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Positive solutions of nonlinear third-order m-point BVP for an increasing homeomorphism and homomorphism with sign-changing nonlinearity[☆]

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ABSTRACT

In this paper, several existence theorems of positive solutions are established for a nonlinear m-point boundary value problem for the following third-order differential equations

$$(\phi(u''))' + a(t)f(t, u(t)) = 0, \quad t \in (0, 1),$$
$$\phi(u''(0)) = \sum_{i=1}^{m-2} a_i \phi(u''(\xi_i)), \qquad u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

where $\phi : R \longrightarrow R$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$. The nonlinear term f may change sign. As an application, an example to demonstrate our results is given.

1. Introduction

In this paper, we study the existence of positive solutions of the following third-order differential equations:

$$(\phi(u''))' + a(t)f(t, u(t)) = 0, \quad t \in (0, 1), \tag{1.1}$$

$$\phi(u''(0)) = \sum_{i=1}^{m-2} a_i \phi(u''(\xi_i)), \qquad u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \tag{1.2}$$

where $\phi : R \longrightarrow R$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$.

A projection $\phi : R \longrightarrow R$ is called an increasing homeomorphism and homomorphism, if the following conditions are satisfied:

(i) if $x \le y$, then $\phi(x) \le \phi(y)$, $\forall x, y \in R$;
(ii) ϕ is a continuous bijection and its inverse mapping is also continuous;
(iii) $\phi(xy) = \phi(x)\phi(y)$, $\forall x, y \in R$.

We will assume that the following conditions are satisfied throughout this paper:

(H_1) $0 < \xi_1 < \cdots < \xi_{m-2} < 1$, $a_i, b_i \in [0, +\infty)$ satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$, and $\sum_{i=1}^{m-2} b_i < 1$, $\sum_{i=1}^{m-2} b_i \ge \sum_{i=1}^{m-2} b_i \xi_i$;

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(H_2) $a(t) \in C([0,1],[0,+\infty))$ and there exists $t_0 \in (0,1)$, such that $a(t_0) > 0$;
(H_3) $f \in C([0,1] \times [0,+\infty),(-\infty,+\infty)), f(t,0) \geq 0$.

Recently, much attention has been paid to the existence of positive solutions for second-order nonlinear boundary value problems, see [13–15,17] and references therein. On the one hand, higher-order nonlinear boundary value problems have been studied extensively, for details, see [2,9,16,20] and references therein. On the other hand, the boundary value problems with the p-Laplacian operator have also been discussed extensively in the literature, for example, see [1,3,6,11,12,18,19]. However, to the best of our knowledge, there are not many results concerning the third-order differential equations of increasing homeomorphism and homomorphism.

In [2], Anderson considered the following third-order nonlinear problem:

$$x'''(t) = f(t, x(t)), \qquad t_1 \leq t \leq t_3, \tag{1.3}$$

$$x(t_1) = x'(t_2) = 0, \qquad \gamma x(t_3) + \delta x''(t_3) = 0. \tag{1.4}$$

He used the Krasnoselskii and Leggett and Williams fixed-point theorems to prove the existence of solutions to the nonlinear problem (1.3) and (1.4).

In [7], Baofang Liu and Jihui Zhang studied the existence of positive solutions of differential equation of the form

$$(\phi(x'))' + q(t)f(x(t)) = 0, \quad t \in (0,1), \tag{1.5}$$

$$x(0) - \beta x'(0) = 0, \qquad x(1) + \delta x'(1) = 0, \tag{1.6}$$

where $\phi : R \longrightarrow R$ is an increasing homeomorphism and positive homomorphism and $\phi(0) = 0$. They prove the existence of one or two positive solutions by using a fixed-point index theorem in cones.

In [9], Shuhong Li considered the existence of single and multiple positive solutions to the nonlinear singular third-order two-point boundary value problem:

$$u'''(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1, \tag{1.7}$$

$$u(0) = u'(0) = u''(1) = 0. \tag{1.8}$$

Under various assumptions on a and f, they established intervals of the parameter λ which yield the existence of at least two, and infinitely many positive solutions of the boundary value problem by using the Krasnoselskii fixed-point theorem of cone expansion–compression type.

In [10], Zeqing Liu discussed the existence of at least one or two nondecreasing positive solutions for the following singular nonlinear third-order differential equation:

$$x'''(t) + \lambda \alpha(t) f(t, x(t)) = 0, \quad a < t < b, \tag{1.9}$$

$$x(a) = x''(a) = x'(b) = 0. \tag{1.10}$$

Green's function and the fixed-point theorem of cone expansion and compression type are utilized in his paper.

In [16], Sun considered the following nonlinear singular third-order three-point boundary value problem:

$$u'''(t) - \lambda a(t) F(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.11}$$

$$u(0) = u'(\eta) = u''(1) = 0. \tag{1.12}$$

He obtained various results on the existence of single and multiple positive solutions to the boundary value problem (1.11) and (1.12) by using a fixed-point theorem of cone expansion–compression type due to Krasnoselskii.

In [20], Zhou and Ma studied the existence and iteration of positive solutions for the following third-order generalized right-focal boundary value problem with the p-Laplacian operator:

$$(\phi_p(u''))'(t) = q(t) f(t, u(t)), \quad 0 \leq t \leq 1, \tag{1.13}$$

$$u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u'(\eta) = 0, \qquad u''(1) = \sum_{i=1}^{n} \beta_i u''(\theta_i). \tag{1.14}$$

They established a corresponding iterative scheme for (1.13) and (1.14) by using the monotone iterative technique.

Agarwal [1] considered the following singular boundary value problem

$$(\phi_p(y'))' + q(t) f(t, y(t)) = 0, \quad t \in (0,1), \tag{1.15}$$

$$y(0) = y(1) = 0, \tag{1.16}$$

by means of the upper and lower solution method, where the nonlinearity f is allowed to change sign.

In [11], the authors studied the singular boundary value problem

$$-(\phi_p(y'))' = q(t) f(t, y(t)), \quad t \in (0,1), \tag{1.17}$$

$$y(0) = y(1) = 0, \qquad \psi(y(1)) + y'(1) = 0, \tag{1.18}$$

where the nonlinearity f is allowed to change sign and ψ may be nonlinear.

In [4], by proving a new fixed-point theorem in cones, Ge and Ren obtained the existence of positive solutions to the nonlinear boundary value problem

$$(\phi_p(y'))' + q(t)f(t, y(t)) = 0, \quad t \in (0, 1), \tag{1.19}$$

$$y(0) - B_0(y'(0)) = 0, \qquad y(1) + B_1(y'(1)) = 0, \tag{1.20}$$

with sign-changing nonlinearity.

