

Unified and extended form of three types of splines

Guozhao Wang, Mei'e Fang*

Institute of Computer and Image Processing and the Department of Mathematics, Zhejiang University, Hangzhou 310027, China

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Abstract

The three types refer to polynomial, trigonometric and hyperbolic splines. In this paper, we unify and extend them by a new kind of spline (UE-spline for short) defined over the space $\{\cos \omega t, \sin \omega t, 1, t, \dots, t^l, \dots\}$, where l is an arbitrary nonnegative integer. ω is a frequency sequence $\{\omega_i = \sqrt{\alpha_i}\}_{i=-\infty}^{+\infty}$, $\alpha_i \in \mathfrak{R}$. Existing splines, such as usual polynomial B-splines, CB-splines, HB-splines, NUAT splines, AH splines, FB-splines and the third form FB-splines etc., are all special cases of UE-splines. UE-splines inherit most properties of usual polynomial B-splines and enjoy some other advantageous properties for modelling. They can exactly represent classical conics, the catenary, the helix, and even the eight curve, a kind of *snake-like* curves etc.

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1. Introduction

B-splines are popularly applied in modelling free-form curves and surfaces because of their unified mathematic representation and many desirable properties. But they still have their limitations. This has been discussed in [10]. So it is important to find more useful splines, which not only keep back the shortcomings of B-splines, but also inherit the desirable properties of polynomial B-splines. There merged many relative literatures [4–6, 11–14]. In recent years, several new splines defined in nonpolynomial spaces have been proposed. CB-splines were introduced in [16, 17]. Exponential B-splines have been studied in [5]. But these bases do not overlap in the cases of high order. Then Lu et al. [8] proposed uniform hyperbolic B-splines in the space $\{\sinh t, \cosh t, t^{k-3}, \dots, t, 1\}$. Mainar and Peña, [9] Chen and Wang [1] did some works on C-Bezier basis in the space $\{\sin t, \cos t, t^{k-3}, \dots, t, 1\}$. In the same space, nonuniform algebraic–trigonometric B-splines (NUAT splines) were constructed in [15]. A subdivision scheme on trigonometric spline was proposed in [2, 3]. Later, hyperbolic splines were also extended to the case of nonuniform knot vector in [7]. That is AH splines.

So many types of splines over different spaces have been proposed. Each type has its own merits. If only there existed a unified form of them over a certain common space! Zhang et al. [19] did a good job for this. They unified CB-splines and HB-splines into FB-splines (Functional B-splines) in [18]. Again, by extending the calculation to complex numbers,

* Corresponding author.

E-mail addresses: fangmeie@163.com, mimi52mimi2002@yahoo.com.cn (M. Fang).

the third form FB-splines with different parameters were proposed in [19]. While the third form FB-splines are limited in the case of uniform knot vector and order 4. This is not sufficient for modelling free-form curves and surfaces. Furthermore, we think that there should exist a unified form of all splines listed above, including nonuniform splines of arbitrary orders. Fortunately, we achieve this idea through introducing a frequency sequence and defining UE-splines by an integral method over the space $\{\cos \omega t, \sin \omega t, 1, t, \dots, t^l, \dots\}$. UE-splines not only unify the above-mentioned splines, but also include much more various forms. And they inherit most of the desirable properties of usual polynomial B-splines too.

The rest of this paper is structured as follows. Section 2 gives the definition and classification of UE-spline bases, proves that the above-mentioned splines are special cases of UE-splines and introduces UE-Bezier. The properties of UE-spline bases are discussed in Section 3. And Section 4 defines UE-spline curves. Their properties and some examples are also shown in this section. Section 5 concludes the paper.

2. UE-spline bases

2.1. The definition of UE-spline bases

Definition 2.1 (UE-spline bases). Let T be a given knot sequence $\{t_i\}_{-\infty}^{+\infty}$ with $t_i \leq t_{i+1}$, ω be a given frequency sequence $\{\omega_i = \sqrt{\alpha_i}\}_{-\infty}^{+\infty}$, where $\alpha_i \in \mathfrak{R}$ and $\alpha_i \leq \min_{j=i,i+1}(\pi/(t_{j+1} - t_j))^2$. $N_{i,k}(t)$ constructed by the following formulae is called a UE-spline basis in the span of $\{\cos \omega t, \sin \omega t, 1, t, \dots, t^l, \dots\}$ for $t \in [t_i, t_{i+k})$.

We first define UE-spline basis functions of order $k = 2$ as follows:

$$N_{i,2}(t) = \begin{cases} \frac{\sin \omega_i(t - t_i)}{\sin \omega_i(t_{i+1} - t_i)}, & t_i \leq t < t_{i+1}, \\ \frac{\sin \omega_i(t_{i+2} - t)}{\sin \omega_i(t_{i+2} - t_{i+1})}, & t_{i+1} \leq t < t_{i+2}, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

For $k \geq 3$, $N_{i,k}(t)$ is defined recursively by

$$N_{i,k}(t) = \int_{-\infty}^t (\delta_{i,k-1} N_{i,k-1}(s) - \delta_{i+1,k-1} N_{i+1,k-1}(s)) ds, \quad (2)$$

where $\delta_{i,k-1} = (\int_{-\infty}^{+\infty} N_{i,k-1}(t) dt)^{-1}$, $i = 0, \pm 1, \dots$.

