



A simple numerical solution for an optimal investment strategy for a DC pension plan in a jump diffusion model

Walter Mudzimbabwe

University of Zimbabwe, P. O Box MP 167, Mount Pleasant, Harare, Zimbabwe



ARTICLE INFO

Article history:

Received 31 October 2017

Received in revised form 21 January 2019

MSC:

49L20

49K15

45J05

Keywords:

Stochastic control

Hamilton–Jacobi–Bellman equation

DC pension plan

Jump diffusion

Integro-differential equation

Bisection method

ABSTRACT

In this paper, we study an optimal investment strategy for a fund manager of a DC pension who wants to maximise the expected exponential utility of the terminal wealth in a market where the stock is a jump diffusion process. Using stochastic control theory, we derive a Hamilton–Jacobi–Bellman equation. Since the market is not complete, the optimal strategy cannot be found in closed as is done in most literature on DC pension plans. We characterise the optimal strategy as a solution of an integro-ordinary differential equation which can easily be solved by a simple numerical method. We investigate the impact of different jump parameters through numerical experiments using a familiar distribution of jumps.

© 2019 Elsevier B.V. All rights reserved.

1. Introduction

There are mainly two forms of pension plans, defined benefit (DB) and defined contribution plans (DC). In defined benefit pension schemes, the benefit is fixed in advance and contributions are adjusted along the way to balance the fund and ensure benefit is met whereas in defined contribution plans the contribution is fixed in advance and the benefit depends on the evolution of the fund. In a DB plan, the investment risk is borne by the sponsor and in DC plans, the risk is shifted to pension members. Due to the development of equity markets, the DC pension plan has become more popular.

In this paper, we consider the problem of optimal investment of the wealth from contributions in a market with a riskless and risky asset. We assume that the pension members are interested in maximising utility of terminal wealth. Several authors have considered a similar problem. In [1], this problem is solved under the constraint that terminal wealth should exceed a minimum guarantee in the presence of stochastic interest rates. In [2], the results are extended to include inflation risk where pension planner can invest in indexed bonds and in [3] they use stochastic dynamic theory in a stochastic volatility framework.

In [4], a stochastic control problem for DC pension is analysed where interest follows generalised Cox and Vasicek processes. They find a nonlinear Hamilton–Jacobi–Bellman equation which they transform to a linear PDE using Legendre transform to find closed form solutions for the investment strategy. In [5], a general utility function is used and the problem is solved using stochastic dynamic approach.

Some authors assume that pension members are interested in maximising return and minimising volatility of fund, i.e., they formulate the DC pension problem as a mean–variance optimisation. In [6], they extend the model in [3] as mean–variance optimisation problem and use stochastic control methods to determine optimal strategies. In [7], a DC pension

E-mail address: mudzmudz@gmail.com.

with return of premiums clauses is studied as a mean–variance control problem. Closed form solutions are found using stochastic control methods. In [8], a mean–variance approach is compared with an expected utility of terminal wealth problem. In [9], time-consistent strategies are found using game theoretic methods since the mean–variance problem cannot be solved using the usual stochastic control methods.

We note that most of the literature above assumes that the market is complete and so the martingale method can be used [1,2]. In this paper we extend the analysis and model the risky asset as a jump diffusion process. In [10], a jump diffusion stock is included in a mean–variance optimal control problem and a time-consistent investment strategy is found using game theoretic methods by extending the Hamilton–Jacobi–Bellman principle. In the present paper we use the usual stochastic control methods and characterise our strategies as solution to integro-differential equations. Most of the literature on DC pension plans mainly seeks to find analytic strategies, [2–4,6,10,11] etc. The methods in the aforementioned papers work for specific situations by taking advantage of linearity of PDEs in [10] or existence of replicating assets in the market e.g., in [3] or the form of utility functions e.g., in [4]. Our method provides a new solution to an extension of Gao paper where we consider exponential utility. The dual method of Gao would not be applicable in the present paper. For more general cases, the above methods may fail. Our main contribution is to solve a general problem using a numerical method. The other added contribution is a detailed analysis of the impact jump parameters on the optimal strategy. In this paper, we consider an example of a typical jump distribution and analyse the impact of jump size, likelihood of jumps and the nature of jumps themselves i.e. positive or negative jumps. This analysis is not provided by some authors see Garrido and Vázquez [12], Courtois and Menoncin [13] or Nkeki [14] for example. In [12], pricing of pension plans is considered but impact of jump parameters is not shown precisely even though a normal distribution for jumps is modelled and in [13], a Levy process is considered with examples but the sensitivity analysis of parameters is not given. In [14], a jump diffusion model for DC pensions is considered but example of jump distribution is not shown with the author focusing on effect of mesh size on price of plan. This paper also provides an alternative to Monte Carlo methods that are increasingly being used for optimal control problems see Ma et al. [15], Rogers [16], Henry–Labordère et al. [17] etc. Together with the Monte Carlo methods, our method is able to handle more general problems with complex utility functions [15] in our case we have an exponential utility.

