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A new approach to the numerical solution of weakly singular Volterra integral equations

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Abstract

We consider linear weakly singular Volterra integral equations of the second kind, with kernels of the form $k(x,v)=|x-v|^{-\alpha}K(x,v)$, $0<\alpha<1$, or $k(x,v)=\log|x-v|K(x,v)$, $K(x,v)$ being a smooth function. The solutions of such equations may exhibit a singular behaviour in the neighbourhood of the initial point of the interval of integration. By a transformation of the unknown function we obtain an equation which is still weakly singular, but whose solution is as smooth as we like. This resulting equation is then solved by standard product integration methods.

MSC: primary 65R20

Keywords: Product integration; Weakly singular Volterra integral equation

1. Introduction

Let us consider the linear Volterra integral equation of the second kind

$$\varphi(x)=g(x)+\int_a^x p(x-v)K(x,v)\varphi(v)\,\mathrm{d}v, \quad a\leqslant x\leqslant X, \tag{1.1}$$

where $K(x,v)$ is a smooth function and $p(x-v)$ is a kernel of the form

$$p(x-v)=|x-v|^{-\alpha}, \quad 0<\alpha<1 \tag{1.2'}$$

or

$$p(x-v)=\log|x-v|. \tag{1.2''}$$

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The numerical treatment of Eq. (1.1) is not simple, because, as it is well known, the solutions of weakly singular Volterra integral equations usually have a weak singularity at $x=a$, even when the inhomogeneous term $g(x)$ is regular.

Specific methods have been proposed by several authors for equations with smooth solution [5] and for equations with nonsmooth solution [1–3,7,9]. It has also been considered the possibility of using a Nyström method on a small interval $[a,b]$ and a step-by-step method when x is "far enough" from the initial point [12].

In this paper, we propose a completely different approach, which allows to solve weakly singular integral equations with nonsmooth solutions by using standard methods for regular integral equations, and hence at a lower computational cost. Our approach consists in the application of a smoothing technique, which has been successfully employed in [11] for the solution of weakly singular Fredholm integral equations of the second kind. Taking advantage of the analysis of the behaviour of the solution of (1.1) as $x \to a$, we introduce in (1.1) a smoothing change of variable and obtain an equation with a perturbed kernel, but with a smoother solution. The resulting equation is still weakly singular, but it has a solution as smooth as we like and can be solved by standard product integration methods of known order of convergence.

In Section 2 we recall some known results about the behaviour of equations of type (1.1), with kernel of type (1.2′) and (1.2″). In Section 3 we introduce the smoothing transformation. In Section 4 the transformed equation is solved by a Nyström method. In Section 5 the same equation is solved by a step-by-step method based on the product Simpson's rule and the product three–eights rule as an end rule. In Section 6 we give some numerical examples.

2. Solution behaviour

Since the rate of convergence of a numerical method depends on the regularity of the solution of (1.1), the knowledge of the behaviour of the solution is very important in the choice of the method; for this reason this section will be devoted to the analysis of the properties of the solution of (1.1).

Given Eq. (1.1) with kernel (1.2′) or (1.2″), it has been proved in [10] that the solution of (1.1) is unique and continuous in $[a,X]$, if $g \in L^1(a,X)$ and $K \in L^\infty$, both as a function of x and of v, $a \leqslant v \leqslant x \leqslant X$. (Strictly speaking in [10] the uniqueness was proved for an equation with $K(x,v) \equiv h(v)$, but the argument is easily generalized to our case.)

The unique solution of (1.1) is usually nonsmooth at $x=a$, even if $g(x)$ is smooth. A deeper insight into this problem is provided by many differentiability results for $\varphi(x)$ obtained by various authors under specific hypotheses on $g(x)$ and $K(x,v)$. We limit ourselves to remember some of them.

Given Eq. (1.1) with kernel of type (1.2′) or (1.2″),

(i) if $g(x)$ and $K(x,v)$ in (1.1) are of class C^1 in $a \leqslant v \leqslant x \leqslant X$, then $\varphi(x) \in C^1(a,X]$ [10, Theorem 1];

(ii) if g, $K \in C^2$, $a \leqslant v \leqslant x \leqslant X$ then $\varphi(x) \in C^2(a,X]$ [10, Theorem 4].

In the special case where the kernel is of type (1.2′)

(i) if $g(x)$ is real analytic in the neighbourhood of $a \leqslant x \leqslant X$, K is analytic on an open set containing (x,v), $a \leqslant x$, $v \leqslant X$, then $\varphi(x)$ is analytic in $\varepsilon < x \leqslant X$, $\varepsilon > a$, ε as close as we like to a, [10, Theorem 6] and, as a corollary, if α is rational, $\alpha = p/q$, then $\varphi(x^q)$ is analytic in a neighbourhood of $x = a$ [10, Corollary 4];

(ii) if $g(x)$ and $K(x, v)$ in (1.1) are of class C^1 in $a \leqslant x \leqslant X$ and in $a \leqslant v \leqslant x \leqslant X$, respectively, then [8]

$$\varphi(x) = g(x) + \frac{K(a,a)\varphi(a)}{1-\alpha}(x-a)^{1-\alpha} + \mathrm{O}((x-a)^{2(1-\alpha)}), \quad x \to a. \tag{2.1}$$

(iii) if $g(x) = g_1(x) + (x-a)^\beta g_2(x)$ and g_1, g_2, $K \in C^m$ for $a \leqslant x$, $v \leqslant X$, then [6,4] $\varphi(x) \in C[a, X] \cap C^m(a, X]$.

