

Asymptotic stability of the fundamental solution method

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Abstract

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In a previous paper (1988) we studied the numerical stability of the fundamental solution method applied to the Dirichlet problem of Laplace's equation, and presented a practical scheme for examining the stability of the method using the singular value decomposition. In the present paper, we give an asymptotic estimate of the stability with respect to the number of collocation points.

Keywords: Asymptotic stability, fundamental solution method, singular value decomposition.

1. Introduction

When we apply the fundamental solution method (FSM) to the Dirichlet problem for Laplace's equation, we inevitably solve an ill-conditioned linear system to determine the coefficients, or charge, to obtain the approximate solution. This situation is studied in detail in [8] making use of the singular value decomposition (SVD), which we shall outline below.

We consider the fundamental solution method to approximate the solution of the Dirichlet problem for Laplace's equation of the form

$$\Delta u = 0 \quad \text{in } \Omega, \quad (1.1)$$

$$u = g \quad \text{on } \partial\Omega, \quad (1.2)$$

where

$$\Omega = \{ \omega \in \mathbb{R}^2 \mid \| \omega \|_2 < \rho \},$$

in which the norm $\| \cdot \|_2$ represents the ordinary Euclidian norm. The fundamental solution method approximates the solution $u(x)$ by

$$u_n(x) = \sum_{k=1}^n c_k G(x, y_k), \quad x \in \Omega, \quad (1.3)$$

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where $G(x, y)$ is the Green's function for (Δ, Ω) , namely,

$$G(x, y) = -\frac{1}{2} \log \|x - y\|_2, \quad x, y \in \mathbb{R}^2.$$

The points y_k , called charge points, are chosen to lie on the boundary of the circle $\Omega_R = \{\omega \in \mathbb{R}^2 \mid \|\omega\|_2 < R\}$ where $R > \rho$, and the c_k 's are constants to be determined. The vector $c = c_1, c_2, \dots, c_n)^t \in \mathbb{R}^n$ is called the charge and is determined in such a way that $u_n(x)$ satisfies the boundary condition

$$u_n(\hat{x}_j) = g(\hat{x}_j), \quad j = 1, 2, \dots, n, \quad (1.4)$$

where the $\hat{x}_j$'s are n collocation points on the boundary $\partial\Omega$.

With the collocation points $\hat{x}_k = \rho e^{2\pi(k-1)i/n}$ and the charge points $y_k = R e^{2\pi(k-1)i/n}$, $k = 1, 2, \dots, n$, the following results stating that the approximate solution u_n converges to the solution u exponentially with respect to n are known.

Theorem 1.1 (Katsurada [7]). (a) Suppose that the harmonic extension of u exists in

$$\Omega_{r_0} = \{\omega \mid \|\omega\|_2 < r_0\} \quad \text{with } \rho < r_0;$$

then we have, for sufficiently large n ,

$$\|u - u_n\|_\infty \leq \sup_{\|x\|_2 = r_0} |u(x)| \frac{2}{1 - \rho/r_0} \left\{ (1 + C(R, \rho)) \left(\frac{\rho}{r_0}\right)^{n/3} + 4 \left(\frac{\rho}{R}\right)^{n/3} \right\}, \quad (1.5)$$

where the norm $\|u\|_\infty$ is defined by $\sup_{x \in \Omega_0} |u(x)|$ and $C(R, \rho)$ is a constant given by

$$C(R, \rho) = \max \left\{ 1, \frac{\log(R^n + \rho^n)}{|\log(R^n - \rho^n)|} \right\}.$$

We call n "sufficiently large" if $(\rho/R)^n \leq \frac{1}{2}$, $n \log R \geq 4(\rho/R)^n$ and $(\rho/R)^{2n/3} \leq n \log R$.

(b) The condition number $\text{Cond}(n, R)$ of the coefficient matrix of the system of linear equations (1.4) which defines the charge c grows exponentially with respect to n , and can be estimated by

$$\text{Cond}(n, R) \sim \frac{1}{2} \log R n \left(\frac{R}{\rho} \right)^{n/2}. \quad (1.6)$$

Note that Theorem 1.1 is only valid for the circular domain and cannot be applied to more complicated regions. This is because the estimates (1.5) and (1.6) follow from the fact that the coefficient matrix for the particular location of $\hat{x}_k$ and y_k is circulant. The result (b) is also obtained in [2]. For the properties of circulant matrices, see, e.g., [3]. Numerical stability of this method is studied in [8]. Some numerical examples of FSM for the Dirichlet problem are also given in [1].

2. Formulation and basic results

The method of Section 1 reduces to the following two steps.