Our method is not without its own shortcomings. Since it is based on the bisection method used in root finding, if the function has a singularity, the method may fail. Also, if the function has many roots, the method can only converge to one of them. This implies that the method is sensitive to initial guess. In this paper, we consider double exponential functions where these issues can be avoided.

The paper is organised as follows. In Section 2, we introduce the dynamics of the assets available for investment and deduce the fund process. In Section 3, we present the stochastic control problem and a verification theorem. Section 4 provides the solution of our control problem together with an existence result for the optimal strategy. Section 5 provides numerical experiments and sensitivity analysis.

2. Market model

The market model in this contribution is a special case of the model introduced in [10]. As in [10], we assume that all processes and random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The market has risky and riskfree assets available to the pension planner to invest in. The riskfree asset whose value $S_0(t)$ at t is given by

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1, \quad (1)$$

where $r > 0$ is the riskfree interest. The price $S_1(t)$ of the risky asset is given by the jump diffusion process

$$dS_1(t) = S_1(t-) \left(\mu dt + \sigma dW(t) + d \sum_{i=1}^{N(t)} J_i \right), \quad S_1(0) = 1, \quad (2)$$

where $W(t)$ is a standard Brownian motion, μ and σ are positive constants $N(t)$ is a homogeneous Poisson process with intensity λ , J is the size of jump drawn from a distribution with mean $\mu_J = E(J)$. We assume that $W(t)$, $N(t)$ and J are mutually independent.

The above dynamics refer to a special case of Sun et al. [10], where r , μ and σ are deterministic functions of t . We have chosen these parameters to be constant but the method used here can be extended to one in [10]. One reason for this is that we want to focus on the impact of jumps. As we will see later, Eq. (15) can be adapted to these functions, e.g. $b(t) = -\gamma \exp\left(\int_t^T r(s)ds\right)$ in this case, by following the argument leading to that equation.

However, if r , μ and σ are stochastic as the case of Gao [4], then the resulting HJB equation becomes complicated and at this stage we have not found a way to solve it.

We assume that the pension planner invests $\pi(t)$ in the risky asset and the rest in the riskfree asset. In addition, we assume pension members contribute premiums at the rate of $c(t)$ at time t . To simplify the analysis, we also assume that the contribution is non-random. The dynamics of the wealth are therefore given by (Gao [4]),

$$dX(t) = \pi(t) \frac{dS_1(t)}{S_1(t)} + (X(t) - \pi(t)) \frac{dS_0(t)}{S_0(t)} + c(t)dt.$$

The dynamics of wealth can be simplified to get

$$dX(t) = (rX(t) + (\mu - r)\pi(t) + c(t))dt + \pi(t)\sigma dW(t) + \pi(t)d\sum_{i=1}^{N(t)} J_i. \quad (3)$$

3. Optimal control problem

We assume a standard Merton problem where the pension member is interested in maximising terminal utility of the wealth since the wealth provides age-old pension to members upon retirement. We assume the utility function is exponential

$$u(x) = -\frac{1}{\gamma}e^{-\gamma x},$$

where $\gamma > 0$ is the coefficient of risk aversion. This form of utility is popular since it provides closed form solution due to its scaling properties [18–20]. The problem can be written as

$$\begin{cases} \max_{\pi \in \Pi} E[u(X(T))], \\ \text{s.t } dX(t) = (rX(t) + (\mu - r)\pi(t) + c(t))dt + \pi(t)\sigma dW(t) \\ + \pi(t)d\sum_{i=1}^{N(t)} J_i. \end{cases} \quad (4)$$

The value function is defined as follows

$$H(t, x) = \sup_{\pi \in \Pi} E[u(X(T)) | X(t) = x], \quad 0 \leq t \leq T. \quad (5)$$

The following theorem (see [21]) provides a Hamilton–Jacobi–Bellman (HJB) equation which characterises a solution to the Merton problem.