In [6], Ji, Feng and Ge have considered the existence of multiple positive solutions for the following BVP:

$$(\phi_p(u'))' + a(t)f(t, u(t)) = 0, \quad t \in (0, 1), \tag{1.21}$$

$$u(0) = \sum_{i=1}^{m} a_i u(\xi_i), \qquad u(1) = \sum_{i=1}^{m} b_i u(\xi_i), \tag{1.22}$$

where $0 < \xi_1 < \cdots < \xi_m < 1$, $a_i, b_i \in [0, +\infty)$ satisfy $0 < \sum_{i=1}^{m-2} a_i, \sum_{i=1}^{m-2} b_i < 1$. The nonlinearity f is allowed to change sign. Using a fixed-point theorem for operators on a cone, they provided sufficient conditions for the existence of (1.21) and (1.22)

In this paper, on the one hand, our work concentrates on the case when the nonlinear term may change sign. We will use the property of the solutions of the BVP (1.1) and (1.2) to overcome the difficulty. On the other hand, we will establish the key conditions in Theorems 3.1 and 3.2 by using a new inequality. At the end of the paper, we will give an example which illustrates that our work is true. The method is motivated by [6].

The rest of the paper is arranged as follows. We state some lemmas and prove several preliminary results in Section 2, Section 3 is devoted to the existence of positive solution of (1.1) and (1.2), the main tool being the fixed-point theorem in cone.

2. Preliminaries and some lemmas

To prove the main results in this paper, we will employ several lemmas. These lemmas are based on the linear BVP

$$(\phi(u''))' + h(t) = 0, \quad t \in (0, 1), \tag{2.1}$$

$$\phi(u''(0)) = \sum_{i=1}^{m-2} a_i \phi(u''(\xi_i)), \qquad u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i). \tag{2.2}$$

Lemma 2.1. *If $\sum_{i=1}^{m-2} a_i \neq 1$ and $\sum_{i=1}^{m-2} b_i \neq 1$, then for $h \in C[0, 1]$ the BVP (2.1) and (2.2) has the unique solution*

$$u(t) = \int_0^t (t - s)\phi^{-1}\left(-\int_0^s h(\tau)\mathrm{d}\tau + A\right)\mathrm{d}s + C, \tag{2.3}$$

where

$$A = -\frac{\sum\limits_{i=1}^{m-2} a_i \int_0^{\xi_i} h(\tau)\mathrm{d}\tau}{1 - \sum\limits_{i=1}^{m-2} a_i},$$

$$C = \frac{\sum\limits_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)\phi^{-1}\left(-\int_0^s h(\tau)\mathrm{d}\tau + A\right)\mathrm{d}s - \int_0^1 (1 - s)\phi^{-1}\left(-\int_0^s h(\tau)\mathrm{d}\tau + A\right)\mathrm{d}s}{1 - \sum\limits_{i=1}^{m-2} b_i}.$$

Proof. Necessity. By taking the integral of the problem (2.1) on $[0, t]$, we have

$$\phi(u''(t)) = -\int_0^t h(\tau)\mathrm{d}\tau + A, \tag{2.4}$$

then

$$u''(t) = \phi^{-1}\left(-\int_0^t h(\tau)\mathrm{d}\tau + A\right). \tag{2.5}$$

By taking the integral of the (2.5) on $[0, t]$, we can get

$$u'(t) = \int_0^t \phi^{-1}\left(-\int_0^s h(\tau)\mathrm{d}\tau + A\right)\mathrm{d}s + B. \tag{2.6}$$

By taking the integral of the (2.6) on $[0, t]$, we can get

$$u(t) = \int_0^t (t-s)\phi^{-1}\left(-\int_0^s h(\tau)\mathrm{d}\tau + A\right)\mathrm{d}s + Bt + C. \tag{2.7}$$

Similarly, letting $t = 0$ on (2.4), we have $\phi(u''(0)) = A$, letting $t = \xi_i$ on (2.4), we have

$$\phi(u''(\xi_i)) = -\int_0^{\xi_i} h(\tau)\mathrm{d}\tau + A.$$

Letting $t = 0$ on (2.6), we have

$$u'(0) = B.$$

Letting $t = 1$ on (2.7), we have

$$u(1) = \int_0^1 (1-s)\phi^{-1}\left(-\int_0^s h(\tau)\mathrm{d}\tau + A\right)\mathrm{d}s + B + C.$$

Similarly, letting $t = \xi_i$ on (2.7), we have

$$u(\xi_i) = \int_0^{\xi_i} (\xi_i-s)\phi^{-1}\left(-\int_0^s h(\tau)\mathrm{d}\tau + A\right)\mathrm{d}s + B\xi_i + C.$$

By the boundary condition (2.2), we can get

$$B = 0, \tag{2.8}$$

$$A = \sum_{i=1}^{m-2} a_i\left(-\int_0^{\xi_i} h(\tau)\mathrm{d}\tau + A\right). \tag{2.9}$$

Solving Eq. (2.9), we get

$$A = -\frac{\sum\limits_{i=1}^{m-2} a_i \int_0^{\xi_i} h(\tau)\mathrm{d}\tau}{1 - \sum\limits_{i=1}^{m-2} a_i}. \tag{2.10}$$

By the boundary condition (2.2), we can obtain

$$\int_0^1 (1-s)\phi^{-1}\left(-\int_0^s h(\tau)\mathrm{d}\tau + A\right)\mathrm{d}s + C = \sum_{i=1}^{m-2} b_i\left[\int_0^{\xi_i} (\xi_i-s)\phi^{-1}\left(-\int_0^s h(\tau)\mathrm{d}\tau + A\right)\mathrm{d}s + C\right].$$

Substituting (2.10) in the above expression, one has

$$C = \frac{\sum\limits_{i=1}^{m-2} b_i \int_0^{\xi_i}(\xi_i-s)\phi^{-1}\left(-\int_0^s h(\tau)\mathrm{d}\tau + A\right)\mathrm{d}s - \int_0^1(1-s)\phi^{-1}\left(-\int_0^s h(\tau)\mathrm{d}\tau + A\right)\mathrm{d}s}{1 - \sum\limits_{i=1}^{m-2} b_i}.$$

Sufficiency. Let u be as in (2.3). Taking the derivative of (2.3), we have

$$u'(t) = \int_0^t \phi^{-1}\left(-\int_0^s h(\tau)\mathrm{d}\tau + A\right)\mathrm{d}s,$$

moreover, we get

$$u''(t) = \phi^{-1}\left(-\int_0^t h(\tau)\mathrm{d}\tau + A\right),$$

and

$$\phi(u'') = -\left(\int_0^t h(\tau)\mathrm{d}\tau - A\right),$$

taking the derivative of this expression yields $(\phi(u''))' = -h(t)$. Routine calculation verifies that u satisfies the boundary value conditions in (2.2), so that u given in (2.3) is a solution of (2.1) and (2.2).

