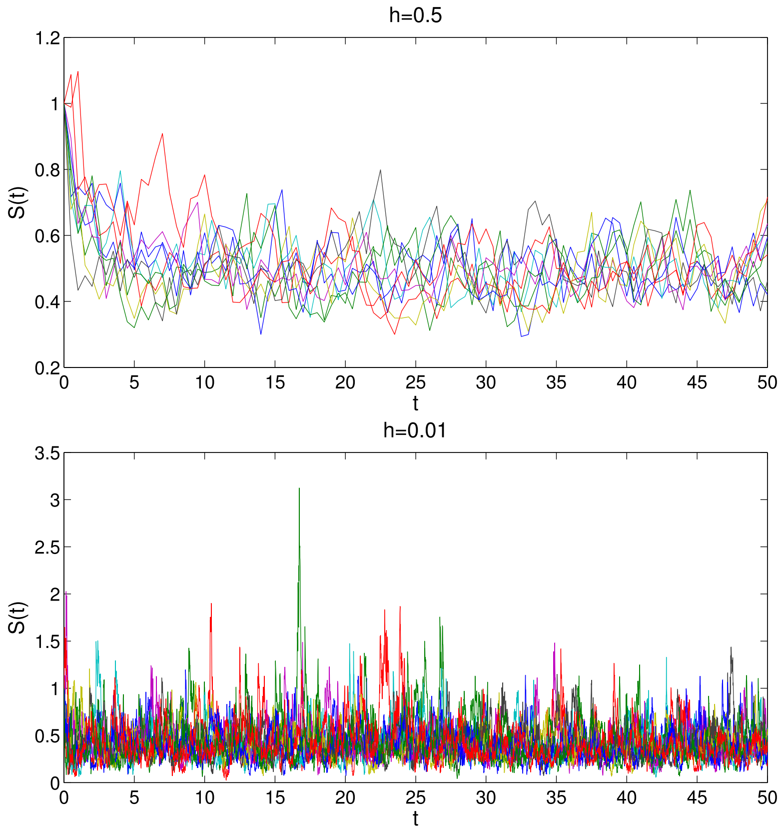

Fig. 1. Ten solution paths of Example 1, approximated by BIM with $C_0 = 10$, $C_2 = 1$ and $\epsilon = 0.01$.

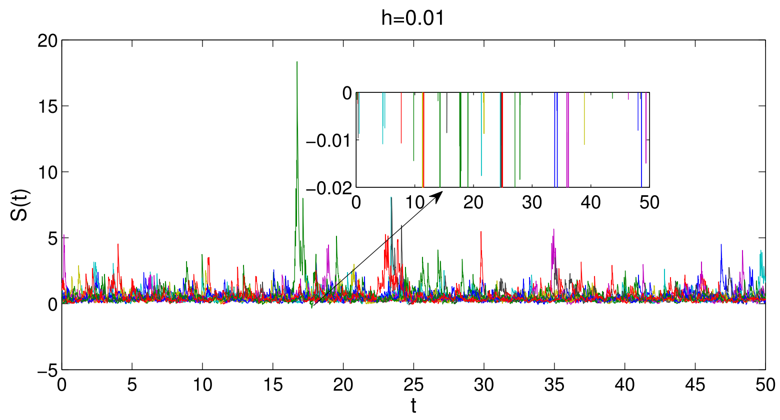

Fig. 2. Ten solution paths of Example 1, approximated by Euler method.

Example 4 (*A Path Dependent Option*)**.** Let $S(t)$ be the solution of Eq. (1) and $\hat{s}(t)$ be the BIM process defined by (10). We consider an up-and-out call option with the expiry time T, the exercise price K and the fixed barrier B. Payoff of this option at expiry time T is $(S(T)-K)^+$, if $S(t)$ never exceeds the fixed barrier B and is zero otherwise. We suppose that the expected payoff is computed from (10). Define