Theorem 1. Assume H in (5) is such that $H(t, x) \in C^{1,2}([0, T] \times \mathbb{R}_+)$. Then

$$\begin{aligned} 0 = \max_{\pi} \{ & H_t + (rx + (\mu - r)\pi + c)H_x + \frac{1}{2}\pi^2\sigma^2H_{xx} \\ & + \lambda E[H(t, x + \pi J) - H(t, x)] \} \end{aligned} \quad (6)$$

with boundary condition

$$H(T, x) = u(x). \quad (7)$$

The proof is standard see Fleming and Soner [21]. The following verification theorem is necessary to ensure that the solution to HJB equation indeed solves the optimisation problem.

Theorem 2. Assume $W(t, x) \in C^{1,2}([0, T] \times \mathbb{R}_+)$ satisfies HJB subject to the boundary condition. Then $H(t, x) = W(t, x)$. If $\pi^*(t)$ satisfies

$$\begin{aligned} 0 = & H_t + (rx + (\mu - r)\pi^* + c)H_x + \frac{1}{2}(\pi^*)^2\sigma^2H_{xx} \\ & + \lambda E[H(t, x + \pi^* J) - H(t, x)] \end{aligned} \quad (8)$$

$\forall (t, x) \in [0, T] \times \mathbb{R}_+$. Then $\pi^*(t)$ is optimal i.e., $W(t, x) = H^{\pi^*}(t, x)$

The proof can also be found in [21].

We remark that as pointed out in this section, the application of stochastic control to deduce HJB equations is standard here. The main difference with [10] and [11], is that the optimal strategy π^* cannot be found in closed form. Later in the paper, we show how the optimal strategy can be found by numerically solving HJB equations. This section serves to ensure a self contained exposition.

4. Solution of optimisation problem

From the boundary condition (7), we try a solution of the form

$$H(t, x) = -e^{a(t)+b(t)x}. \quad (9)$$

This implies that $a(T) = -\log(\gamma)$ and $b(T) = -\gamma$. Therefore

$$H_t = (a' + b'x)H, \quad H_x = bH, \quad H_{xx} = b^2H.$$

Substituting into (6), we obtain

$$0 = \max_{\pi} \{ (a' + b'x) + (rx + (\mu - r)\pi + c)b + \frac{1}{2}\pi^2\sigma^2b^2 + \lambda E[e^{b(t)\pi J} - 1] \}. \quad (10)$$

Comparing coefficients, we deduce that

$$b' + rb = 0, \quad b(T) = -\gamma,$$

whose solution can be found as $b(t) = -\gamma e^{r(T-t)}$. Therefore our trial solution becomes

$$H(t, x) = -e^{a(t) - \gamma e^{r(T-t)}x}. \quad (11)$$

The form of this solution can be compared to the ones in [11,18,20] etc.

We also deduce that a satisfies the differential-integro equation

$$0 = \max_{\pi} \{ a' - ((\mu - r)\pi(t) + c)\gamma e^{r(T-t)} + \frac{1}{2}\pi^2(t)\sigma^2\gamma^2 e^{2r(T-t)} + \lambda E[e^{-\gamma e^{r(T-t)}\pi(t)J} - 1] \}. \quad (12)$$

Let

$$g(\pi) = a' - ((\mu - r)\pi(t) + c)\gamma e^{r(T-t)} + \frac{1}{2}\pi^2(t)\sigma^2\gamma^2 e^{2r(T-t)} + \lambda E[e^{-\gamma e^{r(T-t)}\pi(t)J} - 1].$$

Then our HJB can be written as

$$0 = \max_{\pi} g(\pi). \quad (13)$$

Applying a first order condition $g' = 0$ to the static optimisation problem (13) implies

$$0 = \mu - r - \pi\sigma^2\gamma e^{r(T-t)} + \lambda E[J e^{-\gamma e^{r(T-t)}\pi(t)J}]. \quad (14)$$

The following lemma shows that π^* can be found as a solution to Eq. (14).

