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Construction of parameterizations of masks for tight wavelet frames with two symmetric/antisymmetric generators and applications in image compression and denoising☆

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A B S T R A C T

In this paper, we present a general construction framework of parameterizations of masks for tight wavelet frames with two symmetric/antisymmetric generators which are of arbitrary lengths and centers. Based on this idea, we establish the explicit formulas of masks of tight wavelet frames. Additionally, we explore the transform applicability of tight wavelet frames in image compression and denoising. We bring forward an optimal model of masks of tight wavelet frames aiming at image compression with more efficiency, which can be obtained through SQP (Sequential Quadratic Programming) and a GA (Genetic Algorithm). Meanwhile, we present a new model called Cross-Local Contextual Hidden Markov Model (CLCHMM), which can effectively characterize the intrascale and cross-orientation correlations of the coefficients in the wavelet frame domain, and do research into the corresponding algorithm. Using the presented CLCHMM, we propose a new image denoising algorithm which has better performance as proved by the experiments.

1. Introduction

The tight wavelet frames and orthonormal wavelets, which are developed in parallel, are the two main theories in wavelet analysis. The tight wavelet frames are different from the orthonormal wavelets because of redundancy. By sacrificing orthonormality and allowing redundancy, the tight wavelet frames become much easier to construct than the orthonormal wavelets. The tight wavelet frames have the same computational complexity as orthonormal wavelets and can be applied to image processing. Especially, due to the redundancy, the tight wavelet frames have many desirable properties and are of interest in high-resolution image reconstruction [1], image inpainting [2] and image analysis and synthesis [3].

Here, we give a brief outline of recent researches on construction theories of tight wavelet frames. In [4], the authors studied compactly supported tight frames $\Psi = \{\psi_1, \psi_2, \ldots, \psi_r\}$ for $L^2(R)$ that corresponded to some refinable functions with compact support. They gave a precise existence criterion of Ψ in terms of an inequality condition on the Laurent polynomial symbols of refinable functions, and they showed that this condition was not always satisfied (implying the nonexistence of tight frames via the matrix extension approach). Also the authors gave a constructive proof that when Ψ did exist, two functions with compact support were sufficient to constitute Ψ, while three guaranteed symmetry/antisymmetry, when the given refinable function was symmetric. In [5], the authors studied tight wavelet frames associated with the given

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refinable functions which were obtained with the unitary extension principles. All possible solutions of the corresponding matrix equations were found. It was proved that the problem of the extension might always be solved with two framelets. In particular, if the symbols of refinable functions were polynomials (rational functions), the corresponding framelets with polynomial (rational) symbols could be found. In [6], the author presented a systematic algorithm for constructing symmetric/antisymmetric tight wavelet frames and orthonormal wavelet bases generated by a given refinable function with an integer dilation factor $d \geq 2$. Special attention was paid to the issues of the minimality of a number of framelet generators and the size of the generator support. In [7], the authors used the UEP approach and provided the method of constructing symmetric wavelet tight frames with two generators. In [8], the authors discussed wavelet frames constructed via multiresolution analysis (MRA), with emphasis on tight wavelet frames. In particular, they established general principles and specific algorithms for constructing framelets and tight framelets, and they showed how the methods could be used for the systematic construction of splines, pseudo-spline tight frames, and symmetric bi-frames with short supports and high approximation orders. The connection of these frames with multiresolution analysis guaranteed the existence of fast implementation algorithms, which were discussed briefly as well. In [9], the authors utilized the oblique extension principle (OEP), and presented a necessary and sufficient condition for the construction of symmetric multiresolution analysis tight wavelet frames with two compactly supported generators derived from a given symmetric refinable function. Once such a necessary and sufficient condition was satisfied, an algorithm would be used to construct a symmetric framelet filter bank with two high-pass filters which was of interest in applications such as signal denoising and image processing. In [10,11], the authors studied the compactly supported tight affine frames with integer dilations and maximum vanishing moments. Recently, other novel approaches to constructing compactly supported wavelet frames and dual frames with more than two framelets were found in papers [12–24]. But those researches focused on theoretical analysis, provided very little transforms of tight frames in practical applications.

It should be mentioned that constructing a wavelet frame satisfying too many properties could be very difficult. For example, it is complicated to design UPE-based symmetric compactly supported tight wavelet frames with the minimal number of frame generators and an arbitrary number of vanishing moments. The parameterization of FIR systems is of fundamental importance to design of filters with the desired properties, which is one of the most attractive issues in wavelet frame theory. The purpose of this paper is to realize the parameterizations of masks for tight wavelet frames. We present a novel construction technique for tight wavelet frames with two symmetric/antisymmetric generators, which are of arbitrary length and centers. We particularly emphasize that the presented mask expressions have as many free parameters as possible. Motivated by these applications, we explore the power redundancy of wavelet frames. The wavelet frames theory is applied to image compression at a low bit-rate and image denoising. An optimal model of FIR filters (masks) aiming at image compression is brought forward, and the optimal FIR filters can be got correspondingly through SQP and GA. As is indicated by the experimental results, the efficiency is notable. We present an model, called CLCHMM which can effectively characterize the intrascale and cross-orientation correlations of the coefficients in the wavelet frame domain and do some research into the corresponding algorithm. Applying the presented CLCHMM, we propose a new image denoising algorithm. We carry out a series of experiments to evaluate the suitability of wavelet frames that are based on the constructed masks for the compression of still images and image denoising. The positive effect of redundancy in image compression and denoising has been discovered in our research.

This paper is organized as follows. Section 2 concerns some basic concepts about wavelet frames. In Section 3, we present a general construction framework of parameterizations of masks for tight wavelet frames with two generators, which are of arbitrary lengths and centers, as well as several construction examples. In Section 4, we do some research into applications of tight wavelet frames including image compression and denoising. And Section 5 is the conclusion.

2. Review on concepts concerning wavelet frames

Before we state our main results, we start by reviewing some major concepts concerning wavelet frames.

In the rest of this paper, we use N, N_0 and Z to denote the sets of all natural numbers, nonnegative integers and integers, respectively.

The definition of a frame is provided as below.

Definition 2.1. Let H be a Hilbert space with inner product $\langle ., . \rangle$, and norm $\|.\| = (\langle ., . \rangle)^{\frac{1}{2}}$. A system $X \subset H$ is called a frame of H if there are two positive constants A, B such that

$$A\|f\|^2 \leq \sum_{x \in H} |\langle f, x \rangle|^2 \leq B\|f\|^2, \quad \forall f \in H. \tag{2.1}$$

The constants A and B are called bounds of the frame. If $A = B$, then X is called a tight frame.

We are interested in the study of wavelet frames that are derived from a multiresolution analysis (MRA). Let $\phi \in L^2\{R\}$ be given and D be the operator of dyadic dilation: $(Df)(y) = 2^{\frac{1}{2}} f(2y)$ and T_t be the translation: $(T_t f)(x) = f(x - t)$. Set $V_j = D^j V_0, j \in Z$.

Definition 2.2. $\phi \in L^2(R)$ is said to generate an MRA $\{V_k, k \in Z\}$, if ϕ satisfies the following conditions,

(1) $V_k \subset V_{k+1}, \quad k \in Z$;

(2) $\overline{\bigcup V_k} = L^2(R), \qquad \bigcap V_k = \{0\}$;

(3) $D(V_k) = V_{k+1}, \quad k \in Z$;

(4) $T_1(V_0) = V_0$;

(5) $T_k\phi, \quad k \in Z$ is an orthonormal basis of V_0.

If condition (5) is replaced by $T_k\phi$, $k \in Z$ being a frame of V_0, then we call MRA as FMRA.

In [25], some properties of FMRA are studied. The generator ϕ of an MRA is known as a scaling function or a refinable function. Let $\Psi = \{\psi_1, \psi_2, \dots, \psi_r\}$ be a subset of $L^2(R)$, then the dyadic wavelet system generated by the mother wavelets Ψ is the family

$$X(\Psi) = \{\psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j \cdot -k) : \psi \in \Psi, j, k \in Z\}. \tag{2.2}$$

Definition 2.3 ([8]). A wavelet system $X(\Psi)$ is said to be MRA-based if there exists an MRA $\{V_j, j \subset Z\}$, such that the condition $\Psi \subset V_1$ holds. If, in addition, the system $X(\Psi)$ is a frame, we refer to its elements as framelets.

Definition 2.4 ([8]). Suppose that $V_j, j \in Z$ is an MRA induced by a refinable function ϕ, let $\Psi = \{\psi_1, \psi_2, \dots, \psi_r\}$ be a finite subset of V_1, $\widehat{\phi} = (\tau_0\widehat{\phi})\left(\frac{\cdot}{2}\right)$, $\widehat{\psi_i} = (\tau_i\widehat{\psi_i})\left(\frac{\cdot}{2}\right)$, $i = 1, 2, \dots, r$, we introduce the notation $\tau_0, \tau_1, \dots, \tau_r$ for the combined MRA masks (or filters. We will use but will not distinguish them in the rest of this paper).

In [26], the sufficient condition of construction of wavelet frames is provided.

Lemma 2.1 (*The Unitary Extension Principle (UEP)*). *Let $\tau_0, \tau_1, \dots, \tau_r$ be the combined MRA masks that satisfy Definition 2.4. $\widehat{\phi}(0) = 1$, $|\widehat{\phi}(\omega)| \le c(1 + |\omega|^{-\frac{1}{2}-\varepsilon})$, $\varepsilon > 0$*

$$M(z) = \begin{pmatrix} \tau_0(z) & \tau_0(-z) \\ \tau_1(z) & \tau_1(-z) \\ \vdots & \vdots \\ \tau_r(z) & \tau_r(-z) \end{pmatrix} \quad \text{for } |z| = 1. \tag{2.3}$$

If

$$M^*(z)M(z) = E, \tag{2.4}$$

where we use $M^*(z)$ to represent the complex conjugate of transpose of $M(z)$, then $\Psi = \{\psi_1, \psi_2, \dots, \psi_r\}$ is a wavelet frame that is derived from a multiresolution analysis (MRA), and MRA is generated by ϕ.

It was proved in [5], that the decay assumption of $\widehat{\phi}$ at infinity can be removed. Thus if $\widehat{\phi}(0) = 1$, we have masks τ_i, $i = 0, 1, \dots, r$, such that τ_0 generates a function ϕ in $L^2(R)$ with $\widehat{\phi}(0) = 1$, and $M(z)$ satisfies (2.4), then we have a tight wavelet frame.

3. Construction of parameterizations of masks for tight wavelet frames with two symmetric/antisymmetric generators

In this section, we present a general construction framework of parameterizations of masks for tight wavelet frames with two symmetric/antisymmetric generators. Compared with other accomplishments in the field [12,4,14,15,8,16–20,9,21–23,5–7,24], the new construction has the following characteristics. By constructing several categories of paraunitary matrices with specific characteristics, we realize the construction of symmetric/antisymmetric parameterizations of masks with arbitrary lengths. The mask expressions are obtained by multiplying several paraunitary matrices. The calculation is less complex, and the outcome is easy to achieve, and the masks constructed have high degrees of freedom. [27] gives the expressions of mask parameterizations for tight frames with forms: $h(z) = \sum_{k=0}^{2n-1} h_k z^{-k}$, $g(z) = \sum_{k=0}^{2n-1} g_k z^{-k}$, $f(z) = \sum_{k=0}^{2m-1} f_k z^{-k}$, and $h(z) = \sum_{k=0}^{2n} h_k z^{-k}$, $g(z) = \sum_{k=0}^{2n} g_k z^{-k}$, $f(z) = \sum_{k=0}^{2m} f_k z^{-k}$. Compared with [27], this paper is more inclusive, where more cases under different circumstances are considered.