It is easy to see that BVP $(\phi(u''))' = 0, \phi(u''(0)) = \sum_{i=1}^{m-2} a_i\phi(u''(\xi_i)), u'(0) = 0, u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i)$ has only the trivial solution if $\sum_{i=1}^{m-2} a_i \neq 1, \sum_{i=1}^{m-2} b_i \neq 1$. Thus u in (2.3) is the unique solution of (2.1) and (2.2). The proof is complete. □

Lemma 2.2. *Assume* (H_1) *holds, For* $h \in C[0, 1]$ *and* $h \geq 0$, *then the unique solution* u *of* (2.1) *and* (2.2) *satisfies*

$$u(t) \geq 0, \quad \text{for } t \in [0, 1].$$

Proof. Let

$$\varphi_0(s) = \phi^{-1}\left(-\int_0^s h(\tau)\mathrm{d}\tau + A\right).$$

Since

$$-\int_0^s h(\tau)\mathrm{d}\tau + A = -\int_0^s h(\tau)\mathrm{d}\tau - \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} h(\tau)\mathrm{d}\tau}{1 - \sum_{i=1}^{m-2} a_i} \leq 0,$$

then $\varphi_0(s) \leq 0$.

According to Lemma 2.1, we get

$$\begin{aligned}
u(0) &= C \\
&= \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(\xi_i - s)\varphi_0(s)\mathrm{d}s - \int_0^1(1-s)\varphi_0(s)\mathrm{d}s}{1 - \sum_{i=1}^{m-2} b_i} \\
&\geq \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(1 - s)\varphi_0(s)\mathrm{d}s - \int_0^1(1-s)\varphi_0(s)\mathrm{d}s}{1 - \sum_{i=1}^{m-2} b_i} \\
&= \frac{\sum_{i=1}^{m-2} b_i \left(\int_0^1(1 - s)\varphi_0(s)\mathrm{d}s - \int_{\xi_i}^1(1-s)\varphi_0(s)\mathrm{d}s\right) - \int_0^1(1-s)\varphi_0(s)\mathrm{d}s}{1 - \sum_{i=1}^{m-2} b_i} \\
&= -\frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^1(1-s)\varphi_0(s)\mathrm{d}s}{1 - \sum_{i=1}^{m-2} b_i} - \int_0^1(1-s)\varphi_0(s)\mathrm{d}s \\
&\geq 0
\end{aligned}$$

and

$$\begin{aligned}
u(1) &= \int_0^1(1-s)\varphi_0(s)\mathrm{d}s + C \\
&= \int_0^1(1-s)\varphi_0(s)\mathrm{d}s + \frac{-\int_0^1(1-s)\varphi_0(s)\mathrm{d}s + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(\xi_i - s)\varphi_0(s)\mathrm{d}s}{1 - \sum_{i=1}^{m-2} b_i}
\end{aligned}$$

$$\begin{aligned}
&\geq \int_0^1 (1-s)\varphi_0(s)\mathrm{d}s + \frac{-\int_0^1 (1-s)\varphi_0(s)\mathrm{d}s + \sum\limits_{i=1}^{m-2} b_i \int_0^{\xi_i} (1-s)\varphi_0(s)\mathrm{d}s}{1-\sum\limits_{i=1}^{m-2} b_i} \\
&= \frac{-\sum\limits_{i=1}^{m-2} b_i \int_{\xi_i}^1 (1-s)\varphi_0(s)\mathrm{d}s}{1-\sum\limits_{i=1}^{m-2} b_i} \geq 0.
\end{aligned}$$

If $t \in (0, 1)$, we have

$$\begin{aligned}
u(t) &= \int_0^t (t-s)\varphi_0(s)\mathrm{d}s + \frac{1}{1-\sum\limits_{i=1}^{m-2} b_i}\left[-\int_0^1 (1-s)\varphi_0(s)\mathrm{d}s + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)\varphi_0(s)\mathrm{d}s\right] \\
&\geq \int_0^1 (1-s)\varphi_0(s)\mathrm{d}s + \frac{1}{1-\sum\limits_{i=1}^{m-2} b_i}\left[-\int_0^1 (1-s)\varphi_0(s)\mathrm{d}s + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (1-s)\varphi_0(s)\mathrm{d}s\right] \\
&= \frac{1}{1-\sum\limits_{i=1}^{m-2} b_i}\left[\left(1-\sum_{i=1}^{m-2} b_i\right)\int_0^1 (1-s)\varphi_0(s)\mathrm{d}s - \int_0^1 (1-s)\varphi_0(s)\mathrm{d}s + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (1-s)\varphi_0(s)\mathrm{d}s\right] \\
&= \frac{-1}{1-\sum\limits_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 (1-s)\varphi_0(s)\mathrm{d}s \geq 0.
\end{aligned}$$

So $u(t) \geq 0$, $t \in [0, 1]$. □

By the method of [18], we can prove the following lemma, here, we omit it.