Lemma 1. Eq. (14) has a finite root $\hat{\pi}$.

Proof. Let

$$h(\pi) = \mu - r - \pi\sigma^2\gamma e^{r(T-t)} + \lambda E[J e^{-\gamma e^{r(T-t)}\pi J}].$$

Then

$$\begin{aligned} h'(\pi) &= -\sigma^2\gamma e^{r(T-t)} - \lambda E[\gamma e^{r(T-t)} J^2 e^{-\gamma e^{r(T-t)}\pi J}] \\ &= -\gamma e^{r(T-t)} (\sigma^2 + \lambda E[J^2 e^{-\gamma e^{r(T-t)}\pi J}]) \\ &< 0. \end{aligned}$$

So h is monotone decreasing and $\lim_{\pi \rightarrow -\infty} h(\pi) > 0$ and $\lim_{\pi \rightarrow \infty} h(\pi) < 0$. The result then follows by applying the intermediate value theorem. $\square$

Substituting $\hat{\pi}$ into Eq. (12), we deduce that a satisfies the integro-differential equation

$$a' = ((\mu - r)\hat{\pi}(t) + c)\gamma e^{r(T-t)} - \frac{1}{2}\hat{\pi}^2(t)\sigma^2\gamma^2 e^{2r(T-t)} - \lambda E[e^{-\gamma e^{r(T-t)}\hat{\pi}(t)J} - 1],$$

with boundary condition $a(T) = -\log(\gamma)$. We deduce that if the distribution of jumps J is known then a closed form of a can be found. We summarise the above discussion in the following theorem.

Theorem 3. The optimal investment policy to maximise terminal expected exponential utility is given by the solution to the equation

$$0 = \mu - r - \pi^*(t)\sigma^2\gamma e^{r(T-t)} + \lambda E[J \exp\{-\gamma e^{r(T-t)}\pi^*(t)J\}]. \quad (15)$$

The maximal expected utility is given by

$$H(t, x) = -e^{a(t) - \gamma e^{r(T-t)}x},$$

where $a(t)$ satisfies the integro-differential equation

$$a' = ((\mu - r)\pi^*(t) + c)\gamma e^{r(T-t)} - \frac{1}{2}(\pi^*(t))^2\sigma^2\gamma^2 e^{2r(T-t)} - \lambda E[e^{-\gamma e^{r(T-t)}\pi^*(t)J} - 1],$$

with boundary condition $a(T) = -\log(\gamma)$.

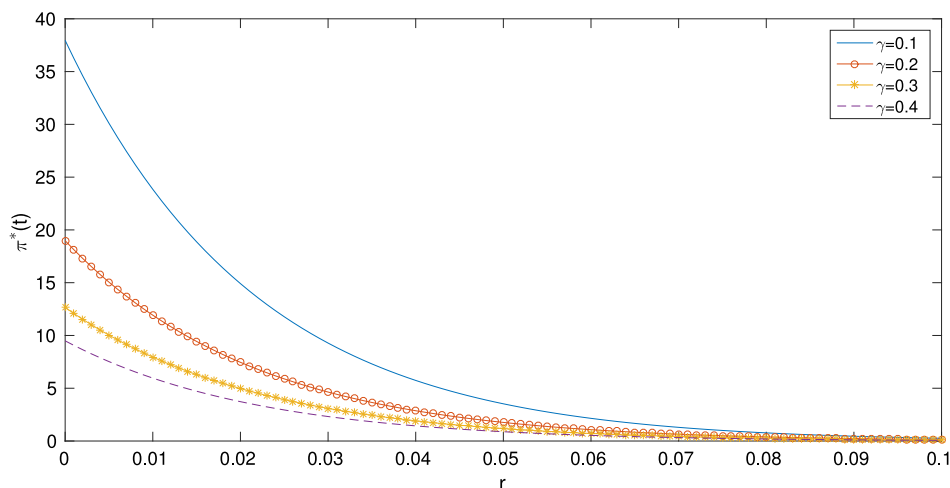


Fig. 1. Effect of γ and r on $\pi^*(t)$.