In the special case where the kernel is of type (1.2″) under the same hypotheses of (2.1), the following analogous asymptotic estimate holds [8]:

$$\varphi(x) = g(x) + K(a,a)\varphi(a)[(x-a)\ln(x-a) - (x-a)] + \mathrm{O}(((x-a)\ln(x-a))^2), \; x \to a. \tag{2.2}$$

3. The smoothing transformation

Following [11] and taking into account the behaviour of the solution $\varphi(x)$ at $x = a$ described in the previous section, we introduce in (1.1) a simple nonlinear transformation $x = \gamma(t)$.

The aim of this change of variable is to obtain an integral equation whose solution does not involve anymore singularities in the first derivatives.

Having this objective in mind, we look for a $\gamma(t)$, with γ sufficiently smooth and monotonic, such that $\gamma(a) = a$ and with a certain number of derivatives vanishing at $t = a$. A simple function satisfying all these conditions is (see [11])

$$\gamma(t) = (t-a)^q + a, \tag{3.1}$$

q an integer.

Suppose we introduce the change $x = \gamma(t)$ into (1.1). We get

$$\varphi(\gamma(t)) = g(\gamma(t)) + \int_a^{\gamma(t)} p(\gamma(t) - v) K(\gamma(t), v)\varphi(v)\,\mathrm{d}v, \quad a \leqslant \gamma(t) \leqslant X. \tag{3.2}$$

Setting $v = \gamma(s)$, we obtain

$$\varphi(\gamma(t)) = g(\gamma(t)) + \int_a^t p(\gamma(t) - \gamma(s)) K(\gamma(t), \gamma(s))\varphi(\gamma(s))\gamma'(s)\,\mathrm{d}s, \tag{3.3}$$

$$a \equiv \gamma^{-1}(a) \leqslant s \leqslant t \leqslant \gamma^{-1}(X).$$

Multiply both sides of (3.3) by $\gamma'(t)$ and set

$$y(t) = \gamma'(t)\varphi(\gamma(t)), \quad f(t) = g(\gamma(t))\gamma'(t), \tag{3.4}$$

we obtain

$$y(t) = f(t) + \int_a^t p(\gamma(t) - \gamma(s)) K(\gamma(t), \gamma(s)) y(s)\gamma'(t)\,\mathrm{d}s, \quad a \leqslant t \leqslant \gamma^{-1}(X). \tag{3.5}$$

Eq. (3.5) has a kernel which is still weakly singular and has a unique continuous solution, since the conditions of Section 2 are still satisfied. However, the smoothness of this solution increases with q (the number of derivatives of $\gamma(t)$ vanishing at $t=a$), which implies that standard numerical methods can be applied for solving (3.5). Once the "transformed" equation (3.5) has been solved, the solution of the "original" equation is given by

$$\varphi(x)=\frac{y(\gamma^{-1}(x))}{\gamma'(\gamma^{-1}(x))},$$

whose denominator vanishes only at $x=a$, where the solution is known.

In the sequel we introduce the transformation (3.1) into Eq. (1.1).

3.1. Kernels of Abel type

In the case of kernels of type (1.2′), the transformed equation (3.5) takes the form

$$y(t)=f(t)+\int_a^t [(t-a)^q-(s-a)^q]^{-\alpha}K[(t-a)^q+a,(s-a)^q+a]y(s)q(t-a)^{q-1}\,\mathrm{d}s. \tag{3.6}$$

Following [11], we define for computational convenience

$$\delta_\alpha(t,s)=\begin{cases}\left|\dfrac{(t-a)^q-(s-a)^q}{t-s}\right|^{-\alpha}, & t\neq s,\\[2mm] [q(s-a)^{q-1}]^{-\alpha}, & t=s\end{cases} \tag{3.7}$$

and rewrite (3.6) as

$$y(t)=f(t)+\int_a^t (t-s)^{-\alpha}\delta_\alpha(t,s)K((t-a)^q+a,(s-a)^q+a)y(s)q(t-a)^{q-1}\,\mathrm{d}s. \tag{3.8}$$

To evaluate the advantage of the smoothing, suppose for example $g,\ K\in C^1,\ a\leqslant v\leqslant x\leqslant X$ in (1.1). In this case the asymptotic expansion (2.1) holds and $\varphi(x)$ is in general only continuous at $x\to a$, while, in force of (3.4),

$$y(t)=q(t-a)^{q-1}\left[g((t-a)^q+a)+\frac{K(a,a)\varphi(a)}{1-\alpha}[(t-a)^q]^{1-\alpha}+\mathrm{O}((t-a)^{2q(1-\alpha)})\right],\quad x\to a. \tag{3.9}$$

It appears from (3.9) that

$$y(t)=\mathrm{O}((t-a)^{2q-1-q\alpha}),\quad x\to a, \tag{3.10}$$

so that the solution of the transformed equation can be made as smooth as one likes by an appropriate choice of q.

3.2. Logarithmic kernels

In this case the transformed equation (3.5) becomes

$$y(t)=f(t)+\int_a^t \ln|(t-a)^q-(s-a)^q|K((t-a)^q+a,(s-a)^q+a)y(s)q(t-a)^{q-1}\,\mathrm{d}s.$$

If we define, as in Eq. (3.6),

$$\delta_0(t,s)=\begin{cases}\ln\left|\dfrac{(t-a)^q-(s-a)^q}{t-s}\right|, & t\neq s,\\ \ln(q(s-a)^{q-1}), & t=s,\end{cases}\tag{3.11}$$

we get

$$\begin{aligned}y(t)=f(t)&+\int_a^t \delta_0(t,s)K((t-a)^q+a,(s-a)^q+a)y(s)q(t-a)^{q-1}\,\mathrm{d}s\\ &+\int_a^t \ln|t-s|K((t-a)^q+a,(s-a)^q+a)y(s)q(t-a)^{q-1}\,\mathrm{d}s.\end{aligned}\tag{3.12}$$

At a first glance Eq. (3.12) looks more complex than Eqs. (1.1)–(1.2″). But this is not the case, because the perturbation kernel

$$\delta_0(t,s)K(\gamma(t),\gamma(s))\gamma'(t)$$

introduced by the transformation has only the fixed singularity at $t=s=a$ and can be made as smooth as we like, by means of the factor $\gamma'(t)$.