Lemma 2.3. *Assume (H_1) holds, if $h \in C[0, 1]$ and $h \geq 0$, then the unique solution u of (2.1) and (2.2) satisfies*

$$\inf_{t\in[0,1]} u(t) \geq \gamma \|u\|,$$

where

$$\gamma = \frac{\sum\limits_{i=1}^{m-2} b_i(1-\xi_i)}{1-\sum\limits_{i=1}^{m-2} b_i\xi_i}, \qquad \|u\| = \max_{t\in[0,1]} |u(t)|.$$

Denote

$$\rho_1 > 0, \qquad \varphi(t) = \min\{t, 1-t\}, \quad t \in (0, 1).$$

Lemma 2.4 (*See [5]*). *Let K be a cone in a Banach space X. Let D be an open bounded subset of X with $D_K = D \cap K \neq \phi$ and $\overline{D_K} \neq K$. Assume that $A : \overline{D_K} \longrightarrow K$ is a completely continuous map such that $x \neq Ax$ for $x \in \partial D_K$. Then the following results hold:*

(1) *If $\|Ax\| \leq \|x\|$, $x \in \partial D_K$, then $i(A, D_K, K) = 1$;*
(2) *If there exists $x_0 \in K \setminus \{\theta\}$ such that $x \neq Ax + \lambda x_0$, for all $x \in \partial D_K$ and all $\lambda > 0$, then $i(A, D_K, K) = 0$;*
(3) *Let U be open in X such that $\overline{U} \subset D_K$. If $i(A, D_K, K) = 1$ and $i(A, U_K, K) = 0$, then A has a fixed point in $D_K \setminus \overline{U}_K$. The same result holds if $i(A, D_K, K) = 0$ and $i(A, U_K, K) = 1$.*

Let the norm on $C[0, 1]$ be the maximum norm. Then the $C[0, 1]$ is a Banach space. Denote

$$K = \{u|u \in C[0, 1], u(t) \geq 0, \inf_{t\in[0,1]} u(t) \geq \gamma\|u\|\},$$

where γ is the same as in Lemma 2.3. It is obvious that K is a cone in $C[0, 1]$.

We define

$$K_\rho = \{u(t) \in K : \|u\| < \rho\}, \qquad K_\rho^* = \{u(t) \in K : \rho\varphi < u(t) < \rho\},$$
$$\Omega_\rho = \{u(t) \in K : \min_{0\le t\le 1} u(t) < \gamma\rho\}.$$

Lemma 2.5 (*See [8]*). *Ω_ρ defined above has the following properties:*

(a) $K_{\gamma\rho} \subset \Omega_\rho \subset K_\rho$;
(b) *Ω_ρ is open relative to K;*
(c) *$x \in \partial\Omega_\rho$ if and only if $\min_{0\le t\le 1} x(t) = \gamma\rho$;*
(d) *If $x \in \partial\Omega_\rho$, then $\gamma\rho \le x(t) \le \rho$ for $t \in [0, 1]$.*

Now, for convenience, we introduce the following notations. Let

$$f_{\gamma\rho}^{\rho} = \min\left\{\min_{0\le t\le 1} \frac{f(t,u)}{\phi(\rho)} : u \in [\gamma\rho, \rho]\right\},$$
$$f_{0}^{\rho} = \max\left\{\max_{0\le t\le 1} \frac{f(t,u)}{\phi(\rho)} : u \in [0, \rho]\right\},$$
$$f_{\varphi(t)\rho}^{\rho} = \max\left\{\max_{0\le t\le 1} \frac{f(t,u)}{\phi(\rho)} : u \in [\varphi(t)\rho, \rho]\right\},$$
$$f^{\alpha} = \limsup_{u\to\alpha} \max_{0\le t\le 1} \frac{f(t,u)}{\phi(u)}, \qquad f_{\alpha} = \liminf_{u\to\alpha} \min_{0\le t\le 1} \frac{f(t,u)}{\phi(u)}, \qquad (\alpha := \infty \text{ or } 0^+),$$

$$m_1 = \left\{\frac{1}{1-\sum_{i=1}^{m-2} b_i}\int_0^1 (1-s)\phi^{-1}\left[\int_0^s a(\tau)\mathrm{d}\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\mathrm{d}\tau}{1-\sum_{i=1}^{m-2} a_i}\right]\mathrm{d}s\right\}^{-1}, \tag{2.11}$$

$$M = \left\{\frac{\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i}{1-\sum_{i=1}^{m-2} b_i}\int_0^1 (1-s)\phi^{-1}\left[\int_0^s a(\tau)\mathrm{d}\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\mathrm{d}\tau}{1-\sum_{i=1}^{m-2} a_i}\right]\mathrm{d}s\right\}^{-1}. \tag{2.12}$$

3. Existence theorems of positive solutions

Theorem 3.1. *Assume* (H_1)–(H_3) *hold, and assume that one of the following conditions holds:*

(H_4) *There exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 < \gamma\rho_2$ such that*

(1) $f(t,u) > 0, \quad t \in [0,1], u \in [\rho_1\varphi(t), +\infty)$,
(2) $f_{\varphi(t)\rho_1}^{\rho_1} \le \phi(m_1), \qquad f_{\gamma\rho_2}^{\rho_2} \ge \phi(M\gamma)$;

(H_5) *There exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 < \rho_2$ such that*

(3) $f(t,u) > 0, \quad t \in [0,1], u \in [\min\{\gamma\rho_1, \rho_2\varphi(t)\}, +\infty)$,
(4) $f_{\gamma\rho_1}^{\rho_1} \ge \phi(M\gamma), \qquad f_{\varphi(t)\rho_2}^{\rho_2} \le \phi(m_1)$.

Then problem (1.1) and (1.2) has a positive solution.

Proof. Assume that (H_4) holds.

$$f^*(t,u) = \begin{cases} f(t,u), & u \ge \rho_1\varphi(t), \\ f(t, \rho_1\varphi(t)), & 0 \le u < \rho_1\varphi(t). \end{cases}$$

It is easy to check that $f^*(t,u) \in C([0,1] \times [0,+\infty), (0,+\infty))$.

Now, we consider the modified problem (3.1) and (3.2)

$$(\phi(u''))' + a(t)f^*(t,u(t)) = 0, \quad t \in (0,1), \tag{3.1}$$

$$\phi(u''(0)) = \sum_{i=1}^{m-2} a_i\phi(u''(\xi_i)), \qquad u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i). \tag{3.2}$$

It is easy to see that the BVP (3.1) and (3.2) has a solution $u = u(t)$ if and only if u is a fixed point of the operator equation

$$(Au)(t) = \int_0^t (t-s)\phi^{-1}\left(-\int_0^s a(\tau)f^*(\tau, u(\tau))\mathrm{d}\tau + \tilde{A}\right)\mathrm{d}s + \tilde{C},$$

where

$$\tilde{A} = -\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f^*(\tau, u(\tau))\mathrm{d}\tau}{1 - \sum_{i=1}^{m-2} a_i},$$

$$\begin{aligned}\tilde{C} \;=\; &\frac{1}{1-\sum_{i=1}^{m-2} b_i}\left[\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)\phi^{-1}\left(-\int_0^s a(\tau)f^*(\tau, u(\tau))\mathrm{d}\tau + \tilde{A}\right)\mathrm{d}s\right.\\ &\left. - \int_0^1 (1-s)\phi^{-1}\left(-\int_0^s a(\tau)f^*(\tau, u(\tau))\mathrm{d}\tau + \tilde{A}\right)\mathrm{d}s\right].\end{aligned}$$

By Lemma 2.3, $A(K) \subset K$. Then $A : K \to K$ is completely continuous.