3.1. Construction of parameterizations of masks in given (anti)symmetric centers

In our previous research [28], we investigated the perfect reconstruction technique of tight wavelet frames and its requirement on the position of symmetry of masks (if the length of a mask is odd, the symmetric center is 0, otherwise it is $\frac{1}{2}$ while the length is even). Based on this work, we provide an approach to the construction of parameterizations of masks for tight wavelet frames with symmetry. The difference with [27] is that the lengths of masks are not under restraint in this paper. We consider the expressions of masks for tight wavelet frames with the following forms:

$$h(z)=\sum_{k=-(2n_1+1)}^{2n_1+1}h_kz^{-k},\qquad g(z)=\sum_{k=-(2n_2+1)}^{2n_2+1}g_kz^{-k},\qquad f(z)=\sum_{k=-(2n_3-1)}^{2n_3-1}f_kz^{-k},\quad \text{and}$$

$$h(z)=\sum_{k=-2n_1}^{2n_1}h_kz^{-k},\qquad g(z)=\sum_{k=-2n_2}^{2n_2}g_kz^{-k},\qquad f(z)=\sum_{k=-2n_3}^{2n_3}f_kz^{-k}.$$

Theorems 3.2, 3.5 and 3.6 propose an efficient method that generates a wide range of explicit expressions of masks for tight wavelet frames.

For any given mask $h(z)$, write $h(z)$ in their polyphase forms

$$h(z)=\sum_k h_kz^k=\sum_k h_{2k}z^{2k}+\sum_{2k+1}h_{2k+1}z^{2k+1}=\frac{\sqrt{2}}{2}h_e(z^2)+\frac{\sqrt{2}}{2}h_o(z^2)z.$$

Define two matrices $M_{h,g,f}$, $P_{h,g,f}$ by

$$M_{h,g,f}=\begin{pmatrix}h(z)&h(-z)\\ g(z)&g(-z)\\ f(z)&f(-z)\end{pmatrix},\qquad P_{h,g,f}=\begin{pmatrix}h_e(z)&h_o(z)\\ g_e(z)&g_o(z)\\ f_e(z)&f_o(z)\end{pmatrix}. \tag{3.1}$$

Observe that

$$M_{h,g,f}=\frac{\sqrt{2}}{2}P_{h,g,f}(z^2)\begin{pmatrix}1&1\\ z&-z\end{pmatrix}, \tag{3.2}$$

or

$$M_{h,g,f}=\frac{\sqrt{2}}{2}P_{h,g,f}(z^2)\begin{pmatrix}1&1\\ z^{-1}&-z^{-1}\end{pmatrix}. \tag{3.3}$$

Thus, the necessary and sufficient condition for $M_{h,g,f}$ to be paraunitary is that $P_{h,g,f}$ is paraunitary.

Let h, g, f be FIR filters. Define h_r, g_r, f_r as follows

$$h_r(z)=g_e(z^{-1})f_o(z^{-1})-f_e(z^{-1})g_o(z^{-1}), \tag{3.4}$$

$$g_r(z)=-h_e(z^{-1})f_o(z^{-1})+f_e(z^{-1})h_o(z^{-1}), \tag{3.5}$$

$$f_r(z)=h_e(z^{-1})g_o(z^{-1})-g_e(z^{-1})h_o(z^{-1}). \tag{3.6}$$

By direct calculation, we have

Lemma 3.1. *Assume that h, g, f are FIR filters. Let h_r, g_r, f_r be defined by (3.4)–(3.6) and set the matrix $A(z)$ by*

$$A(z)=\begin{pmatrix}h_e&h_o&h_r\\ g_e&g_o&g_r\\ f_e&f_o&f_r\end{pmatrix}, \tag{3.7}$$

then, $M_{h,g,f}$ is paraunitary if and only if $A(z)$ is paraunitary.

In the following Lemma 3.2, we construct 3 categories of paraunitary matrices with specific characteristics, which are of great significance in our work. By direct calculation, the following lemma can be proved.

Lemma 3.2. *Define the matrix*

$$V^1(z)=\begin{pmatrix}\frac{1}{2}\cos\alpha(z^{-1}+z)&\frac{1}{2}\sin\alpha(z^{-1}+z)&\frac{1}{2}(z^{-1}-z)\\ -\sin\alpha&\cos\alpha&0\\ \frac{1}{2}\cos\alpha(z^{-1}-z)&\frac{1}{2}\sin\alpha(z^{-1}-z)&\frac{1}{2}(z^{-1}+z)\end{pmatrix}, \tag{3.8}$$

$$V^2(z)=\begin{pmatrix}\frac{1}{2}\cos\alpha(z^{-1}+z)&\frac{1}{2}(z^{-1}-z)&\frac{1}{2}\sin\alpha(z^{-1}+z)\\ \frac{1}{2}\cos\alpha(z^{-1}-z)&\frac{1}{2}(z^{-1}+z)&\frac{1}{2}\sin\alpha(z^{-1}-z)\\ -\sin\alpha&0&\cos\alpha\end{pmatrix}, \tag{3.9}$$

$$V^3(z) = \begin{pmatrix} \frac{1}{2}(z^{-1}+z) & \frac{1}{2}\sin\alpha(z^{-1}-z) & \frac{1}{2}\cos\alpha(z^{-1}-z) \\ \frac{1}{2}(z^{-1}-z) & \frac{1}{2}\sin\alpha(z^{-1}+z) & \frac{1}{2}\cos\alpha(z^{-1}+z) \\ 0 & \cos\alpha & -\sin\alpha \end{pmatrix}, \tag{3.10}$$

where $\alpha \in [-\pi, \pi)$ and V^1, V^2, V^3 are paraunitary, when $|z| = 1$ and satisfy the following matrix equations, respectively.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} V^1(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = V^1(z), \tag{3.11}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V^2(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = V^2(z), \tag{3.12}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V^3(z^{-1}) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = V^3(z). \tag{3.13}$$

The following Theorem 3.1 provides masks of odd lengths that satisfy the matrix transformation proposition.

Theorem 3.1. *Let*

$$h(z) = \sum_{j=-(2k_1+i)}^{2k_1+i} h_j z^j = \frac{\sqrt{2}}{2}(h_e(z^2) + z^{-1}h_o(z^2)), \tag{3.14}$$

$$g(z) = \sum_{j=-(2k_2+i)}^{2k_2+i} q_j z^j = \frac{\sqrt{2}}{2}(g_e(z^2) + z^{-1}g_o(z^2)), \tag{3.15}$$

$$f(z) = \sum_{j=-(2k_3+i)}^{2k_3+i} f_j z^j = \frac{\sqrt{2}}{2}(f_e(z^2) + z^{-1}f_o(z^2)), \quad i = 1, 2, k_1, k_2, k_3 \in N. \tag{3.16}$$

$A(z)$ is defined as (3.7). If h, g are symmetric filters, and f is an antisymmetric filter, here comes the conclusion

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} A(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & -z^{-1} \end{pmatrix} = A(z). \tag{3.17}$$

Proof. If h, g are symmetric filters, and f is an antisymmetric filter, by (3.14)–(3.16), we have,

$$\begin{aligned} &h_e(z^{-1}) = h_e(z), && zh_0(z^{-1}) = h_0(z), \\ &g_e(z^{-1}) = g_e(z), && zg_0(z^{-1}) = g_0(z), \\ &f_e(z^{-1}) = -h_e(z), && zf_0(z^{-1}) = -f_0(z). \end{aligned}$$

Let h_r, g_r, f_r be defined by (3.4)–(3.6), then

$$h_r(z^{-1}) = -zh_r(z), \qquad g_r(z^{-1}) = -zg_r(z), \qquad f_r(z^{-1}) = zf_r(z).$$

As a result, (3.17) can be proved. The proof is completed. □

Taking advantage of Lemma 3.2 and Theorem 3.1, we present a manipulable and simple construction method of tight wavelet frames.

Theorem 3.2. *Let*

$$h(z) = \sum_{j=-(2k_1+1)}^{2k_1+1} h_j z^j = \frac{\sqrt{2}}{2}(h_e(z^2) + z^{-1}h_o(z^2)),$$

$$g(z) = \sum_{j=-(2k_2+1)}^{2k_2+1} q_j z^j = \frac{\sqrt{2}}{2}(g_e(z^2) + z^{-1}g_o(z^2)),$$

$$f(z) = \sum_{j=-(2k_3+1)}^{2k_3+1} f_j z^j = \frac{\sqrt{2}}{2}(f_e(z^2) + z^{-1}f_o(z^2)), \quad k_1, k_2, k_3 \in N.$$

If h, g are symmetric filters, f is an antisymmetric filter and (2.4) holds, they can be factorized in the form of

$$\begin{pmatrix} h(z) \\ g(z) \\ f(z) \end{pmatrix} = \frac{\sqrt{2}}{2} V_n^1(z^2) V_{n-1}^1(z^2) \cdots V_1^1(z^2) P_0(z^2) \begin{pmatrix} 1 \\ z^{-1} \\ 0 \end{pmatrix}, \quad n \in N, \tag{3.18}$$

where

$$P_0(z) = \begin{pmatrix} -\sin\theta & \frac{1}{2}(z+1)\cos\theta & \frac{1}{2}\cos\theta(1-z^{-1}) \\ \cos\theta & \frac{1}{2}(z+1)\sin\theta & \frac{1}{2}\sin\theta(1-z^{-1}) \\ 0 & \frac{1}{2}(1-z) & -\frac{1}{2}(z^{-1}+1) \end{pmatrix} \tag{3.19}$$

and $V^1(z)$ is defined by (3.8) with $\alpha \in [-\pi, \pi)$, V_j^1, $j \in N$, are defined by $V^1(z)$ with $\alpha = \alpha_j$. Where α_j are arbitrary parameters and $\alpha_j \in [-\pi, \pi)$, $j = 1, 2, \ldots, n$.

Proof. Note that

$$h(z) = h_{-1}z^{-1} + h_0 + h_1 z,$$
$$g(z) = g_{-1}z^{-1} + g_0 + h_1 z,$$
$$f(z) = f_{-1}z^{-1} + f_0 + f_1 z.$$

If g, h are symmetric filters, and f is an antisymmetric filter, from (3.1) the matrix $P_{h,g,f}$ is given by

$$P_{h,g,f}(z) = \begin{pmatrix} h_0 & h_{-1} + h_1 z \\ g_0 & g_{-1} + g_1 z \\ f_0 & f_{-1} + f_1 z \end{pmatrix}.$$

If g, h, f are filters, the following statement is satisfied.

$$P_{h,g,f}(z)^* P_{h,g,f}(z) = E.$$

Here come the equations

$$\begin{cases} h_0^2 + g_0^2 + f_0^2 = 1, \\ h_{-1}^2 + g_{-1}^2 + f_{-1}^2 = \frac{1}{2}, \\ h_{-1}^2 + g_{-1}^2 - f_{-1}^2 = 0, \\ h_0 h_{-1} + g_0 g_{-1} + f_0 f_{-1} = 0, \\ h_0 h_{-1} + g_0 g_{-1} - f_0 f_{-1} = 0. \end{cases} \tag{3.20}$$

Solve the equations in (3.20), a group of solutions is given by

$$h_0 = -\sin\theta, \qquad h_1 = \frac{\cos\theta}{2}, \qquad g_0 = \cos\theta, \qquad g_1 = \frac{\sin\theta}{2}, \qquad f_0 = 0, \qquad f_1 = \frac{1}{2}, \quad \theta \in [-\pi, \pi).$$

By (3.7), we have

$$P_0(z) = \begin{pmatrix} -\sin\theta & \frac{1}{2}(z+1)\cos\theta & \frac{1}{2}\cos\theta(1-z^{-1}) \\ \cos\theta & \frac{1}{2}(z+1)\sin\theta & \frac{1}{2}\sin\theta(1-z^{-1}) \\ 0 & \frac{1}{2}(1-z) & -\frac{1}{2}(z^{-1}+1) \end{pmatrix}$$

and $P_0(z)$ satisfies (3.17). Obviously, $B(z) = V_n^1(z)V_{n-1}^1(z) \cdots V_1^1(z)P_0(z)$ is paraunitary. By Lemma 3.2 and Theorem 3.1, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} B(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & -z^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} V_n^1(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} V_{n-1}^1(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} V_{n-2}^1(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdots$$

$$\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} V_1^1(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} P_0(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & -z^{-1} \end{pmatrix} = B(z),$$

thus $B(z)$ satisfies (3.17). By (3.3), it follows that

$$\begin{pmatrix} h(z) & h(-z) \\ g(z) & g(-z) \\ f(z) & f(-z) \end{pmatrix} = \frac{\sqrt{2}}{2} V_n^1(z^2) V_{n-1}^1(z^2) \cdots V_1^1(z^2) P_0(z^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ z^{-1} & -z^{-1} \end{pmatrix},$$

which implies that (3.18) holds. The proof is completed. □

Note that if we select different P_0 in Theorem 3.2, we will obtain different expressions of masks. Theorem 3.2 provides a construction method of P_0. Actually, we may construct many different P_0. Theorem 3.2 give a novel parameterization construction technique. Theorem 3.2 realizes expressions of masks of arbitrary odd lengths for tight wavelet frames, where g, h are symmetric, and f is an antisymmetric and the (anti)symmetric center is 0. In the following Theorem 3.3, we consider the case h, g, f are masks of odd lengths, while h is a symmetric about 0, and g, f are antisymmetric about 0. Under these conditions, we prove that the masks satisfying (2.4) do not exist.