If we suppose $g(x)$ and $K(x,v)$ in (1.1) of class C^1 in $a\leqslant x\leqslant X$ and in $a\leqslant v\leqslant x\leqslant X$, respectively, so that (2.2) holds, then for the solution $y(t)$ of (3.12) we have

$$\begin{aligned}y(t)=q(t-a)^{q-1}&[g((t-a)^q+a)+K(a,a)\varphi(a)[(t-a)^q\ln(t-a)^q-(t-a)]\\ &+\mathrm{O}((t-a)^{2q}\ln^2(t-a)^q)],\quad x\to a,\end{aligned}\tag{3.13}$$

that is

$$y(t)=\mathrm{O}((t-a)^{2q-1-q\alpha_0}),\quad x\to a,\ \ \alpha_0\in R,\tag{3.14}$$

being as small as we like.

4. The Nyström-type method

Since the Nyström-type method is based on whole interval integration rule, we consider a closed integration interval $[a,b]$, replacing the transformation (3.1) by

$$x=(b-a)^{1-q}(t-a)^q+a,\quad q\in N\tag{4.1}$$

mapping $[a,b]$ into $[a,b]$.

4.1. Abel-type kernels

In this case by means of (4.1) we obtain the following equation:

$$y(t)=f(t)+\int_a^t |t-s|^{-\alpha}K_\alpha(t,s)y(s)\,\mathrm{d}s,\quad a\leqslant t\leqslant b,\tag{4.2}$$

where

$$K_\alpha(t,s) = ((b-a)^{1-q})^{1-\alpha} q(t-a)^{q-1}\delta_\alpha(t,s)$$
$$\times K((b-a)^{1-q}(t-a)^q + a, (b-a)^{1-q}(s-a)^q + a)$$

is a smooth function and

$$f(t) = q(t-a)^{q-1} g((b-a)^{1-q}(t-a)^q + a).$$

The smoothness of the solution $y(t) = q(t-a)^{q-1}\varphi((b-a)^{1-q}(t-a)^q + a)$ of (4.2) increases with q.

To define our Nyström method, we first set $z(t) = y((b-a)/2)t + (b+a)/2)$ and rewrite Eq. (4.2) as

$$z(t) = f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right) + \int_{-1}^{t} |t-s|^{-\alpha} H(t,s) z(s)\,\mathrm{d}s, \quad -1 \leqslant t \leqslant 1, \tag{4.3}$$

where

$$H(t,s) = \left(\frac{b-a}{2}\right)^{1-\alpha} K_\alpha\left(\frac{b-a}{2}t + \frac{b+a}{2}, \frac{b-a}{2}s + \frac{b+a}{2}\right).$$

Then we choose $N+1$ distinct points $x_n, n = 0,..,N$ in the interval $[-1,1]$ (in particular, we have chosen the nodes coinciding with the zeros of the Jacobi polynomial $P_N^{(1,0)}(x)$ in addition to the endpoint $x = 1$) and collocate Eq. (4.3) at the nodes

$$z(x_n) = f\left(\frac{b-a}{2}x_n + \frac{b+a}{2}\right) + \int_{-1}^{x_n} |x_n - s|^{-\alpha} H(x_n,s) z(s)\,\mathrm{d}s, \quad n = 0,1,\ldots,N. \tag{4.4}$$

We obtain the following scheme:

$$z_{N,n} = f\left(\frac{b-a}{2}x_n + \frac{b+a}{2}\right) + \sum_{j=0}^{N} w_{n,j} H(x_n,x_j) z_{N,j}, \quad n = 0,1,\ldots,N \tag{4.5}$$

by replacing $H(x_n,s)z(s)$ with the corresponding Lagrange interpolation polynomial associated with the Gauss–Radau nodes $x_n, n = 0,\ldots,N$.

For the computation of the coefficients

$$w_{n,j} = \int_{-1}^{x_n} |x_n - s|^{-\alpha} l_{N,j}(s)\,\mathrm{d}s,$$

we use the method described in [12]. We obtain the approximate solution values $z_{N,n},\ n = 0,1,\ldots,N$, as solution of system (4.5), by means of the linear systems solver F04ARF of the NAG Library. The corresponding values of $\varphi(x)$ can now be deduced and, in particular, setting $\varphi_{N,N} \simeq \varphi(b)$, we have $\varphi_{N,N} = (b-a)^{1-q} y_{N,N}/q$.

4.2. Logarithmic kernels

Let us now consider Eq. (1.1) with kernel (1.2″) in $a \leqslant x \leqslant b$, and assume for the sake of simplicity, and without loss of generality, $K(x,v) \equiv 1$. In this case the introduction of the smoothing transformation (4.1) leads to the new equation

$$y(t) = f(t) + (b-a)^{1-q} q(t-a)^{q-1} \times \left[\int_a^t \delta_0(t,s) y(s)\, \mathrm{d}s + \int_a^t \ln|t-s| y(s)\, \mathrm{d}s \right], \quad a \leqslant t \leqslant b, \tag{4.6}$$

where

$$\delta_0(t,s) = \begin{cases} \ln\left|(b-a)^{1-q} \dfrac{(t-a)^q - (s-a)^q}{t-s}\right|, & t \neq s, \\ \ln|(b-a)^{1-q} q(s-a)^{q-1}|, & t = s. \end{cases}$$

is a smooth function.