In the case $\omega_i = 0$, we set $0/0 = 0$, $\delta_{i,k} N_{i,k}(t) = 0$ if $N_{i,k}(t) = 0$. Otherwise, we compute it by the L'Hospital rule. Additionally, we set $\delta_{i,k} N_{i,k}(t)$ to satisfy

$$\int_{-\infty}^t \delta_{i,k} N_{i,k}(t) dt = \begin{cases} 1, & t \geq t_{i+k}, \\ 0, & t < t_{i+k}, \end{cases}$$

if $N_{i,k}(t) = 0$.

Remark 2.1. For the frequency sequence, ω_i , respectively, takes a value as real nonzero number, zero or a pure imaginary number when $\alpha_i > 0$, $\alpha_i = 0$ or $\alpha_i < 0$. In order to ensure the positivity of UE-spline basis, we limit $\alpha_i \leq \min_{j=i,i+1}(\pi/(t_{j+1} - t_j))^2$ for all ω_i in Definition 2.1. Actually, this condition naturally holds when $\alpha_i = 0$ and $\alpha_i < 0$.

2.2. The classification of UE-splines

In the above definition of UE-spline basis, we see each knot interval $[t_i, t_{i+1})$ corresponds to its unique frequency $\omega_i = \sqrt{\alpha_i}$ ($\alpha_i \in \mathfrak{R}$ and $\alpha_i \leq \min_{j=i,i+1}(\pi/(t_{j+1} - t_j))^2$). In terms of the signs of α_i ($i = 0, \pm 1, \dots$), UE-splines can be classified into four types: polynomial type ($\alpha_i \equiv 0$), trigonometric type ($\alpha_i > 0$ and $\alpha_i \leq \min_{j=i,i+1}(\pi/(t_{j+1} - t_j))^2$),

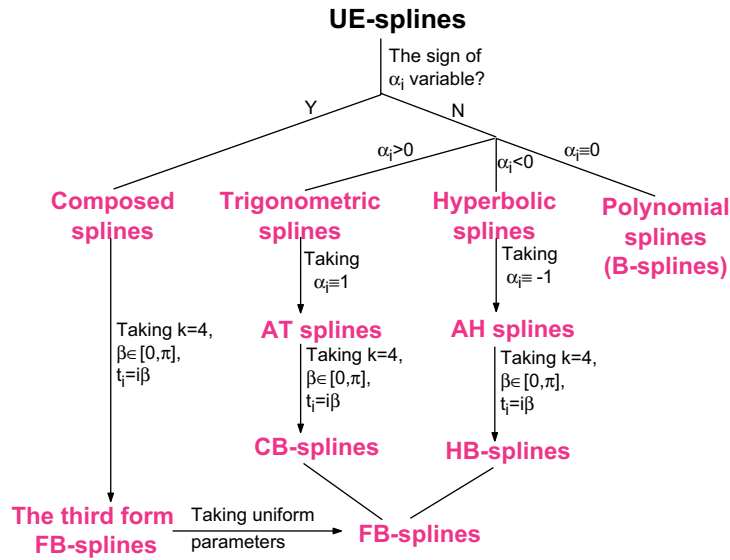


Fig. 1. The classification of UE-splines.

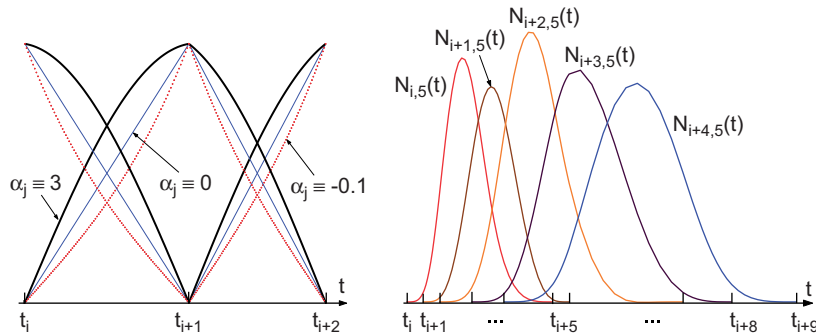


Fig. 2. The UE-spline bases of order 2 and 5.

hyperbolic type ($\alpha_i < 0$) and composed type (α_i involve more than two types of signs). Fig. 1 shows the classification of UE-splines and the relationship of all kinds of existing splines. Fig. 2 (left) illustrates three groups of UE-spline bases of order 2 in the knot interval $[t_i, t_{i+2})$. They belong to trigonometric type ($\{\alpha_j\}_{j=i}^{i+1} \equiv 3$), polynomial type ($\{\alpha_j\}_{j=i}^{i+1} \equiv 0$) and hyperbolic type ($\{\alpha_j\}_{j=i}^{i+1} \equiv -0.1$), respectively. Fig. 2 (right) illustrates a group of UE-spline bases of order 5 in the knot interval $[t_i, t_{i+9})$ with $\{\alpha_j\}_{j=i}^{i+8} = \{9, -4, 1, 2.25, 0, 0.16, 0, -1, 0\}$. α_j ($j = i, \dots, i+8$) involve different types of signs, so these bases belong to composed type.

Now we prove that usual polynomial B-splines, AT splines, AH splines and the third form FB-splines are all special cases of UE-splines in the following propositions.