Theorem 3.3. *Suppose that h, g, f are filters of odd lengths, while h is symmetric about 0, and g, f are antisymmetric about 0. Under these conditions, filters satisfying (2.4) do not exist.*

Proof. Suppose that

$$h(z) = h(z^{-1}), \qquad g(z) = -g(z^{-1}), \qquad f(z) = -f(z^{-1})$$

by (3.1) with $M^*_{h,g,f} M_{h,g,f} = I$, it follows that

$$h(z)^2 - g(z)^2 - f(z)^2 = 1, \tag{3.21}$$

$$h(-z)^2 - g(-z)^2 - f(-z)^2 = 1, \tag{3.22}$$

$$h(z)h(-z) - g(z)g(-z) - f(z)f(-z) = 0. \tag{3.23}$$

By (3.21) and (3.22), we have,

$$g(z) = \sqrt{-1 + h(z)^2}\cos\alpha, \qquad f(z) = \sqrt{-1 + h(z)^2}\sin\alpha, \tag{3.24}$$

$$g(-z) = \sqrt{-1 + h(-z)^2}\cos\alpha, \qquad f(-z) = \sqrt{-1 + h(-z)^2}\sin\alpha. \tag{3.25}$$

Substituting (3.24) and (3.25) into (3.23), we have

$$h(z)^2 + h(-z)^2 = 1. \tag{3.26}$$

Reformulate $M_{h,g,f}$ as

$$M_{h,g,f}(z) = \begin{pmatrix} \alpha \\ A \end{pmatrix}.$$

Then

$$M^*_{h,g,f}(z) M_{h,g,f}(z) = \alpha^*\alpha + A^*A = I. \tag{3.27}$$

By (3.26), we get $\det(I - \alpha^*\alpha) = 0$. So $\det A = 0, f(z) = kg(z), k \in R$. A contradiction is induced then. The proof is completed. □

Next, we discuss the parameterizations of masks of even lengths. The main result are given by Theorem 3.4.

Theorem 3.4. *Assume that*

$$h(z) = \sum_{j=-(2k_1+i)}^{2k_1+i+1} h_j z^j = \frac{\sqrt{2}}{2}(h_e(z^2) + z h_o(z^2)), \tag{3.28}$$

$$f(z) = \sum_{j=-(2k_2+i)}^{2k_2+i+1} f_j z^j = \frac{\sqrt{2}}{2}(f_e(z^2) + z f_o(z^2)), \tag{3.29}$$

$$g(z) = \sum_{j=-(2k_3+i)}^{2k_3+i+1} g_j z^j = \frac{\sqrt{2}}{2}(g_e(z^2) + z g_o(z^2)), \quad i = 0, 1, k_1, k_2, k_3 \in N, \tag{3.30}$$

$$E(z) = \begin{pmatrix} h_e & h_o & h_r \\ g_e & g_o & g_r \\ f_e & f_o & f_r \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.31}$$

If g, h are symmetric filters, and f is an antisymmetric filter, we can deduce that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} E(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E(z). \tag{3.32}$$

If h is a symmetric filter, and f, g are antisymmetric filters we can deduce that

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} E(z^{-1}) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E(z). \tag{3.33}$$

Proof. Suppose that g, h are symmetric filters and f is an antisymmetric filter, by (3.28)–(3.30), the conclusion can be drawn that

$$\begin{aligned} &h_e(z^{-1}) = h_o(z), \qquad h_o(z^{-1}) = h_e(z), \\ &g_e(z^{-1}) = g_o(z), \qquad g_o(z^{-1}) = g_e(z), \\ &f_e(z^{-1}) = -f_o(z), \qquad f_o(z^{-1}) = -f_e(z). \end{aligned} \tag{3.34}$$

By (3.4)–(3.6), we have

$$h_r(z^{-1}) = h_r(z), \qquad g_r(z^{-1}) = g_r(z), \qquad f_r(z^{-1}) = -f_r(z).$$

Since

$$E(z) = \begin{pmatrix} \frac{\sqrt{2}}{2}(h_e + h_o) & \frac{\sqrt{2}}{2}(h_o - h_e) & h_r(z) \\ \frac{\sqrt{2}}{2}(g_e + g_o) & \frac{\sqrt{2}}{2}(g_o - g_e) & g_r(z) \\ \frac{\sqrt{2}}{2}(f_e + f_o) & \frac{\sqrt{2}}{2}(f_o - f_e) & f_r(z) \end{pmatrix},$$

we have

$$E(z^{-1}) = \begin{pmatrix} \frac{\sqrt{2}}{2}(h_e + h_0) & \frac{\sqrt{2}}{2}(h_e - h_o) & h_r(z) \\ \frac{\sqrt{2}}{2}(g_e + g_o) & \frac{\sqrt{2}}{2}(g_e - g_o) & g_r(z) \\ -\frac{\sqrt{2}}{2}(f_e + f_o) & \frac{\sqrt{2}}{2}(f_o - f_e) & -f_r(z) \end{pmatrix}.$$

Thus, (3.32) is proved. So is the other statement.

On the foundation of the above theorem, we propose a manipulable and simple method of construction when the lengths of masks of tight wavelet frames are even.

Theorem 3.5. *Let g, h be symmetric filters, f be an antisymmetric filter and (2.4) holds.*

$$h(z) = \sum_{j=-(2k_1+i)}^{2k_1+i+1} h_j z^j = \frac{\sqrt{2}}{2}(h_e(z^2) + z h_o(z^2)),$$

$$f(z) = \sum_{j=-(2k_2+i)}^{2k_2+i+1} f_j z^j = \frac{\sqrt{2}}{2}(f_e(z^2) + z f_o(z^2)),$$

$$g(z) = \sum_{j=-(2k_3+i)}^{2k_3+i+1} g_j z^j = \frac{\sqrt{2}}{2}(g_e(z^2) + z g_o(z^2)), \quad i = 0, 1, k_1, k_2, k_3 \in N.$$

Define

$$W_1 = \begin{pmatrix} \sin\alpha & 0 & -\cos\alpha \\ \cos\alpha & 0 & \sin\alpha \\ 0 & -1 & 0 \end{pmatrix}, \tag{3.35}$$

and

$$W_2 = \begin{pmatrix} \frac{1}{2}\sin\alpha(z^{-1}+z) & \frac{1}{2}\sin\alpha(z-z^{-1}) & -\cos\alpha \\ \frac{1}{2}\cos\alpha(z^{-1}+z) & \frac{1}{2}\cos\alpha(z-z^{-1}) & \sin\alpha \\ \frac{1}{2}(z^{-1}-z) & -\frac{1}{2}(z+z^{-1}) & 0 \end{pmatrix}. \tag{3.36}$$

Then, h, g, f are factorized as

$$\begin{pmatrix} h(z) \\ g(z) \\ f(z) \end{pmatrix} = \frac{\sqrt{2}}{2} W_i(z^2)V_1^2(z^2)V_2^2(z^2)\cdots V_n^2(z^2) \begin{pmatrix} \frac{\sqrt{2}}{2}(1+z) \\ -\frac{\sqrt{2}}{2}(1-z) \\ 0 \end{pmatrix} \quad i = 1, 2,\ n \in N, \tag{3.37}$$

where $V^2(z)$ is defined by (3.9) with $\alpha \in [-\pi, \pi)$. $V_j^2, j \in N$ are defined by $V^2(z)$ with $\alpha = \alpha_j$. Where α_j are arbitrary parameters and $\alpha_j \in [-\pi, \pi), j = 1, 2, \ldots, n$.

Proof. Assume that

$$\begin{aligned} h(z) &= h_{-1}z^{-1} + h_0 + h_1 z + h_2 z^2 = z(h_{-1}z^{-2} + h_1) + (h_0 + h_2 z^2), \\ g(z) &= g_{-1}z^{-1} + g_0 + g_1 z + g_2 z^2 = z(g_{-1}z^{-2} + g_1) + (g_0 + g_2 z^2), \\ f(z) &= f_{-1}z^{-1} + f_0 + f_1 z + f_2 z^2 = z(f_{-1}z^{-2} + f_1) + (f_0 + f_2 z^2), \end{aligned}$$

then,

$$P_{h,g,f} = \begin{pmatrix} h_{-1}z^{-1} + h_1 & h_0 + h_2 z \\ g_{-1}z^{-1} + g_1 & g_0 + g_2 z \\ f_{-1}z^{-1} + f_1 & f_0 + f_2 z \end{pmatrix}.$$

If $P_{h,g,f}$ is paraunitary, the following conditions are satisfied.

$$\begin{cases} h_{-1}^2 + g_{-1}^2 + f_{-1}^2 + h_1^2 + g_1^2 + f_1^2 = 1, \\ h_{-1}h_0 + g_{-1}g_0 + f_{-1}f_0 = 0, \\ h_0^2 + g_0^2 + f_0^2 + h_2^2 + g_2^2 + f_2^2 = 1, \\ h_0 h_2 + g_0 g_2 + f_0 f_2 = 0, \\ h_1 h_2 + g_1 g_2 + f_1 f_2 = 0, \\ h_{-1}h_0 + g_{-1}g_0 + f_{-1}f_0 = 0, \\ h_{-1}^2 + g_{-1}^2 - f_{-1}^2 = 0, \\ h_0^2 + g_0^2 - f_0^2 = 0. \end{cases}$$

By solving the former equations, two groups of solutions are given by

$$\begin{cases} h_1 = g_1 = f_1 = 0, \\ h_0 = \frac{1}{\sqrt{2}}\sin\alpha, \qquad g_0 = \frac{1}{\sqrt{2}}\cos\alpha, \qquad f_0 = \frac{\sqrt{2}}{2}, \end{cases}$$

$$\begin{cases} h_1 = g_1 = f_1 = 0, \\ h_{-1} = \frac{1}{\sqrt{2}}\sin\alpha, \qquad g_{-1} = \frac{1}{\sqrt{2}}\cos\alpha, \qquad f_{-1} = \frac{\sqrt{2}}{2}. \end{cases}$$

It follows from (3.7) that when $i = 1$

$$P^1_{h,g,f}(z) = \begin{pmatrix} \frac{1}{\sqrt{2}}\sin\alpha & \frac{1}{\sqrt{2}}\sin\alpha & -\cos\alpha \\ \frac{1}{\sqrt{2}}\cos\alpha & \frac{1}{\sqrt{2}}\cos\alpha & \sin\alpha \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

and when $i = 2$

$$p^2_{h,g,f}(z) = \begin{pmatrix} \frac{1}{\sqrt{2}}\sin\alpha(z^{-1}) & \frac{1}{\sqrt{2}}\sin\alpha(z) & -\cos\alpha \\ \frac{1}{\sqrt{2}}\cos\alpha(z^{-1}) & \frac{1}{\sqrt{2}}\cos\alpha(z) & \sin\alpha \\ \frac{\sqrt{2}}{2}z^{-1} & -\frac{\sqrt{2}}{2}z & 0 \end{pmatrix}.$$

From (3.31), we can compute the matrices W_1, W_2 as follows.