To define our Nyström method, we have now to compute the coefficients $wr_{n,j} = \int_{-1}^{x_n} l_{N,j}(s)\, \mathrm{d}s$ corresponding to the first integral in (4.6) and $wl_{n,j} = \int_{-1}^{x_n} \ln|x_n - s| l_{N,j}(s)\, \mathrm{d}s$ corresponding to the second integral in (4.6), where $l_{N,j}(s)$ is the fundamental Lagrange polynomial associated with the Radau nodes $x_n \in [-1, 1]$. To this aim, we have derived algorithms analogous to the ones used for the evaluation of $w_{n,j}$ in (4.5).

The methods derived in Sections 4.1 and 4.2 are both convergent. We do not give here the proof of this since the convergence proof given in [12] obviously is still valid in this case. But what is to be focused here is that the rate of convergence of the Nyström methods in the present case can be chosen as high as one likes: this follows from Theorems 1 and 3 of [12] and from (3.10) and (3.14).

5. Simpson's product integration

5.1. Abel-type kernels

Suppose we have transformed equations (1.1)–(1.2′) into (3.8) as described in Section 3 and put, for notational convenience,

$$K_\alpha(t,s) = q(t-a)^{q-1} K((t-a)^q + a, (s-a)^q + a)\delta_\alpha(t,s).$$

Then, we get

$$y(t) = f(t) + \int_a^t (t-s)^{-\alpha} K_\alpha(t,s) y(s)\, \mathrm{d}s. \tag{5.1}$$

Now, let us define a grid $t_j = a + jh,\ j = 0, \ldots, N;\ Nh = T - a$. By collocating the equation (5.1) on the grid points, we obtain

$$y(t_j) = f(t_j) + \int_a^{t_j} (t_j - s)^{-\alpha} K_\alpha(t_j, s) y(s)\, \mathrm{d}s, \quad j = 1, 2, \ldots, N, \tag{5.2}$$

that is

$$y(t_j) = f(t_j) + \sum_{0}^{m-1} \int_{a+2lh}^{a+(2l+2)h} (t_j - s)^{-\alpha} K_\alpha(t_j, s) y(s)\,\mathrm{d}s \quad \text{if } j = 2m, \text{ even}, \tag{5.3}$$

$$y(t_j) = f(t_j) + \sum_{0}^{m-2} \int_{a+2lh}^{a+(2l+2)h} (t_j - s)^{-\alpha} K_\alpha(t_j, s) y(s)\,\mathrm{d}s$$
$$+ \int_{a+(2m-2)h}^{a+(2m+1)h} (t_j - s)^{-\alpha} K_\alpha(t_j, s) y(s)\,\mathrm{d}s \quad \text{if } j = 2m+1, \text{ odd}. \tag{5.4}$$

For the computation of $y(t_j)$ we use product Simpson's rule in (5.3) and product Simpson's rule ended by a product three–eights rule in (5.4).

By approximating the regular part of the integrands by means of the three points interpolating Lagrange polynomial, and by introducing the change of variable $s = a + 2lh + ph$, we get

$$\begin{aligned} \int_{a+2lh}^{a+(2l+2)h} (t_j - s)^{-\alpha} K_\alpha(t_j, s) y(s)\,\mathrm{d}s \approx h^{1-\alpha} [& K_\alpha(t_j, s_{2l}) y(s_{2l}) b_0(j - 2l) \\ & + K_\alpha(t_j, s_{2l+1}) y(s_{2l+1}) b_1(j - 2l) \\ & + K_\alpha(t_j, s_{2l+2}) y(s_{2l+2}) b_2(j - 2l)] \end{aligned} \tag{5.5}$$

with

$$\begin{aligned} b_0(j-2l) &= \frac{1}{2} \int_0^2 (j - 2l - p)^{-\alpha} (p-1)(p-2)\,\mathrm{d}p, \\ b_1(j-2l) &= -\int_0^2 (j - 2l - p)^{-\alpha} p(p-2)\,\mathrm{d}p, \\ b_2(j-2l) &= \frac{1}{2} \int_0^2 (j - 2l - p)^{-\alpha} p(p-1)\,\mathrm{d}p. \end{aligned} \tag{5.6}$$

By approximating the regular part of the integrand by means of the four points Lagrange interpolating polynomial and by introducing the change of variable $s = a + (2m-2)h + ph$, we get

$$\begin{aligned} & \int_{a+(2m-2)h}^{a+(2m+1)h} (t_j - s)^{-\alpha} K_\alpha(t_j, s) y(s)\,\mathrm{d}s \\ & \quad \approx h^{1-\alpha} [K_\alpha(t_j, s_{2m-2}) y(s_{2m-2}) d_0(3) \\ & \qquad + K_\alpha(t_j, s_{2m-1}) y(s_{2m-1}) d_1(3) + K_\alpha(t_j, s_{2m}) y(s_{2m}) d_2(3) \\ & \qquad + K_\alpha(t_j, s_{2m+1}) y(s_{2m+1}) d_3(3)], \end{aligned} \tag{5.7}$$

with

$$d_0(3) = -\frac{1}{6}\int_0^3 (3-p)^{-\alpha}(p-1)(p-2)(p-3)\,\mathrm{d}p,$$

$$d_1(3) = \frac{1}{2}\int_0^3 (3-p)^{-\alpha}p(p-2)(p-3)\,\mathrm{d}p,$$

$$d_2(3) = -\frac{1}{2}\int_0^3 (3-p)^{-\alpha}p(p-1)(p-3)\,\mathrm{d}p,$$

$$d_3(3) = \frac{1}{6}\int_0^3 (3-p)^{-\alpha}p(p-1)(p-2)\,\mathrm{d}p. \tag{5.8}$$

By substitution of (5.5) and (5.7) into (5.3) and (5.4), following [5], given the starting values $\tilde{y}_i,\ i=0,1$ the discretization method can be written in the form

$$(I + h^{1-\alpha}A_N)y - f = 0, \tag{5.9}$$

where A_N is a triangular matrix containing the weights and $y = (y_0, y_1, \ldots, y_N)^{\mathrm{T}}$ contains the approximate values of $y(t_j)$ and $f = (\tilde{y}_0, \tilde{y}_1, g(t_2), \ldots, g(t_N))^{\mathrm{T}}$. The triangular system is then solved by forward substitution giving the values $y_2, \ldots, y_N$.