In fact, first, we can show that A maps a bounded set into a bounded set.

Assume $c > 0$ is a constant and $u \in \overline{K_c} = \{x \in K : \|u\| \le c\}$. Note that the continuity of f^* guarantees that there is a $c_1 > 0$ such that $f^*(t, u(t)) \le \phi(c_1)$ for $t \in [0, 1]$. So

$$\begin{aligned}\|Au\| \;=\; &\max_{t\in[0,1]} Au(t) \le \tilde{C}\\ \le\; &\frac{-\int_0^1 (1-s)\phi^{-1}\left(-\int_0^s a(\tau)f^*(\tau, u(\tau))\mathrm{d}\tau + \tilde{A}\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2} b_i}\\ \le\; &\frac{-c_1\int_0^1 (1-s)\phi^{-1}\left(-\int_0^s a(\tau)\mathrm{d}\tau - \frac{\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} a(\tau)\mathrm{d}\tau}{1-\sum_{i=1}^{m-2} a_i}\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2} b_i}.\end{aligned}$$

That is, $A\overline{K_c}$ is uniformly bounded.

In addition, notice that: for any $t_1, t_2 \in [0, 1]$, we have

$$\begin{aligned}|Au(t_1) - Au(t_2)| \;=\; &\left|\int_0^{t_1} (t_1 - t_2)\phi^{-1}\left(-\int_0^s a(\tau)f^*(\tau, u(\tau))\mathrm{d}\tau + \tilde{A}\right)\mathrm{d}s\right.\\ &\left.+\int_{t_1}^{t_2} (-t_2 + s)\phi^{-1}\left(-\int_0^s a(\tau)f^*(\tau, u(\tau))\mathrm{d}\tau + \tilde{A}\right)\mathrm{d}s\right|\\ \le\; &-c_1\,|t_1 - t_2|\left[\int_0^1 \phi^{-1}\left(-\int_0^s a(\tau)\mathrm{d}\tau - \frac{\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} a(\tau)\mathrm{d}\tau}{1-\sum_{i=1}^{m-2} a_i}\right)\mathrm{d}s\right.\\ &\left.+\max_{s\in[0,1]}\phi^{-1}\left(\int_0^s -a(\tau)\mathrm{d}\tau - \frac{\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} a(\tau)\mathrm{d}\tau}{1-\sum_{i=1}^{m-2} a_i}\right)\right].\end{aligned}$$

So, by applying the Arzela–Ascoli theorem, we can obtain that $A(K)$ is relatively compact.

Finally, we prove that $A : \overline{K_c} \to K$ is continuous. Suppose that $\{u_n\}_{n=1}^{\infty} \subset \overline{K_c}$ and $u_n(t)$ converges to $u^*(t)$ uniformly on $[0, 1]$. Hence, $\{Au_n(t)\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on $[0, 1]$. The Arzela–Ascoli Theorem tells us that there exists a uniformly convergent subsequence in $\{Au_n(t)\}_{n=1}^{\infty}$. Let $\{Au_{n(m)}(t)\}_{m=1}^{\infty}$ be a subsequence which converges to

$v(t)$ uniformly on [0, 1]. In addition,

$$0 \le Au_n(t) \le \frac{-c_1 \int_0^1 (1-s)\phi^{-1}\left(-\int_0^s a(\tau)\mathrm{d}\tau - \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\mathrm{d}\tau}{1-\sum_{i=1}^{m-2} a_i}\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2} b_i}.$$

Observe the expression of $\{Au_{n(m)}(t)\}$, and then letting $m \to \infty$, we obtain

$$v(t) = \int_0^t (t-s)\phi^{-1}\left(-\int_0^s a(\tau)f^*(\tau, u^*(\tau))\mathrm{d}\tau + \tilde{A^*}\right)\mathrm{d}s + \tilde{C^*},$$

where

$$\tilde{A^*} = -\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f^*(\tau, u^*(\tau))\mathrm{d}\tau}{1-\sum_{i=1}^{m-2} a_i},$$

$$\begin{aligned}\tilde{C^*} = \frac{1}{1-\sum_{i=1}^{m-2} b_i}\Bigg[&\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)\phi^{-1}\left(-\int_0^s a(\tau)f^*(\tau, u^*(\tau))\mathrm{d}\tau + \tilde{A^*}\right)\mathrm{d}s \\ &- \int_0^1 (1-s)\phi^{-1}\left(-\int_0^s a(\tau)f^*(\tau, u^*(\tau))\mathrm{d}\tau + \tilde{A^*}\right)\mathrm{d}s\Bigg]\end{aligned}$$

here we have used Lebesgue's dominated convergence theorem. From the definition of A, we know that $v(t) = Au^*(t)$ on [0, 1]. This shows that each subsequence of $\{Au_n(t)\}_{n=1}^{\infty}$ uniformly converges to $Au^*(t)$. Therefore, the sequence $\{Au_n(t)\}_{n=1}^{\infty}$ uniformly converges to $Au^*(t)$. This means that A is continuous at $u^* \in \overline{K_c}$. So, A is continuous on $\overline{K_c}$ since u^* is arbitrary. Thus, A is completely continuous.

From the condition $(\mathrm{H}_4)(2)$, we have $f^{*\rho_1}_{\varphi(t)\rho_1} \le \phi(m_1), f^{*\rho_2}_{\gamma\rho_2} \ge \phi(M\gamma)$.