(1) We first solve an ill-conditioned linear system to determine the charge c in the form of

$$\Gamma c = g, \quad (2.1)$$

where the given data g may be contaminated by some perturbation Δg , $g \in Y = \mathbb{R}^m$, $c \in X = \mathbb{R}^n$, $\Gamma: X \rightarrow Y$ with

$$(\Gamma c)_j \equiv \sum_{k=1}^n c_k G(\hat{x}_j, y_k) = g(\hat{x}_j), \quad \hat{x}_j \in C(\rho),$$

$$\Gamma = (\gamma_{jk}), \quad \gamma_{jk} = G(\hat{x}_j, y_k), \quad j = 1, 2, \dots, n,$$

and $C(\rho)$ denotes the disc with radius ρ . We note that the results presented in this section are valid for $m \neq n$. We restrict, however, ourselves to the case of $m = n$ in the next section.

(2) We use the intermediate solution c to obtain the final result f (the notation f comes from “final”) from

$$f = \Lambda c, \quad (2.2)$$

where $f \in Y$ and $\Lambda: X \rightarrow Y$ with

$$f_j \equiv \bar{u}_n(x_j) = \sum_{k=1}^n c_k G(x_j, y_k) \equiv (\Lambda c)_j, \quad x_j \in \Omega,$$

$$\Lambda = (\lambda_{jk}), \quad \lambda_{jk} = G(x_j, y_k), \quad j = 1, 2, \dots, n.$$

Due to the ill-conditioning of (2.1), some “large” perturbation Δc may be introduced into the intermediate solution c . One may assume intuitively that the error $\|\Delta f\|$ for any norm in the final result f , where $\Delta f \equiv \Lambda \Delta c$, is on a level with $\|\Delta c\|$ or as large as $\|\Lambda\| \|\Delta c\|$. If this is the case, we cannot accept f as an approximate solution and the method of regularization [6,12] applied to (2.1) may be very effective. But this is not always true. Even if the error $\|\Delta c\|$ is very large, $\|\Delta f\|$ can be very small. In this case, we do not necessarily have to use the regularization method, or in some cases, we may obtain a worse result by using the method. To examine whether the method of regularization is required or not for this class of numerical procedures, we have the following results.

We assume given data $\bar{g} = g + \Delta g$ and the intermediate solution $\bar{c} = c + \Delta c$. We have $\Gamma \bar{c} = \bar{g}$ as well as (2.1). Let $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$ be the singular values of Γ and $\{u_i\}$, $i = 1, 2, \dots, m$, $\{v_j\}$, $j = 1, 2, \dots, n$, be corresponding singular vectors of Γ . Reflecting the ill-conditioning of Γ , we assume that $\sigma_n \rightarrow 0$ as $n \rightarrow \infty$. We can see that $\|\Delta c\|_2 \rightarrow \infty$ as $\sigma_n \rightarrow 0$ from

$$\Delta c = \sum_{i=1}^n \frac{1}{\sigma_i} (\Delta g, u_i) v_i. \quad (2.3)$$

Next, we suppose that final result $\bar{f}$ is given by $\bar{f} = f + \Delta f$. We have $\bar{f} = \Lambda \bar{c}$ as well as (2.2). Let $\hat{\sigma}_1 \geq \hat{\sigma}_2 \geq \dots \geq \hat{\sigma}_n \geq 0$ be singular values of Λ and $\{\hat{u}_i\}$, $i = 1, 2, \dots, m$, $\{\hat{v}_j\}$, $j = 1, 2, \dots, n$, be singular vectors of Λ . As for $\|\Delta f\|$, we have the following result from [8, Theorem 4.1].

Theorem 2.1. *Let $\{u_i, v_i; \sigma_i\}$ and $\{\hat{u}_i, \hat{v}_i; \hat{\sigma}_i\}$, for $i = 1, 2, \dots, n$, be the orthonormal singular systems for Γ and Λ , respectively. Then we have*

$$\|\Delta f\|_2 \leq \|\Xi * \Theta\|_F \|\Delta g\|_2, \quad (2.4)$$

where

$$\Xi = (\xi_{ij}), \quad \xi_{ij} = \frac{\hat{\sigma}_i}{\sigma_j}, \quad (2.5a)$$

$$\Theta = (\theta_{ij}), \quad \theta_{ij} = (\hat{v}_i, v_j), \quad i, j = 1, 2, \dots, n. \quad (2.5b)$$

$\Xi * \Theta$ represents the Hadamard product of the matrices Ξ and Θ (i.e., the ij -th element of $\Xi * \Theta$ is given by $\xi_{ij}\theta_{ij}$), and $\|\cdot\|_F$ denotes the Frobenius norm.