5.2. Logarithmic kernels

For the sake of simplicity set

$$\bar{K}(t,s) = \gamma'(t)K(\gamma(t), \gamma(s))$$

and rewrite Eq. (3.12) in the form

$$y(t) = f(t) + \int_a^t \delta_0(t,s)\bar{K}(t,s)y(s)\,\mathrm{d}s + \int_a^t \ln|t-s|\bar{K}(t,s)y(s)\,\mathrm{d}s. \tag{5.10}$$

Let define a grid $t_j = a + jh,\ j = 0, 1, \ldots, N,\ Nh = T - a$, and collocate (5.10) on these points. We obtain

$$y(t_j) = f(t_j) + \int_a^{t_j} \delta_0(t_j,s)\bar{K}(t_j,s)y(s)\,\mathrm{d}s + \int_a^{t_j} \ln|t_j - s|\bar{K}(t_j,s)y(s)\,\mathrm{d}s, \tag{5.11}$$

that is

$$\begin{aligned} y(t_j) = f(t_j) &+ \sum_0^{m-1} \int_{a+2lh}^{a+(2l+2)h} \ln(t_j - s)\bar{K}(t_j,s)y(s)\,\mathrm{d}s \\ &+ \sum_0^{m-1} \int_{a+2lh}^{a+(2l+2)h} \delta_0(t_j,s)\bar{K}(t_j,s)y(s)\,\mathrm{d}s \quad \text{if } j = 2m, \text{ even,} \end{aligned} \tag{5.12}$$

$$
\begin{aligned}
y(t_j) = f(t_j) &+ \sum_{0}^{m-2} \int_{a+2lh}^{a+(2l+2)h} \ln(t_j - s)\bar{K}(t_j,s)y(s)\,\mathrm{d}s \\
&+ \sum_{0}^{m-2} \int_{a+2lh}^{a+(2l+2)h} \delta_0(t_j,s)\bar{K}(t_j,s)y(s)\,\mathrm{d}s \\
&+ \int_{a+(2m-2)h}^{a+(2m+1)h} \ln(t_j - s)\bar{K}(t_j,s)y(s)\,\mathrm{d}s \\
&+ \int_{a+(2m-2)h}^{a+(2m+1)h} \delta_0(t_j,s)\bar{K}(t_j,s)y(s)\,\mathrm{d}s \quad \text{if } j = 2m+1, \text{ odd.}
\end{aligned} \tag{5.13}
$$

For the approximation of $y(t_j)$, we use product and nonproduct Simpson's rule in (5.12) and product and nonproduct Simpson's rule ended by a product three–eights rule in (5.13). The analogous of (5.5) and (5.7) are

$$
\begin{aligned}
&\int_{a+2lh}^{a+(2l+2)h} \ln(t_j - s)\bar{K}(t_j,s)y(s)\,\mathrm{d}s \\
&\quad\approx h\Bigg[\bar{K}(t_j,s_{2l})y(s_{2l})\left[\frac{1}{3}(\ln h + \delta_0(t_j,s_{2l})) + a_0(j-2l)\right] \\
&\qquad + \bar{K}(t_j,s_{2l+1})y(s_{2l+1})\left[\frac{4}{3}(\ln h + \delta_0(t_j,s_{2l+1})) + a_1(j-2l)\right] \\
&\qquad + \bar{K}(t_j,s_{2l+2})y(s_{2l+2})\left[\frac{1}{3}(\ln h + \delta_0(t_j,s_{2l+2})) + a_2(j-2l)\right]\Bigg]
\end{aligned} \tag{5.14}
$$

with

$$
\begin{aligned}
a_0(j-2l) &= \frac{1}{2}\int_0^2 \ln((j-2l-p)(p-1)(p-2)\,\mathrm{d}p, \\
a_1(j-2l) &= -\int_0^2 \ln(j-2l-p)p(p-2)\,\mathrm{d}p, \\
a_2(j-2l) &= \frac{1}{2}\int_0^2 \ln(j-2l-p)p(p-1)\,\mathrm{d}p
\end{aligned} \tag{5.15}
$$

and

$$
\begin{aligned}
&\int_{a+(2m-2)h}^{a+(2m+1)h} \ln(t_j - s)\bar{K}(t_j,s)y(s)\,\mathrm{d}s \\
&\quad\approx h\Bigg[\bar{K}(t_j,s_{2m-2})y(s_{2m-2})\left[\frac{3}{8}(\ln h + \delta_0(t_j,s_{2m-2})) + f_0(3)\right] \\
&\qquad + \bar{K}(t_j,s_{2m-1})y(s_{2m-1})\left[\frac{9}{8}(\ln h + \delta_0(t_j,s_{2m-1})) + f_1(3)\right]
\end{aligned}
$$

$$+\bar{K}(t_j, s_{2m})y(s_{2m})\left[\frac{9}{8}(\ln h + \delta_0(t_j, s_{2m})) + f_2(3)\right]$$

$$+\bar{K}(t_j, s_{2m+1})y(s_{2m+1})\left[\frac{3}{8}(\ln h + \delta_0(t_j, s_{2m+1})) + f_3(3)\right]\Bigg] \tag{5.16}$$

with

$$f_0(3) = -\frac{1}{6}\int_0^3 \ln(3-p)(p-1)(p-2)(p-3)\,\mathrm{d}p,$$

$$f_1(3) = \frac{1}{2}\int_0^3 \ln(3-p)p(p-2)(p-3)\,\mathrm{d}p,$$

$$f_2(3) = -\frac{1}{2}\int_0^3 \ln(3-p)p(p-1)(p-3)\,\mathrm{d}p,$$

$$f_3(3) = \frac{1}{6}\int_0^3 \ln(3-p)p(p-1)(p-2)\,\mathrm{d}p. \tag{5.17}$$

By substitution of (5.14) and (5.16) into (5.12) and (5.13), knowing the starting values $\tilde{y}_i,\ i=0,1$, the discretization method produces also in this case a triangular linear system, which is easily solved.