Lemma 2.1. $\int_{-\infty}^{+\infty} N_{i,k}^*(t) dt = (t_{i+k} - t_i)/k$, here $N_{i,k}^*(t)$ denote usual polynomial B-spline bases.

Proof. By the difference quotient definition of usual polynomial B-splines: $N_{i,k}^*(t) = (t_{i+k} - t_i)[t_i, \dots, t_{i+k}](x - t)_+^{k-1}$, where $(x - t)_+^{k-1} = (\max\{(x - t), 0\})^{k-1}$ is a truncated power function, we obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} N_{i,k}^*(t) dt &= \int_{-\infty}^{+\infty} (t_{i+k} - t_i)[t_i, \dots, t_{i+k}](x - t)_+^{k-1} dt \\ &= (t_{i+k} - t_i) \cdot \int_{t_i}^{t_{i+k}} [t_i, \dots, t_{i+k}](x - t)_+^{k-1} dt \\ &= \frac{t_{i+k} - t_i}{k} \cdot ([t_i, \dots, t_{i+k}](x - t_i)_+^k - [t_i, \dots, t_{i+k}](x - t_{i+k})_+^k) \\ &= \frac{t_{i+k} - t_i}{k} \cdot [t_i, \dots, t_{i+k}](x - t_i)^k \\ &= \frac{t_{i+k} - t_i}{k}. \quad \square \end{aligned}$$

Proposition 2.1. Taking $\alpha_i \equiv 0$ ($i = 0, \pm 1, \dots$) in the frequency sequence ω in Definition 2.1, we get UE-spline bases equivalent to usual polynomial B-spline bases, i.e., $N_{i,k}(t) = N_{i,k}^*(t)$ ($k \geq 2$) when $\omega \equiv 0$ ($i = 0, \pm 1, \dots$).

Proof. When $k = 2$, we convert formulae (1) into its equivalent form as follows:

$$N_{i,2}(t) = \begin{cases} \frac{\int_{t_i}^t \cos \omega_i(s - t_i) ds}{\int_{t_{i+1}}^{t_i} \cos \omega_i(s - t_i) ds}, & t_i \leq t < t_{i+1}, \\ \frac{\int_{t_{i+1}}^{t_{i+2}+t_{i+1}-t} \cos \omega_i(s - t_{i+1}) ds}{\int_{t_{i+1}}^{t_{i+2}} \cos \omega_i(s - t_{i+1}) ds}, & t_{i+1} \leq t < t_{i+2}, \\ 0, & \text{otherwise.} \end{cases}$$

Then it is easy to demonstrate the conclusion by substituting $\omega_i = \sqrt{\alpha_i} \equiv 0$ into the above formulae. Assume $N_{i,m}(t) = N_{i,m}^*(t)$ when $\alpha_i \equiv 0$ ($i = 0, \pm 1, \dots$). According to Lemma 2.1 and the difference quotient definition of B-splines, we get

$$\begin{aligned} \int_{-\infty}^t \delta_{i,m} N_{i,m}(s) ds &= \left(\int_{-\infty}^{+\infty} N_{i,m}(t) dt \right)^{-1} \int_{-\infty}^t N_{i,m}(s) ds \\ &= \left(\int_{-\infty}^{+\infty} N_{i,m}^*(t) dt \right)^{-1} \int_{-\infty}^t N_{i,m}^*(s) ds \\ &= \frac{m}{t_{i+m} - t_i} \int_{-\infty}^t N_{i,m}^*(s) ds \\ &= \frac{m}{t_{i+m} - t_i} \int_{-\infty}^t (t_{i+m} - t_i)[t_i, \dots, t_{i+m}](x - s)_+^{m-1} ds \\ &= -[t_i, \dots, t_{i+m}](x - t)_+^m. \end{aligned}$$

By the same reason, we have

$$\int_{-\infty}^t \delta_{i+1,m} N_{i+1,m}(s) ds = -[t_{i+1}, \dots, t_{i+m+1}](x - t)_+^m.$$

Then

$$\begin{aligned}
 N_{i,m+1}(t) &= \int_{-\infty}^t (\delta_{i,m} N_{i,m}(s) - \delta_{i+1,m} N_{i+1,m}(s)) \, ds \\
 &= [t_{i+1}, \dots, t_{i+m+1}](x-t)_+^m - [t_i, \dots, t_{i+m}](x-t)_+^m \\
 &= (t_{i+m+1} - t_i)[t_i, \dots, t_{i+m+1}](x-t)_+^m \\
 &= N_{i,m+1}^*(t).
 \end{aligned}$$

At the same time, the definition space is

$$\text{span}\{\cos \omega t, \sin \omega t, 1, t, \dots, t^l, \dots\} = \text{span}\{1, t, \dots, t^{k-1}\}. \quad \square$$

Proposition 2.2. Taking $\alpha_i > 0$ ($i = 0, \pm 1, \dots$) in ω in Definition 2.1, we get trigonometric spline basis over the space $\{\cos \omega t, \sin \omega t, 1, t, \dots, t^{k-3}\}$. Specially, NUAT splines proposed in [15] is just the case of $\omega_i = \sqrt{\alpha_i} = 1$ ($i = 0, \pm 1, \dots$).