$$W_1 = \begin{pmatrix} \sin\alpha & 0 & -\cos\alpha \\ \cos\alpha & 0 & \sin\alpha \\ 0 & -1 & 0 \end{pmatrix},$$

$$W_2 = \begin{pmatrix} \frac{1}{2}\sin\alpha(z^{-1}+z) & \frac{1}{2}\sin\alpha(z-z^{-1}) & -\cos\alpha \\ \frac{1}{2}\cos\alpha(z^{-1}+z) & \frac{1}{2}\cos\alpha(z-z^{-1}) & -\sin\alpha \\ \frac{1}{2}(z^{-1}-z) & -\frac{1}{2}(z+z^{-1}) & 0 \end{pmatrix}.$$

Obviously, (3.32) is satisfied by $W_1(z), W_2(z)$. Set $B_i = W_i(z)V_1^2(z)V_2^2(z)V_3^2(z)\cdots V_n^2(z), i = 1, 2$. By Lemma 3.2, and Theorem 3.4, we have

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} W_i(z^{-1})V_1^2(z^{-1})V_2^2(z^{-1})V_3^2(z^{-1})\cdots V_n^2(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} W_i(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} V_2(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdots \\ &\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} V_n(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = W_i(z)V_1^2(z)V_2^2(z)V_3^2(z)\cdots V_n^2(z). \end{aligned}$$

Consequently, $B_i(z),\ i = 1, 2$ satisfies (3.32). Therefore, by (3.2), $M_{h,g,f}$ can be factorized as

$$\begin{pmatrix} h(z) & h(-z) \\ g(z) & g(-z) \\ f(z) & f(-z) \end{pmatrix} = \frac{\sqrt{2}}{2} W_i(z^2)V_1^2(z^2)V_2^2(z^2)\cdots V_n^2(z^2) \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ z & -z \\ 0 & 0 \end{pmatrix} \quad i = 1, 2, n \in N,$$

which implies that (3.37) holds. The proof is completed. □

Through similar proof processes of Theorem 3.5, the following theorem can be proved.

Theorem 3.6. *Suppose that f is a symmetric filter, and g, h are antisymmetric filters, and (2.4) holds.*

$$h(z) = \sum_{j=-(2k_1+i)}^{2k_1+i+1} h_j z^j = \frac{\sqrt{2}}{2}(h_e(z^2) + zh_o(z^2)),$$

$$f(z) = \sum_{j=-(2k_2+i)}^{2k_2+i+1} f_j z^j = \frac{\sqrt{2}}{2}(f_e(z^2) + zf_o(z^2)),$$

$$g(z) = \sum_{j=-(2k_3+i)}^{2k_3+i+1} g_j z^j = \frac{\sqrt{2}}{2}(g_e(z^2) + z g_o(z^2)), \quad i = 0, 1, k_1, k_2, k_3 \in N.$$

Define the following matrices

$$W_1(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & -\cos\alpha \end{pmatrix}, \qquad W_2(z) = \begin{pmatrix} \frac{1}{2}(z^{-1}+z) & \frac{1}{2}(z-z^{-1}) & 0 \\ \frac{1}{2}\cos\alpha(z^{-1}-z) & -\frac{1}{2}\cos\alpha(z^{-1}+z) & \sin\alpha \\ -\frac{1}{2}\sin\alpha(z^{-1}-z) & \frac{1}{2}\sin\alpha(z^{-1}+z) & -\cos\alpha \end{pmatrix}.$$

Then, h, g, f can be factorized as

$$\begin{pmatrix} h(z) \\ g(z) \\ f(z) \end{pmatrix} = \frac{\sqrt{2}}{2} W_i(z^2) V_1^3(z^2) V_2^3(z^2) \cdots V_n^3(z^2) \begin{pmatrix} \frac{\sqrt{2}}{2}(1+z) \\ -\frac{\sqrt{2}}{2}(1-z) \\ 0 \end{pmatrix} \quad i = 1, 2, \ n \in N, \tag{3.38}$$

where $V^3(z)$ is defined by (3.10) with $\alpha \in [-\pi, \pi)$, V_j^3, $j \in N$, are defined by $V^3(z)$ with $\alpha = \alpha_j$. Where α_j are arbitrary parameters and $\alpha_j \in [-\pi, \pi)$.

Note that Theorems 3.5 and 3.6 provide only a construction method of W_1, W_2. If we select different W_1, W_2 in Theorems 3.5 and 3.6, we will establish different explicit formulas of masks of tight wavelet frames with arbitrary even lengths. We should particularly emphasize that the expressions of masks in Theorems 3.2, 3.5 and 3.6 have as many free parameters as possible. The parameters provide the freedom to optimize the resulting masks with respect to other criteria. For example, we can obtain the best parameters by solving some equations related to the sum rules for scaling function and the vanishing moments for framelets such that the tight frames have a good compression potential. We will explore the transform applicability of the presented library of tight wavelet frames to still image compression and denoising in Section 3.

3.2. Construction of parameterizations of masks in arbitrary (anti)symmetric centers

In this subsection, we provide a general construction framework of parameterizations of masks for tight wavelet frames with two (anti)symmetric generators with arbitrary lengths and (anti)symmetric centers, which have not been discussed in [27]. We consider the expressions of masks with the following forms:

$$h(z) = \sum_{j=-(2k_1+i)}^{2(k_1+k)+i} h_j z^j, \qquad g(z) = \sum_{j=-(2k_1+i)}^{2(k_1+l)+i} q_j z^j, \qquad f(z) = \sum_{j=-(2k_1+i)}^{2(k_1+m)+i} f_j z^j \quad \text{and}$$

$$h(z) = \sum_{j=-(2k_1+i)}^{2(k_1+k)+i} h_j z^j, \qquad g(z) = \sum_{j=-(2k_1+i)}^{2(k_1+l)+i} q_j z^j, \qquad f(z) = \sum_{j=-(2k_1+i)}^{2(k_1+m)+i} f_j z^j.$$

First, we construct categories of matrices with a parameterization as follows

$$U^1(z) = \begin{pmatrix} \frac{1}{2}\cos\alpha(z^{-1}+z) & \frac{1}{2}\sin\alpha z^{k-l}(z^{-1}+z) & \frac{1}{2}z^{k-m}(z^{-1}-z) \\ -\sin\alpha z^{l-k} & \cos\alpha & 0 \\ \frac{1}{2}z^{m-k}\cos\alpha(z^{-1}-z) & \frac{1}{2}z^{m-l}\sin\alpha(z^{-1}-z) & \frac{1}{2}(z^{-1}+z) \end{pmatrix}, \tag{3.39}$$

$$U^2(z) = \begin{pmatrix} \frac{1}{2}(z^{-1}+z) & \frac{1}{2}\cos\alpha z^{k-l}(z^{-1}-z) & \frac{1}{2}\sin\alpha z^{k-m}(z^{-1}-z) \\ 0 & -\sin\alpha & z^{l-m}\cos\alpha \\ \frac{1}{2}z^{m-k}(z^{-1}-z) & \frac{1}{2}\cos\alpha z^{m-l}(z^{-1}+z) & \frac{1}{2}\sin\alpha(z^{-1}+z) \end{pmatrix}, \tag{3.40}$$

$$U^3(z) = \begin{pmatrix} \frac{1}{2}\cos\alpha(z^{-1}+z) & \frac{1}{2}(z^{-1}-z) & \frac{1}{2}\sin\alpha z^{-l-k-m}(z^{-1}+z) \\ \frac{1}{2}\cos\alpha(z^{-1}-z) & \frac{1}{2}(z^{-1}+z) & \frac{1}{2}\sin\alpha z^{-l-k-m}(z^{-1}-z) \\ (-z^{l+k+m})\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \tag{3.41}$$

$$U^4(z)=\begin{pmatrix}\frac{1}{2}(z^{-1}+z) & \frac{1}{2}\sin\alpha(z^{-1}-z) & \frac{1}{2}\cos\alpha z^{-l-k-m}(z^{-1}-z)\\ \frac{1}{2}(z^{-1}-z) & \frac{1}{2}\sin\alpha(z^{-1}+z) & \frac{1}{2}\cos\alpha z^{-l-k-m}(z^{-1}+z)\\ 0 & z^{l+k+m}\cos\alpha & -\sin\alpha\end{pmatrix}, \tag{3.42}$$

where $|z|=1$, $l,m,k\in Z$, $\alpha\in[-\pi,\pi)$. By direct calculation, $U^1(z),U^2(z),U^3(z),U^4(z)$ has the following proposition.

Lemma 3.3. *Assume that $U^1(z),U^2(z),U^3(z),U^4(z)$ are defined by (4.1), (3.39)–(3.42), then $U^1(z),U^2(z),U^3(z),U^4(z)$ are paraunitary matrices when $z=1$ and satisfy the following conditions accordingly.*

$$\begin{pmatrix}z^{2k} & 0 & 0\\ 0 & z^{2l} & 0\\ 0 & 0 & -z^{2m}\end{pmatrix}U^1(z^{-1})\begin{pmatrix}z^{-2k} & 0 & 0\\ 0 & z^{-2l} & 0\\ 0 & 0 & -z^{-2m}\end{pmatrix}=U^1(z), \tag{3.43}$$

$$\begin{pmatrix}z^{2k} & 0 & 0\\ 0 & -z^{2l} & 0\\ 0 & 0 & -z^{2m}\end{pmatrix}U^2(z^{-1})\begin{pmatrix}z^{-2k} & 0 & 0\\ 0 & -z^{-2l} & 0\\ 0 & 0 & -z^{-2m}\end{pmatrix}=U^2(z), \tag{3.44}$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & z^{2l+2k+2m}\end{pmatrix}U^3(z^{-1})\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & z^{-2l-2k-2m}\end{pmatrix}=U^3(z), \tag{3.45}$$

$$\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & z^{2l+2m+2k}\end{pmatrix}U^4(z^{-1})\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & z^{-2l-2m-2k}\end{pmatrix}=U^4(z) \tag{3.46}$$

where $|z|=1$, $l,m,k\in Z$, $\alpha\in[-\pi,\pi)$.

The following theorem provides masks of odd lengths and of arbitrary symmetric centers.

Theorem 3.7. *Suppose that*

$$h(z)=\sum_{j=-(2k_1+i)}^{2(k_1+k)+i}h_jz^j=\frac{\sqrt{2}}{2}(h_e(z^2)+z^{-1}h_o(z^2)), \tag{3.47}$$

$$g(z)=\sum_{j=-(2k_1+i)}^{2(k_1+l)+i}q_jz^j=\frac{\sqrt{2}}{2}(g_e(z^2)+z^{-1}g_o(z^2)), \tag{3.48}$$

$$f(z)=\sum_{j=-(2k_1+i)}^{2(k_1+m)+i}f_jz^j=\frac{\sqrt{2}}{2}(f_e(z^2)+z^{-1}f_o(z^2)),\quad i=0,1,k_1\in N_0,\ k,l,m\in Z. \tag{3.49}$$

$A(z)$ is defined as (3.7), if h,g are symmetric filters and f is an antisymmetric filter, then

$$\begin{pmatrix}z^k & 0 & 0\\ 0 & z^l & 1\\ 0 & & -z^m\end{pmatrix}A(z^{-1})\begin{pmatrix}1 & 0 & 0\\ 0 & z & 0\\ 0 & 0 & -z^{-k-l-m-1}\end{pmatrix}=A(z). \tag{3.50}$$

If h is a symmetric filter, and f,g are antisymmetric filters, then

$$\begin{pmatrix}z^k & 0 & 0\\ 0 & -z^l & 1\\ 0 & & -z^m\end{pmatrix}A(z^{-1})\begin{pmatrix}1 & 0 & 0\\ 0 & z & 0\\ 0 & 0 & z^{-k-l-m-1}\end{pmatrix}=A(z),\quad (k^2+l^2+m^2\neq 0). \tag{3.51}$$

Proof. If h,g are symmetric filters and f is an antisymmetric filter, by (3.47)–(3.49), we have

$$z^kh_e(z^{-1})=h_e(z),\qquad z^{k+1}h_o(z^{-1})=h_0(z),$$
$$z^lg_e(z^{-1})=g_e(z),\qquad z^{l+1}g_o(z^{-1})=g_0(z),$$
$$z^mf_e(z^{-1})=-f_e(z),\qquad z^{m+1}f_o(z^{-1})=-f_0(z),$$

and by (3.4)–(3.6) together with the above equations, we have

$$h_r(z) = -z^{-l-m-1}h_r(z^{-1}), \qquad g_r(z) = -z^{-m-k-1}g_r(z^{-1}), \qquad f_r(z) = z^{-k-l-1}f_r(z^{-1}).$$

Hence,

$$\begin{pmatrix} z^k & 0 & 0 \\ 0 & z^l & 1 \\ 0 & & -z^m \end{pmatrix} A(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & -z^{-k-l-m-1} \end{pmatrix} = A(z).$$

The proof is completed. Similarly, (3.51) can be proved. □

Theorem 3.8 provides masks of even lengths and of arbitrary symmetric centers.