The convergence of product integration methods like those described in Section 5.1 for Eq. (1.1), with a weakly singular kernel of type (1.2′), using three and four points, has been studied by several authors. In particular, it follows from the results of [5] that such methods have order of convergence $4-\alpha$ in the three-points case and 4 in the four-points case, provided that the solution of (1.1) is of class $C^3[a,T]$. Now this case does not frequently occur in practice, but, whatever the behaviour of the solution $\varphi(x)$ of (1.1)–(1.2′) is, the solution $y(t)$ of Eq. (3.8) can be made as regular as one needs, by means of the smoothing technique and by a suitable choice of the parameter q. Therefore, the above maximum order of convergence can always be attained. As a matter of fact, if $g,\ K \in C^1,\ a \leqslant v \leqslant x \leqslant X$, as it was already noticed, the solution $\varphi(x)$ of (1.1) is in general only continuous; to get a transformed equation with solution of class $C^{(k)}$ it is sufficient to take $q \geqslant (k+1)/(2-\alpha)$.

To determine the order of convergence of the above-mentioned 3 and 4 point product integration methods when the kernel is of type (1.2″), it is sufficient to observe that

$$\ln|x-s| = \frac{\ln|x-s||x-s|^\alpha}{|x-s|^\alpha} = h(x,s)|x-s|^\alpha$$

for any $\alpha,\ 0<\alpha<1$, h being a continuous function. As a consequence of this, all the results valid for Abel-type equations are still applicable and it is possible to say that the order of convergence of the present method is $4-\alpha$, with α as small as we like.

6. Numerical results

We have tested the Nyström method of Section 4 and the "step-by-step" method of Section 5 on some weakly singular linear integral equations with kernels of type (1.2′) and of type (1.2″). We give in the following tables the results obtained for them. The equations and their exact solutions are listed below:

$$\varphi(x) = \frac{1}{2}\pi x + \sqrt{x} + \int_0^x -\varphi(v)(x-v)^{-1/2}\,\mathrm{d}v, \quad 0 \leqslant x \leqslant 10,$$

$$\varphi(x) = \sqrt{x}, \tag{6.1}$$

$$\varphi(x) = 1 + \int_0^x -\varphi(v)(x-v)^{-1/2}\,\mathrm{d}v, \quad 0 \leqslant x \leqslant 10,$$

$$\varphi(x) = \exp(\pi x)\,\mathrm{erfc}(\sqrt{\pi x}), \tag{6.2}$$

$$\varphi(x) = 1 - \exp(-x) + \frac{1}{\sqrt{\pi}}\int_0^x -\varphi(v)(x-v)^{-1/2}\,\mathrm{d}v, \quad 0 \leqslant x \leqslant 10,$$

$$\varphi(x) = \frac{1}{2}(\exp(x)\,\mathrm{erfc}(\sqrt{x}) - \exp(-x) + 2\sqrt{\pi}\mathrm{F}(\sqrt{x})), \tag{6.3}$$

$$F(x) = \exp(-x^2)\int_0^x \exp(t^2)\,\mathrm{d}t,$$

$$\varphi(x) = g(x) + \int_{-1}^x -\ln|x-v|\varphi(v)\,\mathrm{d}v, \quad -1 \leqslant x \leqslant 1, \tag{6.4}$$

$$g(x) = \sqrt{x+1} + \tfrac{4}{3}\sqrt{x+1}(x+1)\ln(2\sqrt{x+1}) - \tfrac{16}{9}\sqrt{x+1}(x+1),$$

$$\varphi(x) = \sqrt{x+1}.$$

Tables 1–4 show the relative errors obtained in the original solution $\varphi(x)$ by a 4, 8, 16 and 32-point Nyström method applied to Eqs. (6.1)–(6.4), respectively. Tables 5–8 show the relative

Table 1
Relative errors obtained by Nyström method for Eq. (6.1)

x	N	$q=1$	$q=2$	$q=3$
0.2	4	1.7d-04	6.6d-06	3.1d-06
	8	2.2d-05	1.1d-08	2.5d-08
	16	3.0d-06	5.3d-11	1.5d-11
	32	4.0d-07	2.6d-13	1.3d-14
1	4	2.4d-04	1.8d-05	1.1d-05
	8	2.4d-05	3.4d-08	6.8d-08
	16	2.6d-06	1.2d-10	3.7d-11
	32	3.3d-07	5.0d-13	2.5d-14
10	4	6.9d-04	6.2d-05	4.3d-05
	8	5.5d-05	2.2d-07	2.7d-07
	16	3.1d-06	5.2d-10	1.7d-10
	32	2.0d-07	1.1d-12	1.0d-13