Proof. This conclusion is obvious. Because their recursive formulae for computing basis functions of order k ($k \geq 3$) is the same formulae (2), we only need to demonstrate their basis function of order 2 is equivalent. By formulae (1), we get

$$N_{i,2}(t) = \begin{cases} \frac{\sin \omega_i(t - t_i)}{\sin \omega_i(t_{i+1} - t_i)}, & t_i \leq t < t_{i+1}, \\ \frac{\sin \omega_i(t_{i+2} - t)}{\sin \omega_i(t_{i+2} - t_{i+1})}, & t_{i+1} \leq t < t_{i+2}, \\ 0, & \text{otherwise.} \end{cases}$$

Further substituting α with 1, we see that $N_{i,2}(t)$ is just NUAT spline basis of order 2. At the same time, the definition space is $\{\cos t, \sin t, 1, t, \dots, t^{k-3}\}$. \square

Proposition 2.3. Taking $\alpha_i < 0$ ($i = 0, \pm 1, \dots$) for ω in Definition 2.1, we get hyperbolic spline basis over the space $\{\cosh |\omega|t, \sinh |\omega|t, 1, t, \dots, t^{k-3}\}$, where $|\omega|$ represents the frequency sequence $\{|\omega_i|\}_{-\infty}^{+\infty}$. AH splines developed in [7] is just the case of $\alpha_i \equiv -1$ i.e. $\omega_i \equiv i, i = 0, \pm 1, \dots$.

Proof. This conclusion is based on the following identity in the domain of complex numbers:

$$\sin \omega_i t = \sin(|\omega_i|t \cdot i) = i \cdot \sinh(|\omega_i|t). \quad (3)$$

Similar to the proof of Proposition 2.2, we only need to demonstrate their basis function of order 2 is equivalent. According to the formulae (1) and (3), we have

$$N_{i,2}(t) = \begin{cases} \frac{\sinh |\omega_i|(t - t_i)}{\sinh |\omega_i|(t_{i+1} - t_i)}, & t_i \leq t < t_{i+1}, \\ \frac{\sinh |\omega_i|(t_{i+2} - t)}{\sinh |\omega_i|(t_{i+2} - t_{i+1})}, & t_{i+1} \leq t < t_{i+2}, \\ 0, & \text{otherwise.} \end{cases}$$

When $\alpha_i \equiv -1$, i.e., $\omega_i \equiv i$, $N_{i,2}(t)$ is just AH spline basis of order 2. We see that basis functions of UE-splines still belong to the real space spanned by $\{\cosh |\omega_i|t, \sinh |\omega_i|t, 1, t, \dots, t^{k-3}\}$ even if the values of the ω_i 's take pure imaginary numbers. \square

Proposition 2.4. The third FB-splines developed in [19] are actually UE-splines of order 4 with uniform knot sequence $\{t_i = (i-1)\beta, \beta \in [0, \pi]\}_{-\infty}^{+\infty}$ and $\omega_i = \beta_{i+1}/\beta$.

Proof. Substituting $k = 4$ and $t_i = (i - 1)\beta$ into the formulas (1) and (2), we obtain uniform UE-spline basis $N_{i,4}(t)$ on its nonzero interval $[t_i, t_{i+4})$ as follows.

$$N_{i,4}(t) = \begin{cases} \frac{\omega_i(t - i\beta) - \sin(\omega_i(t - i\beta))}{2\beta\omega_i A_i}, & t_i \leq t < t_{i+1}, \\ \frac{1}{2A_i} + C_1(t - t_{i+1}) + \frac{\sin(\omega_{i+1}(t - (i+1)\beta))}{\beta\omega_{i+1} A_{i+1}} + \frac{\sin(\omega_i(t - (i+2)\beta))}{2\beta\omega_i A_i}, & t_{i+1} \leq t < t_{i+2}, \\ \frac{1}{2A_{i+2}} - C_2(t - t_{i+3}) - \frac{\sin(\omega_{i+1}(t - (i+3)\beta))}{\beta\omega_{i+1} A_{i+1}} - \frac{\sin(\omega_{i+2}(t - (i+2)\beta))}{2\beta\omega_{i+2} A_{i+2}}, & t_{i+2} \leq t < t_{i+3}, \\ \frac{\omega_{i+2}((i+4)\beta - t) - \sin(\omega_{i+2}((i+4)\beta - t))}{2\beta\omega_{i+2} A_{i+2}}, & t_{i+3} \leq t < t_{i+4}, \end{cases}$$

where $A_i = 1 - \cos(\beta\omega_i)$, $C_1 = (1/\beta)(1 - 1/2A_i - 1/A_{i+1})$, $C_2 = (1/\beta)(1 - 1/2A_{i+2} - 1/A_{i+1})$.