Theorem 3.8. *Let*

$$h(z) = \sum_{j=-(2k_1+i)}^{2(k_1+k)+i+1} h_j z^j = \frac{\sqrt{2}}{2}(h_e(z^2) + zh_o(z^2)), \tag{3.52}$$

$$g(z) = \sum_{j=-(2k_1+i)}^{2(k_1+l)+i+1} q_j z^j = \frac{\sqrt{2}}{2}(g_e(z^2) + zg_o(z^2)), \tag{3.53}$$

$$f(z) = \sum_{j=-(2k_1+i)}^{2(k_1+m)+i+1} f_j z^j = \frac{\sqrt{2}}{2}(f_e(z^2) + zf_o(z^2)), \quad i = 0, 1, k_1 \in N_0,\ k, l, m \in Z. \tag{3.54}$$

$E(z)$ is defined as (3.31), if h, g are symmetric filters, and f is an antisymmetric filter, then

$$\begin{pmatrix} z^k & 0 & 0 \\ 0 & z^l & 0 \\ 0 & 0 & -z^m \end{pmatrix} E(z^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & z^{-k-l-m} \end{pmatrix} = E(z). \tag{3.55}$$

If h is a symmetric filter, and f, g are antisymmetric filters, thus

$$\begin{pmatrix} -z^k & 0 & 0 \\ 0 & z^l & 1 \\ 0 & 0 & z^m \end{pmatrix} E(z^{-1}) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-k-l-m} \end{pmatrix} = E(z). \tag{3.56}$$

Proof. If h, g are symmetric filters, and f is an antisymmetric filter, (3.52)–(3.54) together with (3.4)–(3.6), lead to

$$\begin{aligned} &z^k h_e(z^{-1}) = h_0(z), \qquad z^k h_o(z^{-1}) = h_e(z), \\ &z^l g_e(z^{-1}) = g_0(z), \qquad z^l g_o(z^{-1}) = g_e(z), \\ &z^m f_e(z^{-1}) = -f_0(z), \qquad z^k f_o(z^{-1}) = -f_e(z), \\ &h_r(z) = z^{-l-m}h_r(z^{-1}), \qquad g_r(z) = z^{-k-m}g_r(z^{-1}), \\ &f_r(z) = -z^{-l-k}f_r(z^{-1}), \end{aligned} \tag{3.57}$$

then

$$E(z^{-1}) = \begin{pmatrix} \frac{\sqrt{2}}{2}z^{-k}(h_e(z) + h_0(z)) & \frac{\sqrt{2}}{2}z^{-k}(h_e(z) - h_0(z)) & z^{m+k}h_r \\ \frac{\sqrt{2}}{2}z^{-l}(g_e(z) + g_0(z)) & \frac{\sqrt{2}}{2}z^{-l}(g_e(z) - g_0(z)) & z^{m+k}g_r \\ -\frac{\sqrt{2}}{2}z^{-m}(f_e(z) + f_0(z)) & -\frac{\sqrt{2}}{2}z^{-m}(f_e(z) - f_0(z)) & -z^{l+k}f_r \end{pmatrix}$$

which implies (3.55) hold. The other statement (3.56) can be proved similarly. □

By Lemma 3.3 and Theorem 3.7, the following theorem can be proved similarly as the proof of Theorem 3.2.

Theorem 3.9. *Suppose that*

$$\begin{cases} h(z)=\sum_{j=-(2k_1+i)}^{2(k_1+K)+i} h_j z^j=\frac{\sqrt{2}}{2}(h_e(z^2)+z^{-1}h_o(z^2)) \\ g(z)=\sum_{j=-(2k_1+i)}^{2(k_1+L)+i} q_j z^j=\frac{\sqrt{2}}{2}(g_e(z^2)+z^{-1}g_o(z^2)) \\ f(z)=\sum_{j=-(2k_1+i)}^{2(k_1+M)+i} f_j z^j=\frac{\sqrt{2}}{2}(f_e(z^2)+z^{-1}f_o(z^2)) \quad i=0,1\, M=2m,\ L=2l,\ K=2k,k_1,M,L,K\in N_0 \end{cases} \tag{3.58}$$

when $i=1, K=0$ in (3.58), and by (3.7), $W_1(z)$ is defined as follows

$$W_1(z)=\begin{pmatrix} \sum_{j=0}^{K} h_{2j}z^j & \sum_{j=-1}^{K} h_{2j+1}z^{j+1} & h_r^1(z) \\ \sum_{j=0}^{L} g_{2j}z^j & \sum_{j=-1}^{L} g_{2j+1}z^{j+1} & g_r^1(z) \\ \sum_{j=0}^{M} f_{2j}z^j & \sum_{j=-1}^{M} f_{2j+1}z^{j+1} & f_r^1(z) \end{pmatrix} \tag{3.59}$$

with

$$h_r^1(z^{-1})=\left(\sum_{j=0}^{L} g_{2j}z^j\right)\left(\sum_{j=-1}^{M} f_{2j+1}z^{j+1}\right)-\left(\sum_{j=-1}^{L} g_{2j+1}z^{j+1}\right)\left(\sum_{j=0}^{M} f_{2j}z^j\right)$$
$$g_r^1(z^{-1})=\left(\sum_{j=-1}^{K} h_{2j+1}z^{j+1}\right)\left(\sum_{j=0}^{M} f_{2j}z^j\right)-\left(\sum_{j=0}^{K} h_{2j}z^j\right)\left(\sum_{j=-1}^{M} f_{2j+1}z^{j+1}\right)$$
$$f_r^1(z^{-1})=\left(\sum_{j=0}^{K} h_{2j}z^j\right)\left(\sum_{j=-1}^{L} g_{2j+1}z^{j+1}\right)-\left(\sum_{j=-1}^{K} h_{2j+1}z^{j+1}\right)\left(\sum_{j=0}^{L} g_{2j}z^j\right).$$

When $i=0, K=1$ in (3.58), and by (3.7) $W_2(z)$ is defined as follows

$$W_2(z)=\begin{pmatrix} \sum_{j=-1}^{K+1} h_{2j}z^j & \sum_{j=-1}^{K} h_{2j+1}z^{j+1} & h_r^2(z) \\ \sum_{j=-1}^{L+1} g_{2j}z^j & \sum_{j=-1}^{L} g_{2j+1}z^{j+1} & g_r^2(z) \\ \sum_{j=-1}^{M+1} f_{2j}z^j & \sum_{j=-1}^{M} f_{2j+1}z^{j+1} & f_r^2(z) \end{pmatrix} \tag{3.60}$$

with

$$h_r^2(z^{-1})=\left(\sum_{j=-1}^{L+1} g_{2j}z^j\right)\left(\sum_{j=-1}^{M} f_{2j+1}z^{j+1}\right)-\left(\sum_{j=-1}^{L} g_{2j+1}z^{j+1}\right)\left(\sum_{j=-1}^{M+1} f_{2j}z^j\right)$$
$$g_r^2(z^{-1})=\left(\sum_{j=-1}^{K} h_{2j+1}z^{j+1}\right)\left(\sum_{j=-1}^{M+1} f_{2j}z^j\right)-\left(\sum_{j=-1}^{K+1} h_{2j}z^j\right)\left(\sum_{j=-1}^{M} f_{2j+1}z^{j+1}\right)$$
$$f_r^2(z^{-1})=\left(\sum_{j=-1}^{K+1} h_{2j}z^j\right)\left(\sum_{j=-1}^{M} g_{2j+1}z^{j+1}\right)-\left(\sum_{j=-1}^{K} h_{2j+1}z^{j+1}\right)\left(\sum_{j=-1}^{L+1} g_{2j}z^j\right).$$

Suppose that h, g are symmetric and f is antisymmetric. Assume that (2.4) holds. Moreover, if there exist $K, L, M \in N_0$, such that W_1, W_2 are paraunitary, which satisfies (3.50), then h, g, f can be factorized as

$$\begin{pmatrix} h(z) \\ g(z) \\ f(z) \end{pmatrix} = \frac{\sqrt{2}}{2} U_n^1(z^2) U_{n-1}^1(z^2) \cdots U_1^1(z^2) W_i(z^2) \begin{pmatrix} 1 \\ z^{-1} \\ 0 \end{pmatrix} \quad i = 1, 2, n \in N. \tag{3.61}$$

Where U^1 is defined by (3.39) and U_j^1 is defined by U^1 with $\alpha = \alpha_j$. α_j is arbitrary parameter and $\alpha_j \in [-\pi, \pi)$.

Furthermore, suppose that h is symmetric, and f, g are antisymmetric. Assume that (2.4) holds. If there exist $K, L, M \in N_0$, such that W_1, W_2 are paraunitary, which satisfy (3.51), then h, g, f can be factorized as

$$\begin{pmatrix} h(z) \\ g(z) \\ f(z) \end{pmatrix} = \frac{\sqrt{2}}{2} U_n^2(z^2) U_{n-1}^2(z^2) \cdots U_1^2(z^2) W_i(z^2) \begin{pmatrix} 1 \\ z^{-1} \\ 0 \end{pmatrix} \quad (M^2 + L^2 + K^2 \neq 0), \; i = 1, 2, n \in N \tag{3.62}$$

where U^2 is defined by (3.40) and U_j^2 is defined by U^2 with $\alpha = \alpha_j$. α_j is an arbitrary parameter and $\alpha_j \in [-\pi, \pi)$.

In fact, Theorem 3.9 provides a general expression of masks of wavelet frames, which is contingent to W_1 and W_2. In the following corollary, we focus on two specific conditions.

Corollary 3.9.1. *Suppose that the matrices $W_1(z), W_2(z)$ are defined by*

$$W_1(z) = \begin{pmatrix} \cos\alpha & -\frac{\sin\alpha}{2}(1+z) & \frac{\sin\alpha}{2}(z^{-1} - z^{-2}) \\ \sin\alpha & \frac{\cos\alpha}{2}(1+z) & -\frac{\cos\alpha}{2}(z^{-1} - z^{-2}) \\ 0 & \frac{1}{2}(z - z^2) & \frac{1}{2}(1 + z^{-1}) \end{pmatrix}, \quad \forall \alpha \in [-\pi, \pi),$$

and

$$W_2(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(1+z^3) & -\frac{1}{2}(1 - z^{-3}) \\ 0 & \frac{1}{2}(1 - z^3) & \frac{1}{2}(1 + z^{-3}) \end{pmatrix},$$

then

$$\begin{pmatrix} h(z) \\ g(z) \\ f(z) \end{pmatrix} = \frac{\sqrt{2}}{2} U_n^1(z^2) U_{n-1}^1(z^2) \cdots U_1^1(z^2) W_i(z^2) \begin{pmatrix} 1 \\ z^{-1} \\ 0 \end{pmatrix}, \quad i = 1, 2, n \in N \tag{3.63}$$

are FIR filters with odd lengths which satisfy (2.4). When $i = 1$, U^1 is defined by (3.39) with $k = 0, l = 0, m = 1$, and U_j^1 is defined by U^1 with $\alpha = \alpha_j, \alpha_j \in [-\pi, \pi)$. α_j is an arbitrary parameter and $\alpha_j \in [-\pi, \pi), j = 1, 2, \ldots, n$. When $i = 2$, U^1 is defined by (3.39) with $k = 0, l = 1, m = 1$, and U_j^1 is defined by U^1 with $\alpha = \alpha_j, \alpha_j \in [-\pi, \pi)$. α_j is an arbitrary parameter and $\alpha_j \in [-\pi, \pi)$, $j = 1, 2, \ldots, n$.