Table 2
Relative errors obtained by Nyström method for Eq. (6.2)

x	N	$q=1$	$q=2$	$q=3$
0.2	4	2.9d-04	3.4d-05	1.1d-04
	8	4.0d-05	1.9d-07	3.9d-08
	16	5.4d-06	5.3d-09	1.4d-10
	32	7.2d-07	1.0d-10	4.3d-13
1	4	1.5d-03	4.8d-04	8.3d-04
	8	1.6d-04	3.5d-07	1.7d-06
	16	1.9d-05	1.5d-08	2.7d-10
	32	2.3d-06	3.2d-10	1.3d-12
10	4	1.5d-02	6.2d-03	1.8d-03
	8	2.1d-03	1.7d-04	2.1d-05
	16	1.7d-04	1.2d-07	1.9d-09
	32	1.2d-05	6.7d-10	1.1d-12

Table 3
Relative errors obtained by Nyström method for Eq. (6.3)

x	N	$q=1$	$q=2$	$q=3$
0.2	4	4.0d-06	1.2d-05	1.1d-04
	8	1.2d-07	2.2d-09	5.7d-09
	16	3.9d-09	1.2d-13	3.1d-14
	32	1.3d-10	3.3d-16	6.5d-16
1	4	3.8d-05	1.7d-04	1.8d-04
	8	8.3d-07	3.1d-08	5.1d-07
	16	2.2d-08	5.3d-12	2.2d-13
	32	6.6d-10	9.7d-15	4.5d-15
10	4	7.3d-03	1.0d-02	3.0d-03
	8	2.2d-04	6.7d-05	1.6d-04
	16	2.0d-06	2.4d-09	4.2d-08
	32	3.2d-08	3.0d-13	2.2d-14

errors obtained in the original solution $\varphi(x)$ by the step-by-step method on 10, 50, 100, 200 and 300 points applied to the same equations.

We have tested the methods for values of the argument x lying close to the singularity of the solution and for values lying where the solution is well-behaved. Of course, in each table the column $q=1$ is obtained by solving the original equation without any smoothing: in this case the methods converge slowly in presence of a singularity.

In general the results improve as q increases; the case $q=2$ is not significant for Eqs. (6.1) and (6.4), because with this choice of the smoothing parameter the singularity disappears and the

Table 4
Relative errors obtained by Nyström method for Eq. (6.4)

x	N	$q=1$	$q=2$	$q=3$
−0.99	4	4.2d-05	8.3d-10	1.8d-07
	8	7.0d-06	4.0d-12	3.0d-10
	16	1.0d-06	7.2d-15	7.7d-13
	32	1.4d-07	5.2d-17	1.9d-15
0.0	4	3.8d-03	1.0d-05	1.1d-05
	8	5.9d-04	3.2d-08	1.3d-08
	16	8.5d-05	5.8d-11	6.0d-11
	32	1.2d-05	8.1d-14	1.5d-13
1.0	4	5.5d-03	9.3d-05	2.7d-04
	8	5.9d-04	6.7d-08	8.9d-09
	16	7.5d-05	9.9d-11	4.6d-11
	32	9.7d-06	1.3d-13	1.2d-13

Table 5
Relative errors obtained by step-by-step method for Eq. (6.1)

x	N	$q=1$	$q=2$	$q=3$
0.2	10	4.6d-04	6.8d-06	3.7d-05
	50	3.6d-05	1.7d-08	1.3d-07
	100	1.2d-05	1.3d-09	1.1d-08
	200	4.3d-06	1.1d-10	1.0d-09
	300	2.2d-06	2.5d-11	2.4d-10
1.0	10	4.7d-04	1.6d-05	7.3d-05
	50	3.5d-05	3.4d-08	2.5d-07
	100	1.1d-05	2.6d-09	2.2d-08
	200	3.8d-06	2.1d-10	1.9d-09
	300	2.0d-06	5.0d-11	4.7d-10
10	10	2.1d-05	7.5d-05	2.1d-04
	50	2.2d-05	1.0d-07	7.6d-07
	100	6.7d-06	8.1d-09	6.5d-08
	200	2.1d-06	6.7d-10	5.6d-09
	300	1.0d-06	1.5d-10	1.3d-09

solution becomes a polynomial. For these equations the comparison has to be made between the columns $q=1$ and 3; in any case the relative errors obtained with $q=3$ are not significantly worse than those obtained with $q=2$.

In general (see Tables 5–8), the application of a very simple method like Simpson's method to the equations transformed with $q=2$ produces satisfactory relative errors.

Table 6
Relative errors obtained by step-by-step method for Eq. (6.2)

x	N	$q=1$	$q=2$	$q=3$
0.2	10	1.4d-03	4.1d-05	5.2d-06
	50	1.1d-04	4.0d-07	4.2d-08
	100	4.0d-05	5.2d-08	4.4d-09
	200	1.4d-05	6.6d-09	4.2d-10
	300	7.9d-06	1.9d-09	1.0d-10
1.0	10	4.0d-03	1.2d-04	9.3d-06
	50	3.3d-04	1.4d-06	1.8d-07
	100	1.2d-04	1.7d-07	1.6d-08
	200	4.4d-05	2.2d-08	1.4d-09
	300	2.4d-05	6.5d-09	3.6d-10
10	10	3.2d-03	6.1d-05	2.8d-04
	50	7.0d-04	5.3d-06	3.4d-07
	100	3.1d-04	6.8d-07	2.6d-08
	200	1.3d-04	8.5d-08	2.1d-09
	300	7.5d-05	2.5d-08	4.7d-10

Table 7
Relative errors obtained by step-by-step method for Eq. (6.3)

x	N	$q=1$	$q=2$	$q=3$
0.2	10	2.6d-05	1.1d-05	3.2d-05
	50	4.7d-07	5.0d-08	4.9d-08
	100	8.1d-08	4.5d-09	2.9d-09
	200	1.3d-08	4.0d-10	1.7d-10
	300	3.8d-09	8.4d-11	3.0d-11
1.0	10	1.3d-04	5.4d-05	2.0d-04
	50	2.3d-06	1.6d-07	1.0d-06
	100	3.9d-07	1.3d-08	9.2d-08
	200	6.4d-08	1.0d-09	8.2d-09
	300	1.9d-08	2.1d-10	1.9d-09
10	10	3.5d-03	1.2d-04	1.7d-04
	50	7.4d-05	1.4d-07	1.7d-07
	100	1.3d-05	1.6d-08	1.3d-08
	200	2.1d-06	1.5d-09	2.3d-09
	300	6.2d-07	3.4d-10	5.3d-10

A few words are needed about the choice of the smoothing parameter q: even if in theory it was shown that the order of convergence of our methods increases with q, in practice we suggest not to exceed $q=3$.