Reparameterize the above formulas by $\tau = t/\beta$. Then we calculate four nonzero functions $N_{ij}(\tau)$, $j = 0, 1, 2, 3$, on the interval $[0, 1]$. At last, taking $\omega_i = \beta_{i+1}/\beta$, we get

$$N_{i0}(\tau) = \frac{\beta_i(1 - \tau) - \sin(\beta_i(1 - \tau))}{2\beta_i(1 - \cos \beta_i)},$$

$$N_{i3}(\tau) = \frac{\beta_{i+1}\tau - \sin(\beta_{i+1}\tau)}{2\beta_{i+1}(1 - \cos \beta_{i+1})},$$

$$N_{i1}(\tau) = N_{i3}(\tau) - 2N_{i0}(\tau) + (1 - \tau),$$

$$N_{i2}(\tau) = N_{i0}(\tau) - 2N_{i3}(\tau) + \tau.$$

This is just the third form FB-spline basis in C style in [19]. \square

2.3. UE-Bezier

Remark 2.2. When the knot sequence $\{t_i\}_{-\infty}^{+\infty}$ satisfies $t_{i-k+1} = t_{i-k+2} = \dots = t_i < t_{i+1} = \dots = t_{i+k}$, we get UE-Bezier basis $B_{i,k}(t)$ of order k on $[t_i, t_{i+1}]$ over the space $\{\cos \omega_0 t, \sin \omega_0 t, 1, t, \dots, t^l, \dots\}$.

In order to prove the positivity of UE-spline basis in Section 3, we first prove the positivity of UE-Bezier basis as follows.

Lemma 2.2. Let $f = L(\sin \omega_0 t, \cos \omega_0 t, 1, t, \dots, t^{k-3})$, $k \geq 2$, where L is a linear function and t belongs to an arbitrary interval $[t_{i0}, t_{i0+1}]$. If $\omega_0 = \sqrt{\alpha}$ satisfies $\alpha \neq 0$ and $\alpha \leq (\pi/(t_{i0+1} - t_{i0}))^2$, then f has k zeros at most on $[t_{i0}, t_{i0+1}]$.

Proof. Assuming f has $k + 1$ zeros on $[t_{i0}, t_{i0+1}]$, we conclude that $f^{(k-1)}$ has two zeros on the open interval (t_{i0}, t_{i0+1}) according to Rolle's Theorem. Because $\alpha \neq 0$, we only need to prove it impossible in two cases.

- (1) If $\alpha < 0$, $f = L(\sinh \sqrt{|\alpha|}t, \cosh \sqrt{|\alpha|}t, 1, t, \dots, t^{k-3})$. There exist a, b, c_i ($i = 0, \dots, k-3$), $A, B \in \mathfrak{R}$, such that $f = a \sinh \sqrt{|\alpha|}t + b \cosh \sqrt{|\alpha|}t + \sum_{i=0}^{k-3} c_i t^i$ and $f^{(k-1)} = A \sinh \sqrt{|\alpha|}t + B \cosh \sqrt{|\alpha|}t = (A+B)/2e^{\sqrt{|\alpha|}t} + (A-B)/2e^{-\sqrt{|\alpha|}t}$. On the open interval, $f^{(k-1)}$ has one zero at most. This contradicts the assumption.
- (2) If $\alpha > 0$, $f = L(\sin \sqrt{\alpha}t, \cos \sqrt{\alpha}t, 1, t, \dots, t^{k-3})$. There exist a, b, c_i ($i = 0, \dots, k-3$), $A, B, C, \phi \in \mathfrak{R}$, such that $f = a \sin \sqrt{\alpha}t + b \cos \sqrt{\alpha}t + \sum_{i=0}^{k-3} c_i t^i$ and $f^{(k-1)} = A \sin \sqrt{\alpha}t + B \cos \sqrt{\alpha}t = C \sin(\sqrt{\alpha}t + \phi)$. We know $\sin(\sqrt{\alpha}t + \phi)$ has one zero at most on the open interval (t_{i0}, t_{i0+1}) if $\alpha \leq (\pi/(t_{i0+1} - t_{i0}))^2$. This contradicts the assumption too. \square

Lemma 2.3. The UE-Bezier basis are nonnegative on an arbitrary $[t_{i0}, t_{i0+1}]$ and positive on the corresponding open interval with proper α .

Proof. It is easy to conclude that UE-Bezier include three types: $\alpha = 0$, $\alpha > 0$, $\alpha < 0$.

When $\alpha = 0$, it is actually the usual Bezier according to Proposition 2.1 and Remark 2.2. So its positivity is obvious.

When $\alpha \neq 0$ and $\alpha \leq (\pi/(t_{i_0+1} - t_{i_0}))^2$, we consider an arbitrary UE-Bezier basis function $B_{i,k}(t)$, $k \geq 2$, $0 \leq i \leq k$. According to Lemma 2.2, we know $B_{i,k}(t)$ has k zeros on $[t_{i_0}, t_{i_0+1}]$. Again, these k zeros are the i -fold zero at t_{i_0} and the $(k - i)$ -fold zero at t_{i_0+1} by the properties at endpoints of UE-Bezier basis. So on the interval (t_{i_0}, t_{i_0+1}) , $B_{i,k}(t)$ has no zero. Based on the above analysis, it is easy to prove the positivity of UE-Bezier basis by a similar method [1] of proving the positivity of C-Bezier basis on (t_{i_0}, t_{i_0+1}) . \square

3. Properties of UE-spline basis

UE-spline bases enjoy many good properties as follows.