Firstly, we show that

$$i(A, K^*_{\rho_1}, K) = 1.$$

In fact, by (2.11), $f^{*\rho_1}_{\varphi(t)\rho_1} \le \phi(m_1)$ and $u \ne Au$, for $u \in \partial K^*_{\rho_1}$, we have for $\forall u \in \partial K^*_{\rho_1}$,

$$\begin{aligned}-\int_0^s a(\tau)f^*(\tau, u(\tau))\mathrm{d}\tau + \tilde{A} &= -\int_0^s a(\tau)f^*(\tau, u(\tau))\mathrm{d}\tau - \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f^*(\tau, u(\tau))\mathrm{d}\tau}{1-\sum_{i=1}^{m-2} a_i} \\ &\ge -\phi(\rho_1)\phi(m_1)\left[\int_0^s a(\tau)\mathrm{d}\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\mathrm{d}\tau}{1-\sum_{i=1}^{m-2} a_i}\right],\end{aligned}$$

so that

$$\begin{aligned}\varphi(s) &= \phi^{-1}\left(-\int_0^s a(\tau)f^*(\tau, u(\tau))\mathrm{d}\tau + \tilde{A}\right) \\ &\ge -\rho_1 m_1 \phi^{-1}\left[\int_0^s a(\tau)\mathrm{d}\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\mathrm{d}\tau}{1-\sum_{i=1}^{m-2} a_i}\right].\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|Au\| \le \tilde{C} &= \frac{1}{1-\sum\limits_{i=1}^{m-2} b_i}\left(-\int_0^1 (1-s)\varphi(s)\mathrm{d}s + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(\xi_i - s)\varphi(s)\mathrm{d}s\right)\\ &\le \frac{-1}{1-\sum\limits_{i=1}^{m-2} b_i}\int_0^1 (1-s)\varphi(s)\mathrm{d}s\\ &\le \rho_1 m_1 \frac{1}{1-\sum\limits_{i=1}^{m-2} b_i}\int_0^1 (1-s)\phi^{-1}\left[\int_0^s a(\tau)\mathrm{d}\tau + \frac{\sum\limits_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\mathrm{d}\tau}{1-\sum\limits_{i=1}^{m-2} a_i}\right]\mathrm{d}s\\ &= \rho_1 = \|u\|.\end{aligned}$$

This implies that $\|Au\| \le \|u\|$ for $u \in \partial K^*_{\rho_1}$. By Lemma 2.4(1), we have $i(A, K^*_{\rho_1}, K) = 1$.

Secondly, we show that $i(A, \Omega_{\rho_2}, K) = 0$.

Let $e(t) \equiv 1$, for $t \in [0, 1]$; then $e \in \partial K_1$. We claim that $u \ne Au + \lambda e$ for $u \in \partial\Omega_{\rho_2}$, and $\lambda > 0$. In fact, if not, there exist $u_0 \in \partial\Omega_{\rho_2}$ and $\lambda_0 > 0$ such that $u_0 = Au_0 + \lambda_0 e$.

By (2.12), $f^{*\rho_2}_{\gamma\rho_2} \ge \phi(M\gamma)$ and $u \ne Au$ for $u \in \partial\Omega_{\rho_2}$, we have for $t \in [0, 1]$,

$$\begin{aligned}-\int_0^s a(\tau)f^*(\tau, u_0(\tau))\mathrm{d}\tau + \tilde{A}\,|_{u=u_0} &= -\int_0^s a(\tau)f^*(\tau, u_0(\tau))\mathrm{d}\tau - \frac{\sum\limits_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f^*(\tau, u_0(\tau))\mathrm{d}\tau}{1-\sum\limits_{i=1}^{m-2} a_i}\\ &\le -\phi(\rho_2)\phi(M\gamma)\left[\int_0^s a(\tau)\mathrm{d}\tau + \frac{\sum\limits_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\mathrm{d}\tau}{1-\sum\limits_{i=1}^{m-2} a_i}\right],\end{aligned}$$

so that

$$\begin{aligned}\tilde{\varphi}(s) &= \phi^{-1}\left(-\int_0^s a(\tau)f^*(\tau, u_0(\tau))\mathrm{d}\tau + \tilde{A}\,|_{u=u_0}\right)\\ &\le -\rho_2 M\gamma\phi^{-1}\left[\int_0^s a(\tau)\mathrm{d}\tau + \frac{\sum\limits_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\mathrm{d}\tau}{1-\sum\limits_{i=1}^{m-2} a_i}\right].\end{aligned}$$

For ξ_i $(i = 1, \dots, m-2)$, then

$$\int_0^{\xi_i}(\xi_i - s)\tilde{\varphi}(s)\mathrm{d}s \ge \xi_i \int_0^1 (1-s)\tilde{\varphi}(s)\mathrm{d}s.$$

In fact, since

$$\begin{aligned}-\int_0^s a(\tau)f^*(\tau, u_0(\tau))\mathrm{d}\tau + \tilde{A}\,|_{u=u_0} &= -\int_0^s a(\tau)f^*(\tau, u_0(\tau))\mathrm{d}\tau - \frac{\sum\limits_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f^*(\tau, u_0(\tau))\mathrm{d}\tau}{1-\sum\limits_{i=1}^{m-2} a_i}\\ &\le 0,\end{aligned}$$

then $\tilde{\varphi}(s) \le 0$. For $\forall\, t \in (0, 1]$, we have

$$\left(\frac{\int_0^t (t-s)\tilde{\varphi}(s)\mathrm{d}s}{t}\right)' = \frac{t\int_0^t \tilde{\varphi}(s)\mathrm{d}s - \int_0^t (t-s)\tilde{\varphi}(s)\mathrm{d}s}{t^2} \le 0.$$

For $\forall\, t \in (0, 1]$,

$$\frac{\int_0^t (t-s)\tilde{\varphi}(s)\mathrm{d}s}{t} \geq \frac{\int_0^1 (1-s)\tilde{\varphi}(s)\mathrm{d}s}{1}. \tag{3.3}$$

By (3.3), for ξ_i $(i = 1, \ldots, m-2)$, we have

$$\int_0^{\xi_i} (\xi_i - s)\tilde{\varphi}(s)\mathrm{d}s \geq \frac{\xi_i}{1}\int_0^1 (1-s)\tilde{\varphi}(s)\mathrm{d}s. \tag{3.4}$$