Proof. $W_1(z)$ is defined by (3.59) with $K = 0, L = 0, M = 2$

$$W_1(z) = \begin{pmatrix} h_0 & h_{-1}(1+z) & h_r^1(z) \\ g_0 & g_{-1}(1+z) & g_r^1(z) \\ f_0(1-z^2) & f_{-1}(1-z^3) + f_1(z - z^2) & f_r^1(z) \end{pmatrix}.$$

If $W_1(z)^* W_1(z) = E$, we can get the solution $h_0 = \cos\alpha, g_0 = \sin\alpha, f_0 = 0, g_{-1} = \frac{\cos\alpha}{2}, h_{-1} = \frac{-\sin\alpha}{2}, f_{-1} = 0, f_1 = \frac{1}{2}$. By (3.61), (3.63) holds with $i = 1$.

If $W_2(z)$ is defined by (3.59) with $L = 2, M = 2$ as follows

$$W_2(z) = \begin{pmatrix} h_0 & h_{-1}(1+z) & h_r^1(z) \\ g_0(1+z^2) & g_{-1}(1+z^3) + g_1(z + z^2) & g_r^1(z) \\ f_0(1-z^2) & f_{-1}(1-z^3) + f_1(z - z^2) & f_r^1(z) \end{pmatrix}.$$

If $W_2(z)^* W_2(z) = E$, we can get the solution $h_0 = 1, g_0 = g_1 = f_0 = f_1 = 0, g_{-1} = f_{-1} = \frac{1}{2}$. By (3.61), (3.63) holds with $i = 2$. The proof is completed. □

By Lemma 3.3 and Theorem 3.9, the following theorem can be proved similarly to the proof of Theorems 3.5 and 3.6.

Theorem 3.10. *Assume that*

$$\begin{cases} h(z)=\displaystyle\sum_{j=-(2k_1+i)}^{2(k_1+K)+i+1} h_j z^j=\frac{\sqrt{2}}{2}(h_e(z^2)+zh_o(z^2)) \\ g(z)=\displaystyle\sum_{j=-(2k_1+i)}^{2(k_1+L)+i+1} q_j z^j=\frac{\sqrt{2}}{2}(g_e(z^2)+zg_o(z^2)) \\ f(z)=\displaystyle\sum_{j=-(2k_1+i)}^{2(k_1+M)+i+1} f_j z^j=\frac{\sqrt{2}}{2}(f_e(z^2)+zf_o(z^2)), \quad i=0,1,\ M=2m,\ L=2l,\ K=2k, k_1,\ M,L,K\in N_0 \end{cases} \tag{3.64}$$

when $k_1=0, i=1$ in (3.64), and by (3.31), $W_1(z)$ is defined as follows

$$W_1(z)=\begin{pmatrix} \displaystyle\sum_{j=0}^{K+1} h_{2j}z^j & \displaystyle\sum_{j=-1}^{K} h_{2j+1}z^j & h_r^1(z) \\ \displaystyle\sum_{j=0}^{L+1} g_{2j}z^j & \displaystyle\sum_{j=-1}^{L} g_{2j+1}z^j & g_r^1(z) \\ \displaystyle\sum_{j=0}^{M+1} f_{2j}z^j & \displaystyle\sum_{j=-1}^{M} f_{2j+1}z^j & f_r^1(z) \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.65}$$

with

$$h_r^1(z^{-1})=\left(\sum_{j=0}^{L+1} g_{2j}z^j\right)\left(\sum_{j=-1}^{M} f_{2j+1}z^j\right)-\left(\sum_{j=-1}^{L} g_{2j+1}z^j\right)\left(\sum_{j=0}^{M+1} f_{2j}z^j\right)$$

$$g_r^1(z^{-1})=\left(\sum_{j=-1}^{K} h_{2j+1}z^j\right)\left(\sum_{j=0}^{M+1} f_{2j}z^j\right)-\left(\sum_{j=0}^{K+1} h_{2j}z^j\right)\left(\sum_{j=-1}^{M} f_{2j+1}z^j\right)$$

$$f_r^1(z^{-1})=\left(\sum_{j=0}^{K+1} h_{2j}z^j\right)\left(\sum_{j=-1}^{L} g_{2j+1}z^j\right)-\left(\sum_{j=0}^{L+1} g_{2j}z^j\right)\left(\sum_{j=-1}^{K} h_{2j+1}z^j\right).$$

When $k_1=1, i=0$ in (3.64), and by (3.31), $W_2(z)$ is defined as follows

$$W_2(z)=\begin{pmatrix} \displaystyle\sum_{j=-1}^{K+1} h_{2j}z^j & \displaystyle\sum_{j=-1}^{K+1} h_{2j+1}z^j & h_r^2(z) \\ \displaystyle\sum_{j=-1}^{L+1} g_{2j}z^j & \displaystyle\sum_{j=-1}^{L+1} g_{2j+1}z^j & g_r^2(z) \\ \displaystyle\sum_{j=-1}^{M+1} f_{2j}z^j & \displaystyle\sum_{j=-1}^{M+1} f_{2j+1}z^j & f_r^2(z) \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.66}$$

with

$$h_r^2(z^{-1})=\left(\sum_{j=-1}^{L+1} g_{2j}z^j\right)\left(\sum_{j=-1}^{M+1} f_{2j+1}z^j\right)-\left(\sum_{j=-1}^{M+1} f_{2j}z^j\right)\left(\sum_{j=-1}^{L+1} g_{2j+1}z^j\right)$$

$$g_r^2(z^{-1})=\left(\sum_{j=-1}^{K+1} h_{2j+1}z^j\right)\left(\sum_{j=-1}^{M+1} f_{2j}z^j\right)-\left(\sum_{j=-1}^{K+1} h_{2j}z^j\right)\left(\sum_{j=-1}^{M+1} f_{2j+1}z^j\right)$$

$$f_r^2(z^{-1})=\left(\sum_{j=-1}^{L+1} h_{2j}z^j\right)\left(\sum_{j=-1}^{M+1} g_{2j+1}z^j\right)-\left(\sum_{j=-1}^{L+1} h_{2j+1}z^j\right)\left(\sum_{j=-1}^{L+1} g_{2j}z^j\right).$$

Suppose that h, g are symmetric, f is antisymmetric and (2.4) holds. If there exist $K, L, M \in N_0$, such that W_1, W_2 are paraunitary, which satisfies (3.55), then h, g, f can be factorized as

$$\begin{pmatrix} h(z) \\ g(z) \\ f(z) \end{pmatrix} = \frac{\sqrt{2}}{2} W_i(z^2) U_1^3(z^2) U_2^3(z^2) \cdots U_n^3(z^2) \begin{pmatrix} \frac{\sqrt{2}}{2}(1+z) \\ -\frac{\sqrt{2}}{2}(1-z) \\ 0 \end{pmatrix} \quad i = 1, 2, n \in N \tag{3.67}$$

where U^3 is defined by (3.41) and U_j^3 is defined by U^3 with $\alpha = \alpha_j$. α_j is arbitrary parameter and $\alpha_j \in [-\pi, \pi)$, $j = 1, 2, \ldots, n$.

Consequently, if h is symmetric and f, g are antisymmetric, and if there exist $K, L, M \in N_0$, such that W_1, W_2 are paraunitary, which satisfies (3.56), then h, g, f can be factorized as

$$\begin{pmatrix} h(z) \\ g(z) \\ f(z) \end{pmatrix} = \frac{\sqrt{2}}{2} W_i(z^2) U_1^4(z^2) U_2^4(z^2) \cdots U_n^4(z^2) \begin{pmatrix} \frac{\sqrt{2}}{2}(1+z) \\ -\frac{\sqrt{2}}{2}(1-z) \\ 0 \end{pmatrix} \quad i = 1, 2, n \in N \tag{3.68}$$

where U^4 is defined by (3.42) and U_j^4 is defined by U^4 with $\alpha = \alpha_j$. α_j is arbitrary parameter and $\alpha_j \in [-\pi, \pi)$, $j = 1, 2, \ldots, n$.

Corollary 3.10.1. *Suppose that $W_1(z), W_2(z)$ are defined as follows*

$$W_1(z) = \begin{pmatrix} \frac{1}{2}(z+z^{-1})\cos\alpha & \frac{1}{2}(z^{-1}-z)\cos\alpha & -z^{-2}\sin\alpha \\ \frac{1}{2}(z^2+1)\sin\alpha & -\frac{1}{2}\sin\alpha(z^2-1) & z^{-1}\cos\alpha \\ \frac{1}{2}(z^2-1) & -\frac{1}{2}(z^2+1) & 0 \end{pmatrix}, \quad \alpha \in [-\pi, \pi),$$

$$W_2(z) = \begin{pmatrix} \frac{1}{2}(z+z^{-1}) & \frac{1}{2}(z^{-1}-z) & 0 \\ \frac{1}{2}(z^2-1)\cos\alpha & -\frac{1}{2}(z^2+1)\cos\alpha & z^{-1}\sin\alpha \\ \frac{1}{2}(z^2-1)\sin\alpha & -\frac{1}{2}(z^2+1)\sin\alpha & -z^{-1}\cos\alpha \end{pmatrix}, \quad \alpha \in [-\pi, \pi),$$

then

$$\begin{pmatrix} h(z) \\ g(z) \\ f(z) \end{pmatrix} = \frac{\sqrt{2}}{2} W_1(z^2) U_1^3(z^2) U_2^3(z^2) \cdots U_n^3(z^2) \begin{pmatrix} \frac{\sqrt{2}}{2}(1+z) \\ -\frac{\sqrt{2}}{2}(1-z) \\ 0 \end{pmatrix} \quad n \in N \tag{3.69}$$

are FIR filters of even lengths which satisfy (2.4), and h, g are symmetric, f is antisymmetric, where U^3 is defined by (3.41) with $l = 1, m = 1$ and U_j^3 is defined by U^3 with $\alpha = \alpha_j$. Furthermore,

$$\begin{pmatrix} h(z) \\ g(z) \\ f(z) \end{pmatrix} = \frac{\sqrt{2}}{2} W_2(z^2) U_1^4(z^2) U_2^4(z^2) \cdots U_n^4(z^2) \begin{pmatrix} \frac{\sqrt{2}}{2}(1+z) \\ -\frac{\sqrt{2}}{2}(1-z) \\ 0 \end{pmatrix} \quad i = 1, 2, n \in N \tag{3.70}$$

are FIR filters which satisfy (2.4), of even lengths and h is symmetric, g, f are antisymmetric, where U^4 is defined by (3.42) and U_j^4 is defined by U^4 with $l = 1, m = 1, \alpha = \alpha_j$.

Proof. (3.1) together with (3.65), and define $P_{h,g,f}(z)$ as follows

$$P_{h,g,f}(z) = \begin{pmatrix} h_0 + h_2 z & h_2 z^{-1} + h_0 \\ g_0 + g_2 z + g_4 z^2 + g_6 z^3 & g_6 z^{-1} + g_4 + g_2 z + g_0 z^2 \\ f_0 + f_2 z + f_4 z^2 + f_6 z^3 & -f_6 z^{-1} - f_4 - f_2 z - f_0 z^2 \end{pmatrix}.$$

By $P^*_{h,g,f}(z)P^*_{h,g,f}(z) = E$, we can get the solutions

$$h_0 = 0, \qquad h_2 = \frac{\sqrt{2}}{2}\cos\alpha, \qquad g_0 = g_2 = g_6 = 0,$$
$$f_0 = f_2 = f_6 = 0, \qquad g_4 = \frac{\sqrt{2}}{2}\sin\alpha, \qquad f_4 = \frac{\sqrt{2}}{2}$$

then

$$P_{h,g,f}(z) = \begin{pmatrix} \frac{\sqrt{2}}{2} z\cos\alpha & \frac{\sqrt{2}}{2} z^{-1}\cos\alpha \\ \frac{\sqrt{2}}{2} z^2\sin\alpha & \frac{\sqrt{2}}{2}\sin\alpha \\ \frac{\sqrt{2}}{2} z^2 & -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

Moreover, by (3.65), (3.67), it follows that (3.69) holds, similarly (3.70) can be proved. The proof is completed. □

Theorem 3.9, Corollary 3.9.1, Theorem 3.10 and Corollary 3.10.1 propose an efficient method that generates a wide range of new symmetric tight wavelet frames. The presented expressions of masks have many free parameters, and a relatively high degree of freedom as well. By adjusting the parameters, we can design the masks with special properties which is valuable for image processing.