- (1) *Shape adjustable property*: Given a knot sequence, the basis functions of order k will be completely determined for those splines proposed previously. But for UE-splines, it is not the case. Fixing the knot sequence, the shape of UE-spline basis function can still be adjusted by changing the frequency sequence. Fig. 3 illustrated different UE-spline basis functions of order 3 with the same knot sequence but different frequency sequences. In the left figure, frequency sequences are generated randomly. In the right one, the frequency sequence corresponding to each basis function is constant sequence. α_i are marked in the figure.
- (2) *Differential*: $N_{i,k}(t)$ is $(k - r_j - 1)$ time continuously differential at the knot t_j with r_j the number of times t_j appears in the knot sequence $(t_j)_{i=1}^{i+k}$.
- (3) *Derivative*: $N'_{i,k}(t) = \delta_{i,k-1} N_{i,k-1}(t) - \delta_{i+1,k-1} N_{i+1,k-1}(t)$. When $N_{i,k-1} = 0$, $\delta_{i,k-1} N_{i,k-1} = 0$.
- (4) *Local support*: $N_{i,k}(t) = 0$, $t \notin [t_i, t_{i+k}]$.
- (5) *Zero function*: $N_{i,k}(t) \equiv 0$ if and only if $t_i = t_{i+1} = \dots = t_{i+k}$.
- (6) *Positivity*: $N_{i,k}(t) > 0$, $t \in (t_i, t_{i+k})$ and $t_{i+k} > t_i$.
- (7) *Partition of unity*: $\sum_i N_{i,k}(t) = 1$, $k \geq 3$.
- (8) *Linear independence*: $N_{i,k}(t)$, $i = 0, \pm 1, \dots$, are linearly independent on $(-\infty, +\infty)$ if the multiplicity of each knot of T is less than $(k + 1)$, that is, there is no zero function in $N_{i,k}(t)$, $i = 0, \pm 1, \dots$.
- (9) *Knot insertion*: Let $T = \{t_i\}_{i=-\infty}^{+\infty}$ be a knot sequence, $T^1 = \{t_i^1\}_{i=-\infty}^{+\infty}$ be a new knot sequence obtained by inserting a new knot u into T with $t_i \leq u < t_{i+1}$. $N_{i,k}(t)$ and $N_{i,k}^1(t)$ are defined as in formulae (2) for the knot sequence T and T^1 , respectively. Then all $j, k \geq 2$,

$$N_{j,k}(t) = \zeta_{j,k} N_{j,k}^1(t) + \eta_{j+1,k} N_{j+1,k}^1(t), \quad (4)$$

where for $0 \leq r < k$,

$$\zeta_{j,k} = \begin{cases} 1, & j \leq i - k, \\ \frac{\delta_{j,k-1}}{\delta_{j,k-1}^1} \cdot \zeta_{j,k-1}, & i - k < j < i - r + 1, \\ 0, & j \geq i - r + 1, \end{cases}$$

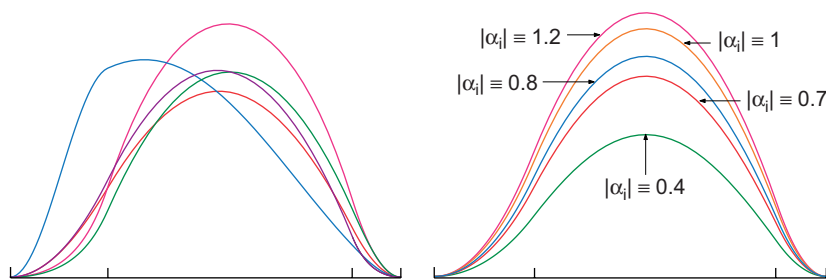


Fig. 3. Different UE-spline basis functions with the same knot sequence but different frequency sequences.

$$\eta_{j,k} = \begin{cases} 0, & j \leq i - r + 1, \\ \frac{\delta_{j,k-1}}{\delta_{j+1,k-1}^1} \cdot \eta_{j,k-1}, & i - k + 1 < j < i - r + 2, \\ 1, & j \geq i - r + 2, \end{cases}$$

and for $r \geq k$,

$$\zeta_{j,k} = \begin{cases} 1, & j \leq i - k, \\ 0, & j > i - k, \end{cases} \quad \eta_{j,k} = \begin{cases} 0, & j \leq i - k + 1, \\ 1, & j > i - k + 1, \end{cases}$$

and

$$\zeta_{j,2} = \begin{cases} 1, & j \leq i, \\ \frac{\sin \omega_i(u - t_i)}{\sin \omega_i(t_{i+1} - t_i)}, & j = i, \\ 0, & j \geq i + 1, \end{cases} \quad \eta_{j,2} = \begin{cases} 1, & j \leq i, \\ \frac{\sin \omega_{i-1}(t_{i+1} - u)}{\sin \omega_{i-1}(t_{i+1} - t_i)}, & j = i, \\ 0, & j \geq i + 1. \end{cases}$$

If $t_i < u < t_{i+1}$ and $s \geq k$, we have $\zeta_{i-k+1,k} = 1$. Furthermore, for all $i, k \geq 3$, we get $\zeta_{i,k} + \eta_{i,k} = 1$. Then the formulae (4) can also be denoted as

$$N_{j,k}(t) = \zeta_{j,k} N_{j,k}^1(t) + (1 - \zeta_{j+1,k}) N_{j+1,k}^1(t), \quad 0 \leq \zeta_{i,k} \leq 1. \quad (5)$$

Besides shape adjustable property, other properties are inherited from usual polynomial B-spline basis. So most of them can be easily deduced by methods similar to the deduction of corresponding properties of B-spline basis. We only prove linear independence and positivity in Propositions 3.1 and 3.2, respectively.