5. Numerical examples

In this section, we present some numerical experiments based on a particular distribution of jumps. We also perform sensitivity analysis of $\pi^*(t)$ on various parameters. Assume jumps have a double exponential distribution with density

$$f_j(x) = p\eta_1 e^{-\eta_1 x} \mathbf{1}_{\{x \geq 0\}} + q\eta_2 e^{\eta_2 x} \mathbf{1}_{\{x < 0\}}, \quad (16)$$

where $p, q \geq 0, p + q = 1$. Unless otherwise stated, the following parameters are used: $T = 40, t = 0, \lambda = 3, p = 0.5, \eta_1 = 1/0.02, \eta_2 = 1/0.03, r = 0.01, \gamma = 0.4, \mu = 0.1, \sigma = 0.2$. We have taken these parameters from Kou [22] and Kou and Wang [23] where jump diffusion models are introduced to option pricing. Some of these parameters are also used in [10]. According to Kou and Wang [23], these parameters reflect typical financial applications.

Eq. (15) now takes the form

$$0 = \mu - r - \pi^*(t)\sigma^2\gamma e^{r(T-t)} + \frac{\lambda p\eta_1}{(\eta_1 + \gamma e^{r(T-t)}\pi^*(t))^2} - \frac{\lambda q\eta_2}{(\eta_2 - \gamma e^{r(T-t)}\pi^*(t))^2}. \quad (17)$$

This equation can be solved using a simple Bisection method.

Let $r \in [0, 0.1]$ and $\gamma = 0.1, 0.2, 0.3, 0.4$. With these parameters we solve for $\pi^*(t)$ in (17) to obtain Fig. 1. From Fig. 1 we see that $\pi^*(t)$ is decreasing in γ . This is intuitive, as γ is a measure of risk aversion meaning that the larger the value of γ is, the less the members are willing to take more risk meaning that they will invest little in the risky asset. This conclusion is also reached in [10]. The added value here is that we are able to analyse the impact of γ and r simultaneously. Although the conclusion is the same, it is remarkable that we still get the same result working with a more general utility function namely the exponential. Here, using an ansatz or guess will not lead to a closed form for $\pi^*(t)$ on using the first order condition in the HJB, as in [10] for example. The methodology here allows one to work with complicated utilities where analytic forms for $\pi^*(t)$ are unlikely to be found.

Fig. 1 also shows that $\pi^*(t)$ decreases with r . An increase in r implies an increase in the return of the riskless asset. This entails that members will invest less in the risky asset and more in the riskless asset.

Let $\mu \in [0.1, 0.5]$ and $\sigma = 0.1, 0.3, 0.5$. With these parameters we solve for $\pi^*(t)$ in (17) to obtain Fig. 2. From Fig. 2 we see that $\pi^*(t)$ is decreasing in σ , the volatility of risky asset. The larger the value σ , the higher the risk of investing in risky asset hence pension members will likely invest less in risky asset. We also see that $\pi^*(t)$ is increasing in μ . Since μ is the expected return of risky asset, an increase in this value means DC members can get more by investing in risky asset.

We now consider the effect of jump parameters. Let $\lambda = 0.01, 0.1, 1, 3$ and $p \in [0, 1]$. With these parameters we solve for $\pi^*(t)$ in (17) to obtain Fig. 3. From Fig. 3 we see that $\pi^*(t)$ is firstly decreasing then increasing in λ . When p is small, a large λ means an increase in the likelihood of a downward jump (by considering the density (16)), which implies increased risk of investing in risky asset. Pension members respond by investing less in risky asset. When p is large and λ is large also, this implies an increase in the likelihood of a positive jump resulting in an increase of expected return from risky asset. Pension members respond by allocating more in risky asset. We also observe that $\pi^*(t)$ increases in p . Since p is the probability of a positive jump implying an increase in expected return from risky asset, this entails