$$V := E[(S(T)-K)^+ 1_{0\le S(t)\le B, 0\le t\le T}];$$

and

$$\hat{V}_h := E[(\hat{s}(T)-K)^+ 1_{0\le \hat{s}(t)\le B, 0\le t\le T}];$$

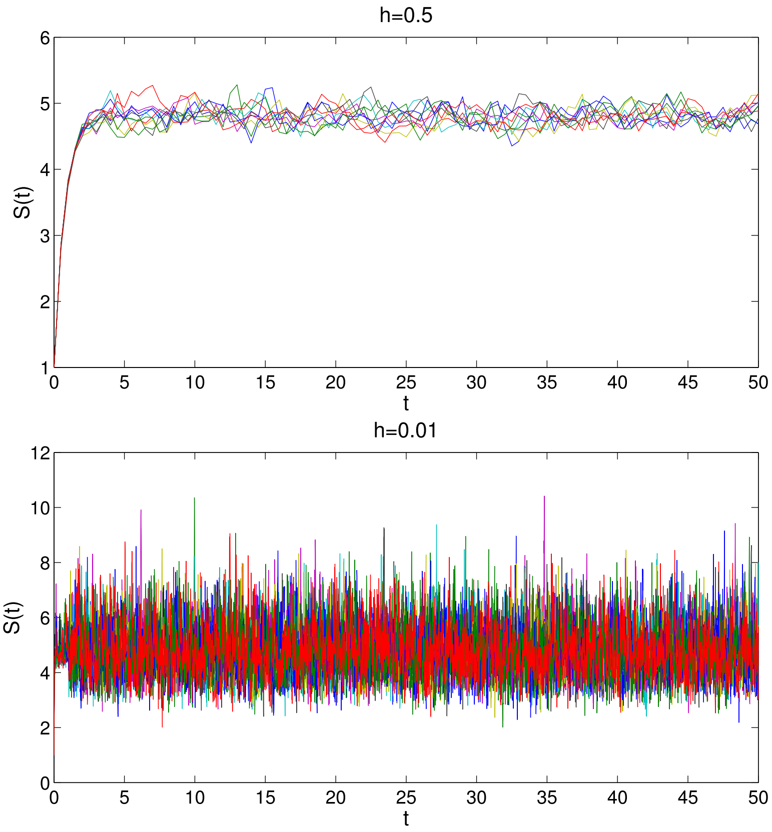

Fig. 3. Ten solution paths of Example 2, approximated by BIM with $C_0 = 200,\ C_2 = 5$ and $\epsilon = 0.01$.

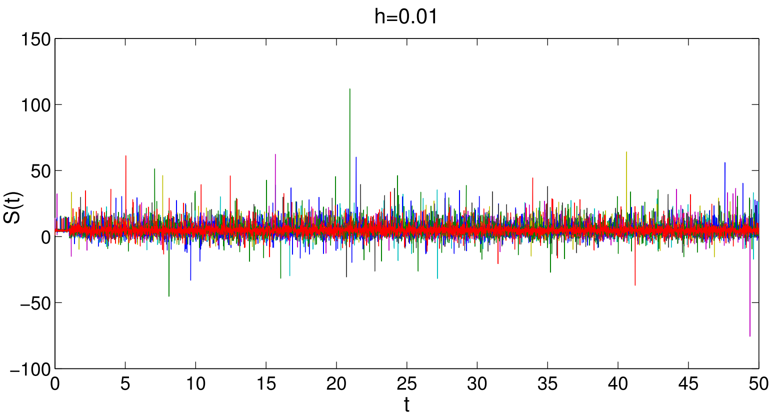

Fig. 4. Ten solution paths of Example 2, approximated by Euler method.

where K and B are constants. We have

$$\lim_{h\to 0} |V - \hat{V}_h| = 0.$$

The proof is the same to that of Theorem 5.1 in [40].

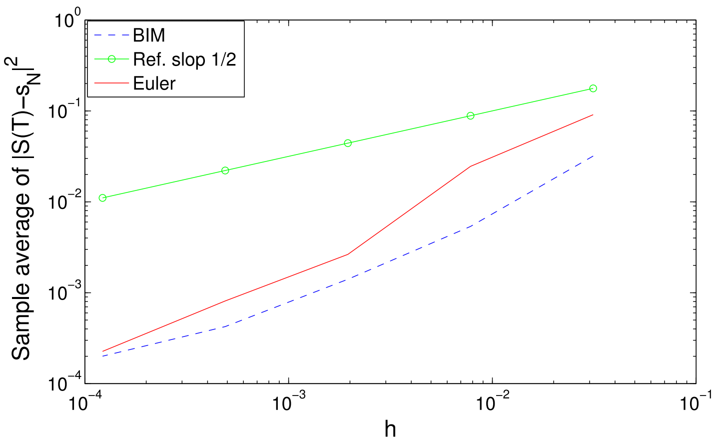

Fig. 5. Strong error of the BIM with $C_0 = 10,\ C_2 = 1$ and $\epsilon = 0.01$, applied to Example 1.

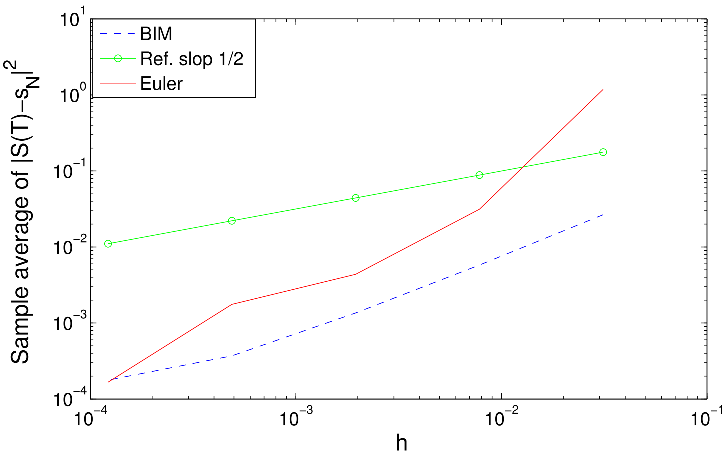

Fig. 6. Strong error of the BIM with $C_0 = 200,\ C_2 = 5$ and $\epsilon = 0.01$, applied to Example 2.