Proposition 3.1. All nonzero UE-spline basis functions $N_{i,k}(t)$, $i=0, \pm 1, \dots$, are linearly independent on $(-\infty, +\infty)$.

Proof. Let $t_{i-r_i+1} = \dots = t_i < t_{i+1} < \dots < t_{i+l_i}$, where $l_i \geq 1$, $r_i \leq k$ is the multiplicity of t_i . At first, we prove that $N_{j,k}(t)$ ($j = i - k + 1, \dots, i + l_i - 1$) are linearly independent on the interval $[t_{i-r_i+1}, t_{i+l_i})$ by induction.

When $k = 2$, the conclusion is obvious. Assume the conclusion holds for $k = m$. When $k = m + 1$, let $l(t) = \sum_{j=i-m}^{i+l_i-1} c_j N_{j,m+1}(t) \equiv 0$, $t \in [t_{i-r_i+1}, t_{i+l_i})$.

Then $l'(t) = \sum_{j=i-m+1}^{i+l_i-1} (c_j - c_{j-1}) \delta_{j,m} N_{j,m}(t)$.

By assumption, we obtain $c_j - c_{j-1} = 0$, $j = i - m + 1, \dots, i + l_i - 1$. That is, $c_{i-m} = c_{i-m+1} = \dots = c_{i+l_i-1}$. Then $l(t) = c_{i+l_i-1} \sum_{j=i-m}^{i+l_i-1} N_{j,m+1}(t) = c_{i+l_i-1} = 0$. We get $c_j = 0$, $j = i - m, \dots, i + l_i - 1$. So $N_{j,k}(t)$ ($j = i - k + 1, \dots, i + l_i - 1$) are linearly independent on the interval $[t_{i-r_i+1}, t_{i+l_i})$.

Let $(-\infty, +\infty) = \bigcup_i I_i$, $I_i = [t_{i-r_i+1}, t_{i+l_i})$, $i = 0, \pm 1, \dots$, and $I_i \cap I_j = \emptyset$ ($i \neq j$). Assume

$$L(t) = \sum_i c_i N_{i,k}(t) \equiv 0. \quad (6)$$

When $t \in I_i$, the formulae (6) becomes $\sum_{j=i-k+1}^{i+l_i+1} c_j N_{j,k}(t) \equiv 0$. At this time, $N_{j,k}(t)$ are nonzero basis functions on I_i . So we get $c_j = 0$, $j = i - k + 1, \dots, i + l_i + 1$. After t covering all the intervals I_i , $i = 0, \pm 1, \dots$, we obtain $c_j = 0$, $j = 0, \pm 1, \dots$. In formula (6), each c_i is actually a certain c_j or the sum of several c_j . Hence $c_i = 0$, $i = 0, \pm 1, \dots$. The proposition is proved. \square

Proposition 3.2. $N_{i,k}(t) \geq 0$ for all t .

Proof. Because of the local support property of UE-spline basis, we only need to prove that the conclusion is correct on its local support interval $[t_i, t_{i+k}]$. We know $N_{i,k}(t) \equiv 0$ if $t_i = t_{i+k}$ (zero function property). So we assume $t_i < t_{i+k}$ in the following proof.

Inserting a series of new knots into the knot sequence T such that the multiplicity of each knot t_j , $j = i, \dots, i + k$, is exactly k , we obtain a new knot sequence T^1 . Let $N_{i,k}^1(t)$ be the new splines for T^1 . By Remark 2.2, $N_{i,k}^1(t)$ is actually

a UE-Bezier basis. Its positivity has been proved in Lemmas 2.2 and 2.3. According to the formulae (5), we know $N_{i,k}(t)$ is actually a convex combination of $N_{i,k}^1(t)$. So $N_{i,k}(t)$ is also positive. Now we can conclude that $N_{i,k}(t) \geq 0$ for all t . \square

4. UE-spline curves

4.1. The definition of UE-spline curves

Definition 4.1 (UE-spline curves). Let $\{P_i\}_{i=0}^n \in \mathbb{R}^3$, $N_{i,k}(t)$ be UE-spline bases of order k corresponding to the partition $T = \{t_j\}_{j=-\infty}^{+\infty}$ of the parameter axis t . Then $P(t) = \sum_{i=0}^n N_{i,k}(t)P_i$, $(t_{k-1} \leq t \leq t_{n+1}, n \geq k-1)$ is called a UE-spline curve of order k corresponding to the knot vector T . P_i ($i = 0, 1, \dots, n$) are control points. The control polygon is obtained by connecting control points with line segment in turn. Fig. 4 illustrates a UE-spline curve of order 4. The frequency sequence is marked in the figure.

4.2. Properties of UE-spline curves

Due to the properties of UE-spline basis and Definition 4.1, it is easy to demonstrate the following properties of UE-spline curves.

- (1) *Differential*: A piece of UE-spline curve $P(t)$ is $(k - r - 1)$ continuously differential at a knot of multiplicity r .
- (2) *Derivative*:

$$\frac{d}{dt}P(t) = \sum_{i=1}^n \delta_{i,k-1} \Delta P_i N_{i,k-1}, \quad t \in [t_{k-1}, t_{n+1}],$$

where $\Delta P_i = P_i - P_{i-1}$.