Applying (3.4), it follows that

$$\begin{aligned}
u_0(t) &= Au_0(t) + \lambda_0 e(t)\\
&\geq \int_0^1 (1-s)\tilde{\varphi}(s)\mathrm{d}s + \frac{1}{1-\sum_{i=1}^{m-2} b_i}\left(-\int_0^1 (1-s)\tilde{\varphi}(s)\mathrm{d}s + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)\tilde{\varphi}(s)\mathrm{d}s\right) + \lambda_0\\
&= \frac{-\sum_{i=1}^{m-2} b_i}{1-\sum_{i=1}^{m-2} b_i}\int_0^1 (1-s)\tilde{\varphi}(s)\mathrm{d}s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(\xi_i - s)\tilde{\varphi}(s)\mathrm{d}s}{1-\sum_{i=1}^{m-2} b_i} + \lambda_0\\
&\geq \frac{-\sum_{i=1}^{m-2} b_i}{1-\sum_{i=1}^{m-2} b_i}\int_0^1 (1-s)\tilde{\varphi}(s)\mathrm{d}s + \frac{\sum_{i=1}^{m-2} b_i\xi_i}{T(1-\sum_{i=1}^{m-2} b_i)}\int_0^1 (1-s)\tilde{\varphi}(s)\mathrm{d}s + \lambda_0\\
&= \frac{-\sum_{i=1}^{m-2} b_i + \sum_{i=1}^{m-2} b_i\xi_i}{1-\sum_{i=1}^{m-2} b_i}\int_0^1 (1-s)\tilde{\varphi}(s)\mathrm{d}s + \lambda_0\\
&\geq \gamma\rho_2 M\frac{\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i}{1-\sum_{i=1}^{m-2} b_i}\int_0^1 (1-s)\phi^{-1}\left[\int_0^s a(\tau)\mathrm{d}\tau + \frac{\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} a(\tau)\mathrm{d}\tau}{1-\sum_{i=1}^{m-2} a_i}\right]\mathrm{d}s + \lambda_0\\
&= \gamma\rho_2 + \lambda_0.
\end{aligned}$$

This implies that $\gamma\rho_2 \geq \gamma\rho_2 + \lambda_0$, a contradiction. Hence, by Lemma 2.4(2), it follows that

$$i(A, \Omega_{\rho_2}, K) = 0.$$

By Lemma 2.5(a) and $\rho_1 < \gamma\rho_2$, we have $\overline{K}_{\rho_1} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2}$. It follows from Lemma 2.4(3) that A has a fixed point u_1 in $\Omega_{\rho_2} \setminus \overline{K^*}_{\rho_1}$, we note that $f^*(t, u) = f(t, u)$ if $u \geq \rho_1\varphi(t)$. Thus, we can get that problem (1.1) and (1.2) has a positive solution. The proof is similar when (H_5) holds, and we omit it here. The proof is complete. □

Theorem 3.2. *Assume* (H_1)–(H_3) *hold, and suppose that one of the following conditions holds:*

(H_6) *There exist* $\rho_1, \rho_2, \rho_3 \in (0, +\infty)$ *with* $\rho_1 < \gamma\rho_2$ *and* $\rho_2 < \rho_3$ *such that*

$(1)\, f(t, u) > 0, \quad t \in [0, 1], u \in [\rho_1\varphi(t), +\infty)$,

$(2)\, f_{\varphi(t)\rho_1}^{\rho_1} \leq \phi(m_1), \qquad f_{\gamma\rho_2}^{\rho_2} \geq \phi(M\gamma), \qquad u \neq Au, \quad \forall\, u \in \partial\Omega_{\rho_2}, \quad f_{\varphi(t)\rho_3}^{\rho_3} \leq \phi(m_1)$;

(H_7) *There exist* $\rho_1, \rho_2, \rho_3 \in (0, +\infty)$ *with* $\rho_1 < \rho_2 < \gamma\rho_3$ *such that*

$(3)\, f(t, u) > 0, \quad t \in [0, 1], u \in [\min\{\gamma\rho_1, \rho_2\varphi(t)\}, +\infty)$,

$(4)\, f_{\gamma\rho_1}^{\rho_1} \geq \phi(M\gamma), \qquad f_{\varphi(t)\rho_2}^{\rho_2} \leq \phi(m_1), \qquad u \neq Au, \quad \forall\, u \in \partial\Omega_{\rho_2}, \quad f_{\gamma\rho_3}^{\rho_3} \geq \phi(M\gamma)$.

Then problem (1.1) *and* (1.2) *has two positive solutions.*

Corollary 3.3. *Assume* (H_1)–(H_3) *holds, if there exist* $\rho', \rho \in (0, +\infty)$, *with* $\rho' < \gamma\rho$ *such that one of the following conditions holds:*

(H_8)

$$(1)\, f(t,u) > 0, \quad t \in [0,1], u \in [\rho'\varphi(t), +\infty),$$
$$(2)\, f^{\rho'}_{\varphi(t)\rho'} \le \phi(m_1), \qquad f^{\rho}_{\gamma\rho} \ge \phi(M\gamma), \qquad u \ne Au, \quad \forall u \in \partial\Omega_\rho, \quad 0 \le f^\infty < \phi(m);$$

(H_9) *There exist* $\rho', \rho \in (0, +\infty)$ *with* $\rho' < \rho$ *such that*

$$(3)\, f(t,u) > 0, \quad t \in [0,1], u \in [\min\{\gamma\rho', \rho\varphi(t)\}, +\infty),$$
$$(4)\, f^{\rho'}_{\gamma\rho'} \ge \phi(M\gamma), \qquad f^{\rho}_{\varphi(t)\rho} \le \phi(m_1), \qquad u \ne Au, \quad \forall u \in \partial\Omega_\rho, \quad \phi(M) < f_\infty \le \infty.$$

Then problem (1.1) *and* (1.2) *has two positive solutions.*

4. Example

In the section, we present a simple example to explain our results.

Let $f(t,0) \equiv 0$. Consider the following BVP

$$(\phi(u''))' + f(t,u(t)) = 0, \quad t \in (0,1), \tag{4.1}$$

$$\phi(u''(0)) = \frac{1}{4}\phi\left(u''\left(\frac{1}{3}\right)\right), \qquad u'(0) = 0, \qquad u(1) = \frac{1}{2}u\left(\frac{1}{3}\right), \tag{4.2}$$

where

$$\phi(u) = \begin{cases} -u^2, & u \le 0, \\ u^2, & u > 0, \end{cases}$$

$$f(t,u) = \begin{cases} \dfrac{1}{5}(1+t)\left(u(t) - \dfrac{\varphi(t)}{2}\right)^{31}, & (t,u) \in [0,1] \times (0,2], \\[2ex] \dfrac{1}{5}(1+t)\left(2 - \dfrac{\varphi(t)}{2}\right)^{31}, & (t,u) \in [0,1] \times (2,+\infty). \end{cases}$$