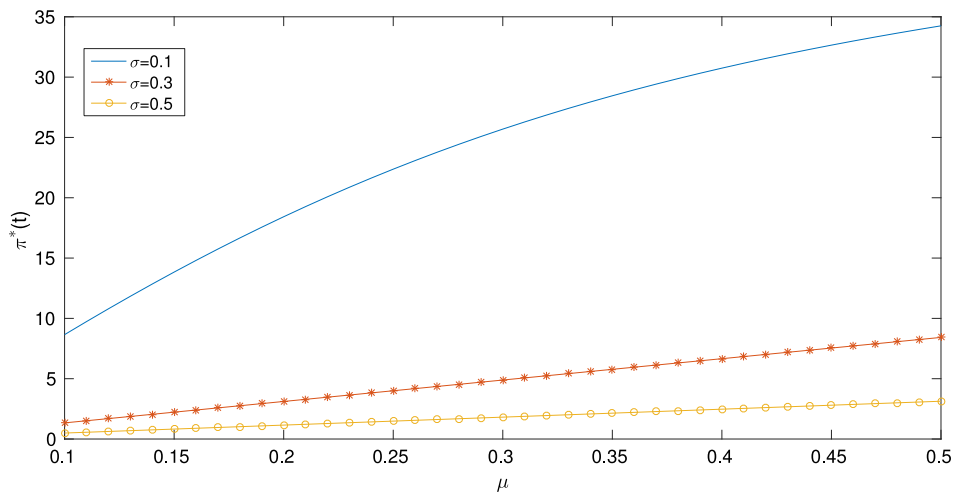


Fig. 2. Effect of σ and μ on $\pi^*(t)$.

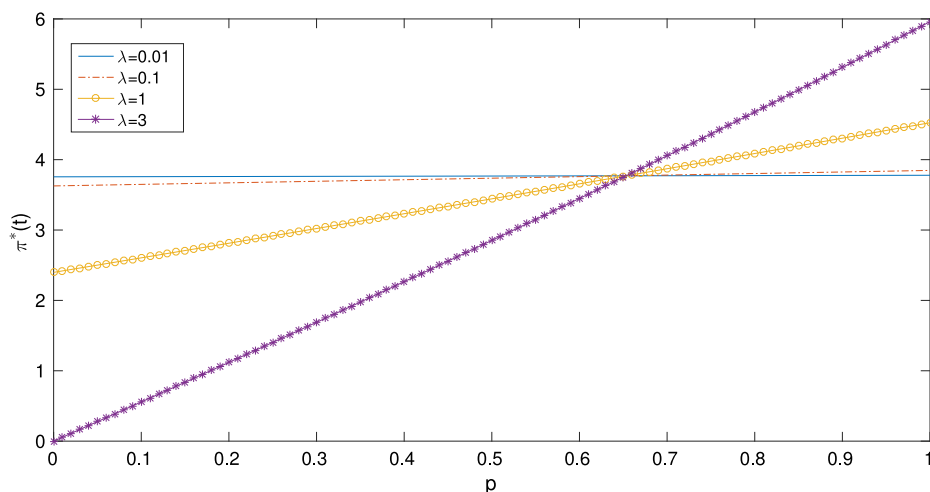


Fig. 3. Effect of p and λ on $\pi^*(t)$.

a higher allocation in risky asset. The results are also found in [24]. The observation adds value to pension modelling literature by analysing the impact of jump parameters on the optimal allocation which is not done in [12], [13] or [14] for example. In [12], pricing of pension plans which is an indirect way of determining optimal allocation is considered but impact of jump parameters is not shown precisely even though a normal distribution for jumps is modelled and in [13], a Levy process for surplus is considered with examples later given of exponential jumps but the sensitivity analysis of parameters is not given. In [14], a jump diffusion model for DC pensions is considered but example of jump distribution is not shown.

Although not all papers deal with analytic strategies, there are very few that deal with numerical methods on pension models. The closest would be Garrido and Vazquez where a normal distribution is used for the jumps. In order to compare we could replace $f_j(x)$ i.e. (16) by a normal density.

6. Conclusion

In this paper, we considered a defined contribution pension plan, where the pension planner has to maximise utility of terminal wealth by deciding how much to invest in a risky asset driven by a jump-diffusion process. We used stochastic control theory to derive the investment strategy as a solution to an integro-differential equation which can be solved by a simple root finding technique given the distribution of jumps.

Our findings confirm existing theories. Further, as we have noted in analysis of results, our approach enables one to obtain a detailed analysis of the impact of jump parameters on the optimal allocation by considering typical models of

jumps. We have added value in this direction by analysing the nature of jumps and the likelihood of a jump happening in an interval of time. For example, we are able to investigate the probability of positive and negative jump occurring and number of jumps in a unit time interval in a single analysis. As we have explained, this analysis is not usually present in most articles on DC pensions.

The allocation strategy depends on risk preferences of pension members. We found that risk averse members will invest less in risky asset. The likelihood of the stock having positive (negative) jumps results in more (less) being allocated to risky asset.