- (3) *Geometric invariance*: The shape of UE-spline curve is independent of the choice of coordinate system because $P(t)$ is an affine combination of the control points.
- (4) *Convex hull property*: $P(t)$ lies inside the convex hull of the corresponding control polygon.
- (5) *Convexity preserving property*: If the control polygon of $P(t)$ is convex, then this curve is also convex.
- (6) *Local control property*: Local adjustment can be made without disturbing the rest of the curve because change of one control points will alter at most k segments of the original UE-spline curve of order k .
- (7) *Variation diminishing property*: No plane has more intersections with the curve $P(t)$ than with its control polygon.

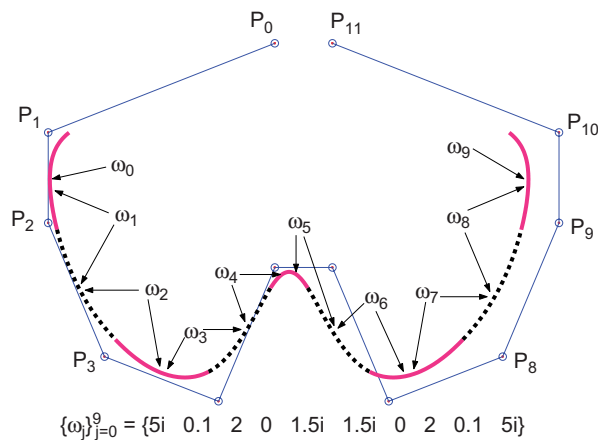


Fig. 4. A UE-spline curve of order 4.

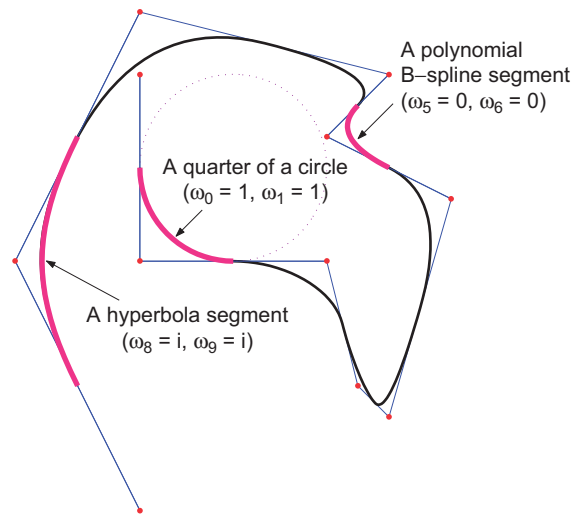


Fig. 5. A UE-spline curve of order 3 including a hyperbola segment, a quarter of a circle and a segment of usual B-spline.

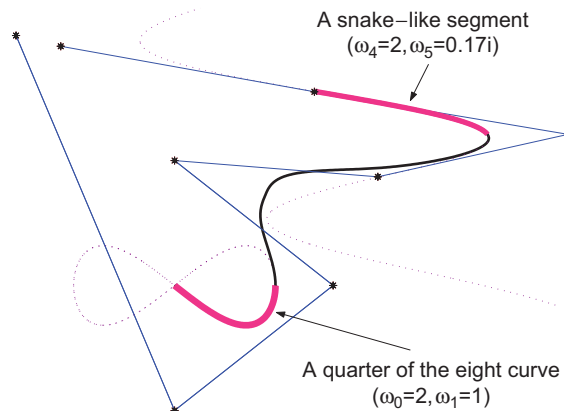


Fig. 6. A UE-spline curve of order 4 including a quarter of the eight curve and a snake-like segment.

Besides classical properties mentioned above, UE-spline curves also have some special properties, which are advantageous for modelling.

- (1) *Shape adjustable property*: The shape of $P(t)$ can be adjusted by changing frequencies in case the control polygon does not change.
- (2) *Local control property relative to frequencies*: Each piece of UE-spline curve is relative to two frequencies. And each frequency affects two pieces of UE-spline curve as illustrated in Fig. 4. That is, change of one frequency can locally adjust two segments of the original UE-spline curve without disturbing the rest of the curve.
- (3) *Variety property*: UE-splines can represent not only classical curves, such as circular arc, hyperbola, helix, catenary etc., but also more particular curves like eight curve and *snake-like* curve. A UE-spline curve can even be composed of pieces of different types. In Fig. 5, the UE-spline curve simultaneously includes segments of usual polynomial, trigonometric, hyperbolic B-spline pieces (marked with thick solid curve) and some composed-type segments (marked with thinner solid curve). In Fig. 6, a quarter of the eight curve C_1 and a segment of the *snake-like* curve C_2 are included in a UE-spline curve of order 4, where $C_1 : (x(t), y(t)) = (-1 + \sin t, -\frac{1}{2} + \frac{1}{2} \sin 2t)$, $C_2 : (x(t), y(t)) = (0.2 \cos 2t - 0.2t + 0.7, \sinh(0.17t) + 1.4)$.

5. Conclusion

UE-splines provide a unified form over a common space for those existing splines defined over different spaces. And due to the introduction of variable frequencies, UE-splines include more plentiful types. The integral definition of UE-spline bases is very simple too. Just as in the construction of B-spline tensor product surfaces from B-spline curves, UE-spline surfaces can be constructed from UE-spline curves easily. And many properties of the curves can be extended to the surfaces. Whether from a theoretical perspective or from a practical standpoint of modelling free-form curves and surfaces, the unification and extension are of important meaning.

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