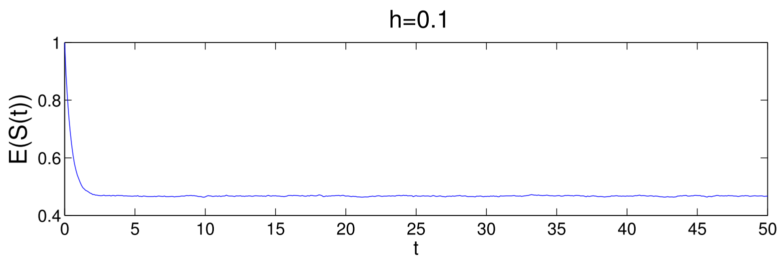

Fig. 7. $E(S(t))$ vs. t of the BIM with $C_0 = 10,\ C_2 = 1$ and $\epsilon = 0.01$ with step size $h = 0.1$, for Example 1.

5. Conclusions

In this work, we have demonstrated convergence and non-negativity properties of the numerical solution obtained by the BIM for delay CIR model with jump. First, we have chosen control functions of the BIM such that this method can preserve non-negativity of solution of the model. Then, we have studied the moments boundedness and convergence of the solution of BIM by the determined control functions. Some numerical experiments have been included which illustrate the theoretical results obtained in this paper.

In the future, we would like to obtain order of convergence of the BIM for the jump CIR model with delay. We also would like to obtain optimal ϵ in numerical experiments.

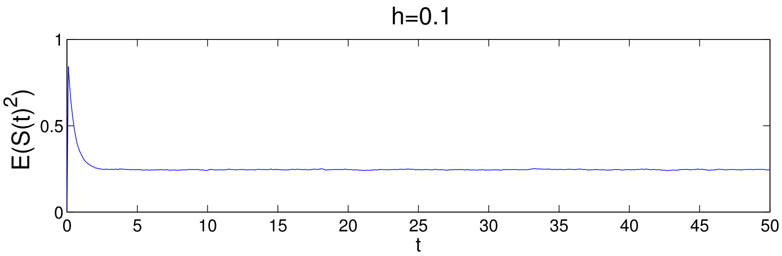

Fig. 8. $E(S(t)^2)$ vs. t of the BIM with $C_0 = 10$, $C_2 = 1$ and $\epsilon = 0.01$ with step size $h = 0.1$, for Example 1.

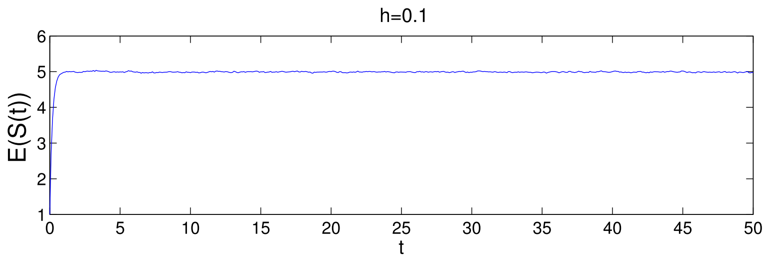

Fig. 9. $E(S(t))$ vs. t of the BIM with $C_0 = 200$, $C_2 = 5$ and $\epsilon = 0.01$ with step size $h = 0.1$, for Example 2.

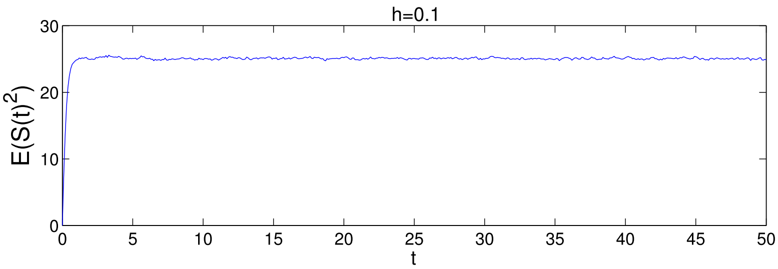

Fig. 10. $E(S(t)^2)$ vs. t of the BIM with $C_0 = 200$, $C_2 = 5$ and $\epsilon = 0.01$ with step size $h = 0.1$, for Example 2.

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