It is easy to check that $f : [0,1] \times [0,+\infty) \longrightarrow (-\infty,+\infty)$ is continuous. In this case, $a(t) \equiv 1$, $a_1 = \frac{1}{4}$, $b_1 = \frac{1}{2}$, $\xi_1 = \frac{1}{3}$, it follows from a direct calculation that

$$\begin{aligned} m_1 &= \left[\frac{1}{1-b_1}\int_0^1 (1-s)\phi^{-1}\left(s + \frac{a_1\xi_1}{1-a_1}\right) \mathrm{d}s\right]^{-1} \\ &= \left[\frac{1}{1-\frac{1}{2}}\int_0^1 (1-s)\left(s + \frac{\frac{1}{4}\cdot\frac{1}{3}}{1-\frac{1}{4}}\right)^{\frac{1}{2}} \mathrm{d}s\right]^{-1} \\ &\approx 1.5565, \end{aligned}$$

$$\gamma = \frac{b_1(1-\xi_1)}{1-b_1\xi_1} = \frac{\frac{1}{2}(1-\frac{1}{3})}{1-\frac{1}{2}\cdot\frac{1}{3}} = \frac{2}{5}.$$

$$\begin{aligned} M &= \left[\frac{b_1 - b_1\xi_1}{1-b_1}\int_0^1 (1-s)\phi^{-1}\left(s + \frac{a_1\xi_1}{1-a_1}\right) \mathrm{d}s\right]^{-1} \\ &= \left[\frac{\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{3}}{1-\frac{1}{2}}\int_0^1 (1-s)\left(s + \frac{\frac{1}{4}\cdot\frac{1}{3}}{1-\frac{1}{4}}\right)^{\frac{1}{2}} \mathrm{d}s\right]^{-1} \\ &\approx 4.6694. \end{aligned}$$

Choose $\rho_1 = 1$, $\rho_2 = 5$, it is easy to check that $1 = \rho_1 < \gamma\rho_2 = \frac{2}{5} \times 5 = 2$, $f(t,u) > 0$, $t \in [0,1]$, $u \in [\varphi(t), +\infty)$,

$$\begin{aligned} f^{\rho_1}_{\rho_1\varphi(t)} &= \max\left\{\max_{0\le t\le 1} \frac{\frac{1}{5}(1+t)(u(t) - \frac{\varphi(t)}{2})^{31}}{1^2}\right\} \\ &= \frac{\frac{1}{5}(1+1)1^{31}}{1^2} = \frac{2}{5} \\ &\le \phi(m_1) = m_1^2 = (1.5565)^2, \end{aligned}$$

$$
\begin{aligned}
f_{\gamma\rho_2}^{\rho_2} &= \min\left\{\min_{0\le t\le 1}\frac{\frac{1}{5}(1+t)(2-\frac{\varphi(t)}{2})^{31}}{5^2}\right\}\\
&= \frac{\frac{1}{5}(1+0)(2-\frac{1}{2})^{31}}{5^2} = \frac{3^{31}}{2^{31}5^3} \approx 16.7420\\
&\ge \phi(M\gamma) = (M\gamma)^2 = \left(4.6694\cdot\frac{2}{5}\right)^2 \approx 3.4885.
\end{aligned}
$$

It follows that f satisfies the conditions (H_4) of Theorem 3.1, then problem (1.1) and (1.2) has at least a positive solution.

References

[1] R.P. Agarwal, H. Lü, D. O'Regan, Existences theorems for the one-dimensional singular *p*-Laplacian equation with sign changing nonlinearities, Appl. Math. Comput. 143 (2003) 15–38.

[2] D.R. Anderson, Green's function for a third-order generalized right focal problem, J. Math. Anal. Appl. 288 (2003) 1–14.

[3] C. Bai, J. Fang, Existence of multiple positive solutions for nonlinear *m*-point boundary value problems, J. Math. Appl. Appl. 81 (2003) 76–85.

[4] W. Ge, J. Ren, Fixed point theorems in double cones and their applications to nonlinear boundary value problems, J. Contemp. Math. 27 (2006) 155–168.

[5] D. Guo, V. Lakshmikanthan, Nonlinear Problems in Abstract Cones, Academic Press, San Diego, 1988.

[6] D. Ji, M. Feng, W. Ge, Multiple positive solutions for multipoint boundary value problems with sign changing nonlinearity, Appl. Math. Comput. 196 (2008) 515–520.

[7] B.F. Liu, J.H. Zhang, The existence of positive solutions for some nonlinear boundary value problems with linear mixed boundary conditions, J. Math. Anal. Appl. 309 (2005) 505–516.

[8] K.Q. Lan, Multiple positive solutions of semilinear differential equations with singularities, J. London Math. Soc. 63 (2001) 690–704.

[9] S.H. Li, Positive solutions of nonlinear singular third-order two-point boundary value problem, J. Math. Anal. Appl. 323 (2006) 413–425.

[10] Z. Liu, J.S. Ume, D.R. Anderson, S.M. Kang, Twin monotone positive solutions to a singular nonlinear third-order differential equation, J. Math. Anal. Appl. 334 (2007) 299–313.

[11] H. Lü, D. O'Regan, R.P. Agarwal, Existences theorems for the one-dimensional singular *p*-Laplacian equation with a nonlinear boundary condition, J. Comput. Appl. Math. 182 (2005) 188–210.

[12] D. Ma, Z. Du, W. Ge, Existence and iteration of monotone positive solutions for multipoint boundary value problem with *p*-Laplacian operator, Comput. Math. Appl. 50 (2005) 729–739.

[13] R. Ma, Multiplicity of positive solutions for second-order three-point boundary value problems, Comput. Math. Appl. 40 (2000) 193–204.

[14] R. Ma, Positive solutions of nonlinear *m*-point boundary value problems, Comput. Math. Appl. 42 (2001) 755–765.

[15] R. Ma, Positive solutions of nonlinear three-point boundary value problems, J. Math. Anal. Appl. 279 (2003) 216–227.

[16] Y.P. Sun, Positive solutions of singular third-order three-point boundary value problem, J. Math. Anal. Appl. 306 (2005) 589–603.

[17] Y. Sun, L. Liu, Solvability for a nonlinear second-order three-point boundary value problem, J. Math. Anal. Appl. 296 (2004) 265–275.

[18] Y. Wang, C. Hou, Existence of multiple positive solutions for one-dimensional *p*-Laplacian, J. Math. Anal. Appl. 315 (2006) 144–153.

[19] Y. Wang, W. Ge, Positive solutions for multipoint boundary value problems with one-dimensional *p*-Laplacian, Nonlinear Appl. 66 (6) (2007) 1246–1256.

[20] C.L. Zhou, D.X. Ma, Existence and iteration of positive solutions for a generalized right-focal boundary value problem with *p*-Laplacian operator, J. Math. Anal. Appl. 324 (2006) 409–424.