References

- [1] G. Deelstra, M. Grasselli, P.F. Koehl, Optimal investment strategies in the presence of a minimum guarantee, *Insurance Math. Econom.* 33 (1) (2003) 189–207.
- [2] N. Han, M. Hung, Optimal asset allocation for DC pension plans under inflation, *Insurance Math. Econom.* 51 (2012) 172–181.
- [3] G. Guan, Z. Liang, Optimal management of DC pension plan in a stochastic interest rate and stochastic volatility framework, *Insurance Math. Econom.* 57 (2014) 58–66.
- [4] J. Gao, Stochastic optimal control of DC pension funds, *Insurance Math. Econom.* 42 (2008) 1159–1164.
- [5] M. Di Giacinto, S. Federico, F. Gozzi, Pension funds with a minimum guarantee: a stochastic control approach, *Finance Stoch.* 15 (2011) 297–342.
- [6] G. Guan, Z. Liang, Mean-variance efficiency of DC pension plan under stochastic interest rate and mean-reverting returns, *Insurance Math. Econom.* 61 (2015) 99–109.
- [7] L. He, Z. Liang, Optimal investment strategy for the DC plan with return of premiums clauses in a mean-variance framework, *Insurance Math. Econom.* 53 (3) (2013) 643–649.
- [8] E. Vigna, On efficiency of mean-variance based portfolio selection in defined contribution pension schemes, *Quant. Finance* 14 (2) (2014) 237–258.
- [9] H. Wu, L. Zhang, H. Chen, Nash equilibrium strategies for a defined contribution pension management, *Insurance Math. Econom.* 62 (2) (2015) 202–214.
- [10] J. Sun, Z. Li, Y. Zeng, Precommitment and equilibrium investment strategies for defined contribution pension plans under a jump-diffusion model, *Insurance Math. Econom.* 67 (2016) 158–172.
- [11] H. Yang, L. Zhang, Optimal investment for insurer with jump-diffusion risk process, *Insurance Math. Econom.* 37 (2005) 615–634.
- [12] M.C. Calvo-Garrido, C. Vázquez, Pricing pension plans under jump-diffusion models for the salary, *Comput. Math. Appl.* 68 (2014) 1933–1944.
- [13] O. Le Courtois, F. Menoncin, Portfolio optimisation with jumps: Illustration with a pension accumulation scheme, *J. Bank. Financ.* 60 (2015) 127–137.
- [14] C.I. Nkeki, Optimal pension fund management in a jump-diffusion environment: theoretical and empirical studies, *J. Comput. Appl. Math.* 330 (2018) 228–252.
- [15] J. Ma, W. Li, H. Zheng, Dual control Monte-Carlo method for tight bounds of value function in regime switching utility maximization, *European J. Oper. Res.* 262 (2017) 851–862.
- [16] L.C.G. Rogers, Pathwise stochastic optimal control, *SIAM J. Control Optim.* 46 (3) (2007) 1116–1132.
- [17] P. Henry-Labordre, C. Litterer, Z. Ren, A dual algorithm for stochastic control problems: Applications to uncertain volatility models and CVA, *SIAM J. Financial Math.* 7 (1) (2016) 159–182.
- [18] S. Browne, Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin, *Math. Oper. Res.* 20 (4) (1995).
- [19] M. Ludkovski, Q. Shen, European option pricing with liquidity shocks, *Int. J. Theor. Appl. Finance* 16 (7) (2013) 135–143.
- [20] X. Zeng, Y. Wang, J.M. Carson, Dynamic portfolio choice with stochastic wage and life insurance, *N. Am. Actuar. J.* 19 (4) (2015) 256–272.
- [21] W.H. Fleming, H.M. Soner, *Controlled Markov processes and viscosity solutions*, Springer, Berlin, 1993.
- [22] S.G. Kou, A jump diffusion model for option pricing, *Manag. Sci.* 48 (2002) 1086–1101.
- [23] S.G. Kou, H. Wang, First passage times of a jump diffusion process, *Adv. Appl. Probab.* 35 (2003) 504–531.
- [24] X. Lin, P. Yang, Optimal investment and reinsurance in a jump diffusion risk model, *ANZIAM J.* 52 (2011) 250–262.