Making use of the singular value decomposition, we can construct the matrices Θ and Ξ numerically. Then we can examine the numerical stability of the procedures which consist of two steps denoted in the form of (2.1) and (2.2).

The elements ξ_{ij} of Ξ , which we called the explosive factor matrix in [8,9], represent the upper bound of the magnification of the u_i -component $(\Delta g, u_i)$ of the perturbation Δg to $\hat{u}_j$ -component $(\Delta f, \hat{u}_j)$ of Δf . To clarify what “magnification of $(\Delta g, u_i)$ to $(\Delta f, \hat{u}_j)$ ” means, we define it precisely.

Definition 2.2. We call the partial derivative

$$\frac{\partial(\Delta f, \hat{u}_j)}{\partial(\Delta g, u_i)}$$

the *magnification* of $(\Delta g, u_i)$ to $(\Delta f, \hat{u}_j)$.

An important fact is that the elements of the matrix $\Xi * \Theta$, i.e., $\xi_{ij} \times \theta_{ij}$, give the exact magnification ratio of $(\Delta g, u_i)$ to $(\Delta f, \hat{u}_j)$ [10, Theorem 5.3].

This certainly gives a clear-cut idea of the mechanism of propagation of the error to the final solution f . For instance, the largest element $\xi_{1n} = \hat{\sigma}_1/\sigma_n$ coincides with the straightforward upper bound in the spectral norm $\|\cdot\|_s$ of the matrix given by $\|\Delta f\|_2 \leq \|\Gamma^{-1}\|_s \|\Lambda\|_s \|\Delta g\|_2$, since $\|\Gamma^{-1}\|_s \|\Lambda\|_s = \hat{\sigma}_1/\sigma_n$. On the other hand, the element θ_{ij} of Θ represents the actual ratio of propagation of $(\Delta g, u_i)$ to $(\Delta f, \hat{u}_j)$. The actual magnification of propagation of $(\Delta g, u_i)$ to $(\Delta f, \hat{u}_j)$ is given by $\xi_{ij} \times \theta_{ij}$ and the upper bound of the total propagation of Δg to Δf is given by the square root of the sum of squares of $\xi_{ij} \times \theta_{ij}$, or $\|\Xi * \Theta\|_F$ (Theorem 2.1).

Each element of ξ_{ij} and θ_{ij} can easily be computed numerically by using the SVD (Singular Value Decomposition [4,5]). It is desired, however, to have an asymptotic estimate of $\|\Delta f\|_2$ with respect to n . Since an analytical estimate of $\|\Delta f\|_2$ for the arbitrary domain is difficult to obtain, we restrict ourselves to the domain of a disk with radius ρ .

As the essential result of the paper, we have a simple estimate of

$$\|\Delta f\|_2 \leq O(n) \|\Delta g\|_2.$$

This estimate is far better than the straightforward estimates of $\|\Lambda\| \|\Gamma\| \leq (R/\rho)^{n/2}$.

This means that the fundamental solution method is quite stable for practical usage in which n is relatively small. This result also implies that the ill-conditioning of the equation (2.1) to determine the coefficients whose condition number grows exponentially is not reflected to the final solution.

3. Asymptotic stability of FSM

In the rest of the paper we consider the case with the collocation points $\hat{x}_k = \rho e^{2\pi(k-1)i/n}$ and the charge points $y_k = R e^{2\pi(k-1)i/n}$, $k = 1, 2, \dots, n$. Namely, we restrict ourselves to the case of

$m = n$ in this section. The fact we use is that the matrix $\Gamma(\hat{x}_j, y_k)$ for the disk region is circulant. We list basic results related to the circulant matrix and the matrices Γ and Λ .

Definition 3.1. A matrix $A = (a_{ij})$ is *circulant* if $a_{i+1,j} = a_{i,j-1}$ and $a_{i+1,1} = a_{i,n}$ for $i, j = 2, 3, \dots, n$, for given $a_{11}, a_{12}, \dots, a_{1n}$.

Hereafter we define $\Lambda(\theta) = G(x_j(\theta), y_k)$ with $x_j(\theta) = \rho e^{(2\pi(j-1)/n+\theta)i}$. We note that by varying θ from 0 to $2\pi/n$, we can cover the whole boundary of Ω . We can prove that the matrix $\Lambda(\theta)$ is circulant.

Lemma 3.2. For arbitrary $\theta \in [0, 2\pi/n]$, $\Lambda(\theta)$ is circulant.