3.3. Examples

In this subsection, we construct frames based on parameterizations of masks h, g, f provided in Theorems 3.2 and 3.5. Here, we construct masks with two parameters. We will carry out a series of experiments to evaluate the suitability of the wavelet frames based on the Examples 3.1 and 3.2 for the compression of still images and image denoising in the next section.

First, we change the variables in order to simplify the mathematic expressions.

$$\sin\alpha = \frac{2t}{1+t^2}, \qquad \cos\alpha = \frac{1-t^2}{1+t^2}.$$

Example 3.1. In Theorem 3.2, let $n = 1$, then,

$$h_3 = \frac{\sqrt{2}}{4}\frac{(2t_1t_0 + t_1^2t_0^2 + 1)}{(1+t_0^2)(1+t_1^2)}, \qquad h_2 = \frac{\sqrt{2}}{4}\frac{(2t_1^2t_0 - 2t_0 - 2t_1t_0^2 + 2t_1)}{(1+t_0^2)(1+t_1^2)},$$
$$h_1 = \frac{\sqrt{2}}{4}\frac{(-t_0^2 - t_1^2 + 2t_0t_1)}{(1+t_0^2)(1+t_1^2)}, \qquad h_0 = 0, \qquad h_{-1} = h_1, \qquad h_{-2} = h_2, \qquad h_{-3} = h_3.$$
$$g_1 = -\frac{\sqrt{2}}{2}\frac{(t_1 - t_1t_0^2 - t_0 + t_0t_1^2)}{(1+t_1^2)(1+t_0^2)}, \qquad g_0 = -\frac{\sqrt{2}}{2}\frac{(-4t_1t_0 - 1 + t_0^2 + t_1^2 - t_0^2t_1^2)}{(1+t_1^2)(1+t_0^2)}$$
$$g_{-1} = g_1,$$
$$f_3 = -\frac{\sqrt{2}}{4}\frac{(2t_1t_0 + t_1^2t_0^2 + 1)}{(1+t_0^2)(1+t_1^2)}, \qquad f_2 = -\frac{\sqrt{2}}{4}\frac{(2t_1^2t_0 - 2t_0 - 2t_1t_0^2 + 2t_1)}{(1+t_0^2)(1+t_1^2)},$$
$$f_1 = -\frac{\sqrt{2}}{4}\frac{(-t_0^2 - t_1^2 + 2t_0t_1)}{(1+t_0^2)(1+t_1^2)}, \qquad f_0 = 0, \qquad f_{-i} = -f_i, \quad i = 1, 2, 3.$$

Where t_0, t_1 are arbitrary real numbers.

Example 3.2. In Theorem 3.5, let $n = 1$, then,

$$h_3 = \frac{-t_0t_1^2}{(1+t_0^2)(1+t_1^2)}, \qquad h_2 = \frac{t_0}{(1+t_0^2)(1+t_1^2)}, \qquad h_1 = \frac{-t_1 + t_1t_0^2}{(1+t_0^2)(1+t_1^2)}$$
$$h_0 = h_1, \qquad h_{-1} = h_2, \qquad h_{-2} = h_3$$
$$g_3 = \frac{1}{2}\frac{t_0^2t_1^2 - t_1^2}{(1+t_0^2)(1+t_1^2)}, \qquad g_2 = \frac{1}{2}\frac{1-t_0^2}{(1+t_0^2)(1+t_1^2)}, \qquad g_1 = -2\frac{t_0t_1}{(1+t_0^2)(1+t_1^2)},$$

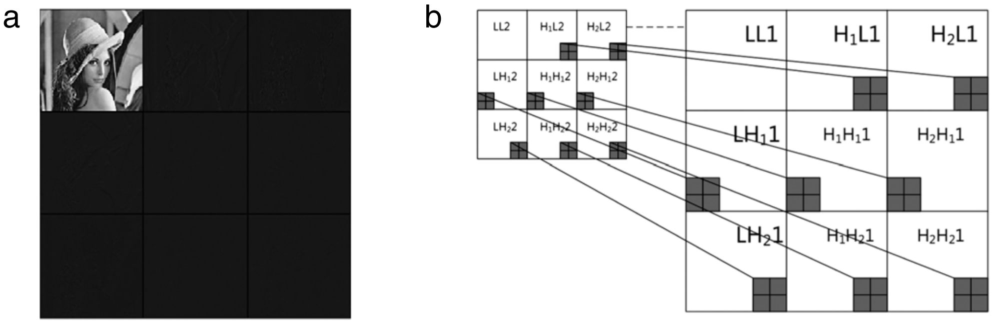


Fig. 1. (a) Wavelet frame transform of a image. (b) Two-dimensional two-scale WFT. The tree structure is shown by the link of the solid lines. The WFT of *LL*1 is shown by the link of the dashed line.

$$g_0 = g_1, \qquad g_{-1} = g_2, \qquad g_{-2} = g_3,$$
$$f_3 = -\frac{1}{2}\frac{t_1^2}{1+t_1^2}, \qquad f_2 = \frac{1}{2}(1+t_1^2), \qquad f_{-1} = -f_2, \qquad f_{-2} = -f_3, \qquad f_1 = f_0 = 0.$$

Where t_0, t_1 are arbitrary real numbers.

4. Research on applications of tight wavelet frames

The redundant representation offered by wavelet frames has already been put to good use in high-resolution image reconstruction [1], image inpainting [2], image analysis and synthesis [3] and is currently explored for image compression and denoising. In this section, we will explore the power of redundancy and strong robustness of wavelet frames for image compression and image denoising. The positive effect has been discovered in our research.

4.1. Application of tight wavelet frames (I): image denoising based on cross-local contextual hidden Markov model of wavelet frame domain

In this subsection, we investigate into the power of redundancy of tight wavelet frame transformations for image denoising. We propose the Cross-Local Contextual Markov Model based on this transform, and study the corresponding algorithms. In order to testify to the effectiveness of the model, we apply the model to image denoising, and subsequently prove its effect in experiments.

Let the image be $\{f_{i,j},\ i, j = 1, 2, \ldots, N\}$, where N is some integer power of 2. The two-dimensional wavelet frame transform (WFT) represents the image with both the spatial and frequency characteristics as shown in Fig. 1(a). Suppose that the low-pass filter of WFT is L, and high-pass filters of WFT are H_1, H_2; it is usual to label the subbands of the wavelet frame transform as in Fig. 1(b).

If a set of wavelet frame coefficients $w_{k,i,j,b}$ from the J-scale WFT of a image of $N \times N$ is given, there are $N_j \times N_j$ $\left(N_j = \frac{N}{2^j}\right)$ coefficients in the jth scale and bth subband, where $k, i = 1, 2, \cdot, N_j, j = 1, 2, \cdot, J$ and $b \in \{H_1L, H_2L, LH_1, H_1H_1, H_2H_1, LH_2, H_1H_2, H_2H_2\}$.

Gaussian mixture models can closely approximate the distribution of wavelet frame coefficients $w_{k,i,j,b}$ for most real-world images. We associate each wavelet frame coefficient $w_{k,i,j,b}$ with a hidden state variable $S_{k,i,j,b} \in \{0, 1\}$. The state 0 corresponds to a zero-mean, low-variance Gaussian, capturing the peakiness around zero. The other state 1 corresponds to a zero-mean, high-variance Gaussian, capturing the heavy tails. Thus the probability density function of $w_{k,i,j,b}$ is determined by

$$f(w_{k,i,j,b}) = p_{S_{k,i,j,b}}(0)f(w_{k,i,j,b}|S_{k,i,j,b} = 0) + p_{S_{k,i,j,b}}(1)f(w_{k,i,j,b}|S_{k,i,j,b} = 1) \tag{4.1}$$

where

$$f(w_{k,i,j,b}|S_{k,i,j,b} = m) = \frac{1}{\sqrt{2\pi\sigma^2_{(k,i,j,b),m}}} \exp\frac{w^2_{k,i,j,b}}{\sigma^2_{(k,i,j,b),m}} = g(w_{k,i,j,b}; 0, \sigma^2_{(k,i,j,b),m}) \quad (m = 0, 1)$$

$p_{S_{k,i,j,b}}(m)$ is the state value probability mass function for $S_{k,i,j,b} = m$. $p_{S_{k,i,j,b}}(0) = 1 - p_{S_{k,i,j,b}}(1)$ and $\sigma^2_{(k,i,j,b),0} < \sigma^2_{(k,i,j,b),0}$. $S_{k,i,j,b} = 0$ and $S_{k,i,j,b} = 1$ express two kinds of different states: $f(w_{k,i,j,b}|S_{k,i,j,b} = 0)$ expresses the condition density function of $w_{k,i,j,b}$ in the condition of state 0, and $f(w_{k,i,j,b}|S_{k,i,j,b} = 1)$ expresses the condition density function of $w_{k,i,j,b}$ in the condition of state 1.

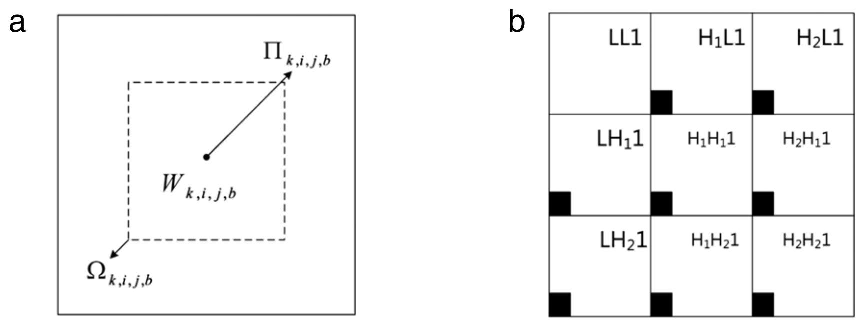


Fig. 2. (a) The LGMM where $w_{k,i,j,b}$ is associated with $\prod_{k,i,j,b}$. (b) A set in the CLCHMM. The black node denotes the random variable W in the same scale and orientation.

We know that the relative magnitude of a wavelet frame coefficient is closely related to the magnitude of its neighborhood. Thus, we propose the Local Gaussian Mixture Model (LGMM). The LGMM assumes that each wavelet frame coefficients $w_{k,i,j,b}$ follows a local Gaussian Mixture distribution, and a LGMM parameterized by $\prod_{k,i,j,b} = \{p_{S_{k,i,j,b}}(m), \sigma^2_{(k,i,j,b),m} | m = 0, 1\}$. $\prod_{k,i,j,b}$ can be estimated by the neighborhood of $w_{k,i,j,b}$, $\Omega_{k,i,j,b}$ which is selected by a square window of $2C_j + 1$ and centered at $w_{k,i,j,b}$ as shown in Fig. 6(a), i.e., $\Omega_{k,i,j,b} = \{w_{x,y,j,b} | x = k - C_j, \cdot, k + C_j; y = i - C_j, \cdot, i + C_j\}$.