Table 8
Relative errors obtained by step-by-step method for Eq. (6.4)

x	N	$q=1$	$q=2$	$q=3$
−0.99	10	6.7d-03	6.2d-08	3.5d-06
	50	6.7d-04	8.8d-11	8.1d-09
	100	3.9d-06	5.3d-12	5.7d-10
	200	1.4d-06	3.3d-13	3.9d-11
	300	7.6d-07	6.4d-14	8.3d-12
0.0	10	8.5d-03	7.9d-05	6.3d-04
	50	8.7d-04	9.1d-08	1.9d-06
	100	3.1d-04	5.0d-09	1.4d-07
	200	1.1d-04	2.9d-10	1.0d-08
	300	6.1d-05	5.8d-11	2.2d-09
1.0	10	7.1d-03	4.5d-05	1.7d-03
	50	7.2d-04	9.4d-08	5.2d-06
	100	2.6d-04	6.5d-09	3.9d-07
	200	9.3d-05	4.4d-10	2.9d-08
	300	5.1d-05	9.6d-11	6.3d-09

Table 9
Comparison with [12]

Eq.	N2	NS
(6.1)	2.5d-14	3.0d-08
(6.2)	1.3d-12	2.0d-07
(6.3)	4.5d-15	1.8d-09
(6.4)	1.2D-13	2.6D-04

As a matter of fact, higher values of q make the solution of the transformed equation more "flat", as $t \to a$, (see Eq. (3.1)); this implies an initial loss of accuracy for the very small values of N. Of course an optimal value of q could exist for each particular problem, and this value would depend on the behaviour of the solution of the problem itself; but the results obtained by $q=2$ and 3 are more than satisfactory.

On the other side, we observe that with the recommended moderate values of q, the inverse transformation of Section 3 allows to compute the solution of the original Eq. (1.1) very close to the singularity $x=a$. As an example, we were able to compute accurately the solution of the test equations at points lying at a distance of 10^{-8} from a.

The comparison of our results with those obtained by means of some of the methods cited in Section 1 is encouraging. In [12] one of the authors proposed to couple Nyström method with Simpson's method and apply them to Eq. (1.1). In Table 9, we compare the relative errors obtained for the Eqs. (6.1)–(6.4) at $x=1$ by means of 32-points Nyström method applied to equations transformed with $q=3$ (N2) and by means of 32-point Nyström method plus 80-points Simpson's method applied to the original equation (NS): the advantage of the new approach is evident.

Table 10
Comparison between Simpson's, Nyström's and Lubich's methods

x	N	Eq. (6.2)			Eq. (6.3)		
		Simpson	Nyström	Lubich	Simpson	Nyström	Lubich
0.2	4		1.1d-04	1.9d-08		1.1d-04	3.2d-09
	8		3.9d-08	1.8d-08		5.7d-09	3.2d-09
	16		1.4d-10	6.2d-09		3.1d-14	1.0d-09
	32		4.3d-13	1.1d-09		6.5d-16	1.7d-10
	64	1.8d-08	2.8d-15	1.4d-10	1.7d-08		2.1d-11
	128	1.0d-09		1.3d-11	1.0d-09		3.1d-12
1	4		8.3d-04	4.4d-06		1.8d-04	1.0d-07
	8		1.7d-06	3.1d-06		5.1d-07	1.2d-07
	16		2.7d-10	1.1d-06		2.2d-13	6.8d-08
	32		1.3d-12	2.2d-07		4.5d-15	1.4d-08
	64	7.8d-08	9.1d-15	2.9d-08	4.3d-07		1.8d-09
	128	7.0d-09		2.9d-09	3.9d-08		1.7d-10
10	4		1.8d-03	8.9d-04		3.0d-03	1.7d-03
	8		2.1d-05	6.2d-04		1.6d-04	3.8d-04
	16		1.9d-09	2.6d-04		4.2d-08	6.9d-05
	32		1.1d-12	6.8d-05		2.2d-14	2.8d-05
	64	1.3d-07	2.5d-14	1.3d-05	4.4d-09		3.0d-06
	128	1.0d-08		1.7d-06	7.5d-09		1.7d-07

For equations with kernel of type (2.1′), with $\alpha = \frac{1}{2}$, we were able to compare our method with the method introduced in [9], which is considered the most efficient for such equations, but that cannot be straightly extended to different kernels. We have applied Lubich's method to Eqs. (6.2) and (6.3) and the Nyström method and the step-by-step method to the same equations transformed with $q = 3$.

Looking at Table 10, we can observe that for $N = 4$ the fractional linear method gives very good approximations, but the convergence is very slow, in particular when the solution is singular. The Nyström method combined with the smoothing procedure has a much higher rate of convergence. While it gives lower accuracy for $N = 4$, it becomes rapidly superior for the larger values of N.

To make our analysis more reliable we have also computed the time requested by the different methods to achieve a prescribed accuracy of the solution. Also from this point of view we can affirm that Nyström method applied after smoothing is the most efficient, because it results to be from three (close to the singularity) to ten and more times (going far from the singularity) faster than the others.

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