Proof. From the definition, we have

$$(\Lambda(\theta))_{jk} = G(x_j(\theta), y_k) = \log_e |\rho e^{(2\pi(j-k)/n+\theta)i} - R|.$$

Hence it is sufficient to show that the matrix $E = (\epsilon_{jk})$, where

$$\epsilon_{jk} = e^{(2\pi(j-k)/n+\theta)i},$$

is circulant (i.e., $\epsilon_{j+1,k} = \epsilon_{j,(k-1) \bmod n}$ for $j = 1, 2, \dots, n-1$, and $k = 1, 2, \dots, n$), which is an easy exercise. $\square$

Lemma 3.3. If the domain Ω is a disc with radius ρ , then the matrix $\Gamma(\hat{x}_j, y_k)$ is circulant.

Proof. The proof is quite similar to that of Lemma 3.2. $\square$

Lemma 3.4. Let $\Theta = (\theta_{ij})$ with $\theta_{ij} = (v_i, \hat{v}_j)$, where $\{v_i\}$ and $\{\hat{v}_j\}$ are the singular vectors of Γ and $\Lambda(\theta)$, respectively. We have

$$\theta_{ij} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Proof. We note that both Γ and $\Lambda(\theta)$, for every θ , are circulant matrices as is stated in Lemmas 3.2 and 3.3. Since for every circulant matrix the eigenvectors are given by the Fourier matrix which is orthonormal (see, e.g., [3]), we have

$$\theta_{ij} = (v_i, \hat{v}_j) = (v_i, v_j) = \delta_{ij},$$

which implies the lemma. $\square$

The next lemma is given in [7].

Lemma 3.5. Let the eigenvalues of $\Lambda(\theta)$ be $\hat{\lambda}_j$. We have

$$\hat{\lambda}_j = \frac{n}{4\pi} \sum_{k=1}^n \frac{1}{kn+j} \left(\frac{\rho}{R} \right)^{kn+j} e^{i(kn+j)\theta}.$$

Proof. It is well known that the eigenvalues λ_j , $j = 1, 2, \dots, n$, of a circulant matrix $\text{circ}(a_1, a_2, \dots, a_n)$ whose j th row is given by $(a_j, a_{j+1}, \dots, a_n, a_1, a_2, \dots, a_{j-1})$ are given by

$$\lambda_j = \sum_{k=1}^n a_k \omega^{(j-1)k},$$

where ω is the n th primitive root of unity. The proof follows from this fact and the definition of $\Lambda(\theta)$. A proof in detail is given in [7]. $\square$

Theorem 3.6. *For the fundamental solution method, ξ_{kk} , the diagonal elements of the explosive factor matrix Ξ , are less than or equal to unity.*

Proof. From the definition of Ξ , we have $\xi_{kk} = \hat{\lambda}_k(\theta)/\lambda_k$. From Lemma 3.5 we have

$$\left| \frac{\hat{\lambda}_k(\theta)}{\lambda_k} \right| = \left| \frac{\sum_{p=1}^n \frac{1}{pn+k} \left(\frac{\rho}{R} \right)^{pn+k} e^{i(pn+k)\theta}}{\sum_{p=1}^n \frac{1}{pn+k} \left(\frac{\rho}{R} \right)^{pn+k}} \right|.$$

Since the numerator satisfies

$$\begin{aligned} \left| \sum_{p=1}^n \frac{1}{pn+k} \left(\frac{\rho}{R} \right)^{pn+k} e^{i(pn+k)\theta} \right| &\leq \sum_{p=1}^n \left| \frac{1}{pn+k} \left(\frac{\rho}{R} \right)^{pn+k} \right| |e^{i(pn+k)\theta}| \\ &= \left| \sum_{p=1}^N \frac{1}{pn+k} \left(\frac{\rho}{R} \right)^{pn+k} \right|, \end{aligned}$$

it follows that

$$\left| \frac{\hat{\lambda}_k(\theta)}{\lambda_k} \right| = \xi_{kk} \leq 1. \quad \square$$

From these results, we have the final theorem.

Theorem 3.7. *Let Δf be the error introduced in the approximate solution of the fundamental solution method. Then*

$$\|\Delta f\|_2 \leq O(n) \|\Delta g\|_2.$$

Proof. The proof immediately follows from Theorem 2.1. Lemma 3.3 and Theorem 3.6. $\square$

This theorem asserts that the effect of ill-conditioning of Γ , whose condition number grows exponentially as is stated in Theorem 1.1(b), is not reflected in the final solution. The method of the fundamental solution is stable for relatively small n , which is usually the case with the method because the rapid convergence of the method is guaranteed by Theorem 1.1(a).

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