The dependencies across subbands are useful for frame domain characterization. To capture the dependency, we propose to group frame coefficients at the same location and scale into a set $w_{k,i,j} = \{w_{k,i,j,b} | b \in B\}$, as shown in Fig. 2(b), where $B = \{H_1L, H_2L, LH_1, H_1H_1, H_2H_1, LH_2, H_1H_2, H_2H_2\}$. We define the random variable of $w_{k,i,j,b}$ by $w_{k,i,j}$, whose value is $w_{k,i,j} = 1$, if $w^2_{k,i,j,b} > \sigma^2_{k,i,j,b}$, or $w_{k,i,j} = 0$ if $w^2_{k,i,j,b} \le \sigma^2_{k,i,j,b}$, where $\sigma^2_{k,i,j,b}$ is the average energy of frame coefficients of the set $w_{k,i,j}$. By conditioning (4.1) on $w_{k,i,j}$ and using the LGMM, we propose the Cross-Local Contextual Hidden Markov model (CLHMM) based on frame domain for $w_{k,i,j,b}$ as

$$f(w_{k,i,j,b} | w_{k,i,j} = w) = \sum_{m=0}^{1} p_{S_{k,i,j,b}|w_{k,i,j}}(m | w_{k,i,j} = w) g(w_{k,i,j,b}; 0, \sigma^2_{(k,i,j,b),m}) \tag{4.2}$$

where

$$p_{S_{k,i,j,b}|w_{k,i,j}}(m | w_{k,i,j} = w) = \frac{p_{S_{k,i,j,b}}(m) p_{w_{k,i,j}|S_{k,i,j,b}(w|m)}}{\sum_{m=0}^{1} p_{S_{k,i,j,b}}(m) p_{V_{k,i,j,b}|S_{k,i,j,b}(w|m)}}.$$

A two-state, zero mean CLCHMM based on the wavelet frames domain is shown as follows

$$\begin{aligned} &\theta_{k,i,j,b} = \{p_{S_{k,i,j,b}}(m), p_{w_{k,i,j}|S_{k,i,j,b}(w|m)}, \sigma^2_{(k,i,j,b),m} | w, m = 0, 1\} \\ &j = 1, \ldots, J \quad \text{and} \quad k, i = 1, 2, \ldots, N_j \\ &b \in \{H_1L, H_2L, LH_1, H_1H_1, H_2H_1, LH_2, H_1H_2, H_2H_2\}. \end{aligned} \tag{4.3}$$

The parameters of CLCHMM based on the wavelet frame domain can be got by EM algorithm [29–31].

The problem of suppressing noise in digital images is based on the model

$$g = f + \varepsilon \tag{4.4}$$

where f denotes the true noise-free pixel values, g the observed noisy pixels, and ε the noise. And ε are independent and identically distributed (i.i.d.) as normal $N(0, \sigma^2)$ and independent of f. We wish to estimate the noise-free image f. Translated into the frame domain, the problem is as follows:

$$\text{Given } G = F + \acute{\varepsilon}, \text{ estimate } F \tag{4.5}$$

where $\acute{\varepsilon}$ are also i.i.d.$N(0, \sigma^2)$.

If F is a mixture of zero-mean Gaussian, then G is also a mixture of zero-mean Gaussian—the addition of zero-mean independent Gaussian noise $\acute{\varepsilon}$ increases the variance of each mixture component by σ^2, but leaves the other parameters unaffected. Hence, if the parameters on the noisy image G are $\{p_{S_{k,i,j,b}}(m), p_{w_{k,i,j,b}|S_{k,i,j,b}(w|m)}, \gamma^2_{(k,i,j,b),m} | w, m = 0, 1\}$, then the parameters on the original image F are $\{p_{S_{k,i,j,b}}(m), p_{w_{k,i,j,b}|S_{k,i,j,b}(w|m)}, \sigma^2_{(k,i,j,b),m} | w, m = 0, 1\}$, where

$$\sigma^2_{(k,i,j,b),m} = \gamma^2_{(k,i,j,b),m} - \sigma^2_n. \tag{4.6}$$

Table 1
The PSNR of the noisy image of Lena and the denoised images with different denoising methods and different noise levels.

σ	Noise image	Wiener 2	Visu Shrink	Sure Shrink	Bayes Shrink	Adaptive Bayes Shrink	HMT	Denoising Algorithm 4.1
10	28.12	32.67	30.34	33.34	33.32	33.18	33.84	33.50
15	24.60	31.26	28.52	31.40	31.41	32.39	31.76	31.73
20	22.11	30.00	27.24	30.09	30.17	31.07	30.39	30.47
25	20.15	28.86	26.34	29.12	29.22	30.70	29.24	29.43
30	18.60	27.82	26.26	28.34	28.48	28.81	28.35	28.58

Table 2
The PSNR of the noisy image of Barbara and the denoised images with different denoising methods and different noise levels.

σ	Noise image	Wiener 2	Visu Shrink	SURE Shrink	Bayes Shrink	Adapt Bayes Shrink	HMT	Denoising Algorithm 4.1
10	28.14	28.02	27.29	31.90	30.86	31.37	31.36	32.08
15	24.62	27.12	25.01	29.52	28.51	29.96	29.23	30.04
20	22.12	26.24	23.65	27.86	27.13	28.36	27.80	28.57
25	20.15	25.43	22.83	26.67	26.01	27.23	25.99	27.54
30	18.60	24.70	22.26	25.67	25.16	25.55	25.11	26.67

Table 3
Image denoising results by denoising Algorithm 4.2 for some test images with additive white Gaussian noise of $\sigma = 20$.

w	Barbara	Lena	Woman	Pepper	Bridge	Boat
0	28.57	30.46	33.03	30.38	26.26	28.87
0.3	28.63	30.51	33.06	30.44	26.33	28.95
0.6	28.70	30.56	33.10	30.52	26.38	29.03
0.8	28.72	30.64	33.21	30.60	26.30	29.05
0.9	28.71	30.71	33.34	30.68	26.17	29.06

If the parameters of CLCHMM are given, we can estimate $F_{k,i,j,b}$ as the conditional mean as

$$\hat{w}(F[k,i,j,b]) = \sum_{m=0}^{1} p_{s_{k,i,j,b}|w_{k,i,j,b}}(m|w_{k,i,j,b} = w)\frac{\sigma^2_{(k,i,j,b),m}}{\sigma^2_{(k,i,j,b),m} + \sigma_n^2} G_{k,i,j,b} \tag{4.7}$$

$b \in \{H_1L, H_2L, LH_1, H_1H_1, H_2H_1, LH_2, H_1H_2, H_2H_2\}$.

The algorithm can be summarized as follow:

Algorithm 4.1 (*Denoising Algorithm*). **Step 1**. 2-dimension tight wavelet frames decompose the observation image g up to level J.

Step 2. The parameters θ of CLCHMM based on frame domain is calculated by EM algorithm.

Step 3. For each subband (except the lowpass residual), estimate F using (4.7).

Step 4. Invert tight wavelet frames to obtain the denoised image $\hat{f}$ from the processed subbands and the lowpass LLJ residual.

We can improve Algorithm 4.1 as follows: Let w be a weight between 0 and 1. The linear combination $(1-w)f + w\hat{f}$ will be considered as a new noisy image. Using Algorithm 4.1, we obtain a new denoised image. We vary the weight w and use Algorithm 4.1 iteratively as follows:

Algorithm 4.2. Weight denoising algorithm

Step 1. Set the weight vector $w = [0, 0.3, 0.6, 0.8, 0.9]$.

Step 2. For $k = 1, \ldots, \text{length}(w)$ do: Replace f by $(1-w)f + w\hat{f}$. Apply Algorithm 4.1 to f to obtain $\hat{f}$.

Step 3. Output $\hat{f}$.

We perform our experiments on the well-known images Lena and Barbara. The noisy images with different noise levels are generated by adding Gaussian white noise to the original noise-free images. For comparison, we implement the famous denoising methods: Visu Shrink, Sure Shrink, Bayes Shrink, Adaptive Bayes Shrink, HMT (based on wavelet transform) and Wiener 2 [32]. The experimental results in PSNR are shown in Tables 1–3. The results indicate that our algorithms have a better PSNR. The experimental results prove that the presented CLCHMM can effectively characterize the intrascale and cross-orientation correlations of the coefficients in wavelet frames domain and have a positive effect on image denoising. Figs. 3 and 4 are presented for visual inspection of these results. The original noise-free image, the noisy image, the denoised image with Algorithm 4.1 for Lena and Barbara are shown from the left of the figure to the right, respectively. From the figures we can see that Algorithm 4.1 retains the edges and detail of the images.

Fig. 3. (a) Original Lena image. (b) Noisy image with $\sigma = 20$. (c) Denoising image.

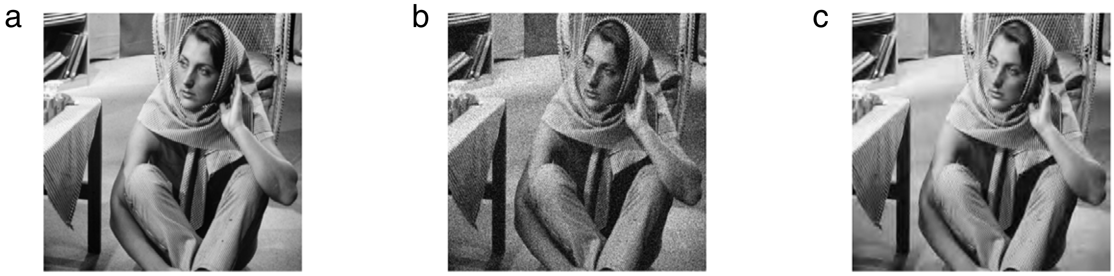
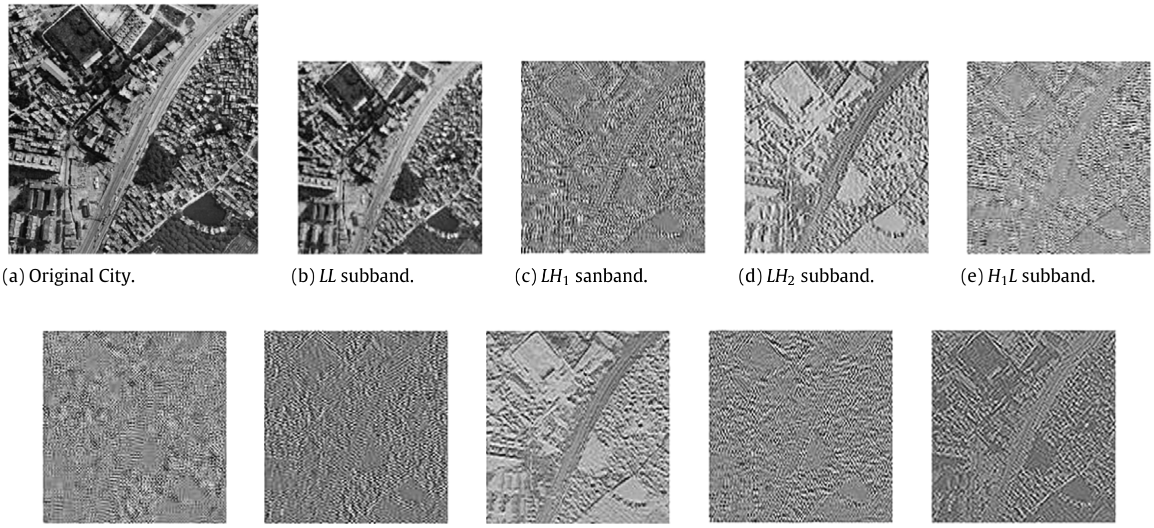


Fig. 4. (a) Original Barbara image, (b) Noisy image with $\sigma = 20$, (c) Denoising image.



(a) Original City. (b) LL subband. (c) LH_1 sanband. (d) LH_2 subband. (e) H_1L subband.

(f) H_1H_1 subband. (g) H_1H_2 subband. (h) H_2L subband. (i) H_2H_1 subband. (j) H_2H_2 subband.

Fig. 5. Decomposition of image of the City.

4.2. Application of tight wavelet frames (II): image compression

In this subsection, the devised transforms are applied to achieve compression for still images including a remote sensing image at a low rate. The optimization model of FIR filters aiming at the characteristics of a remote sensing image is presented, along with some experimental results.

The image City is shown as follows. From Fig. 5, we can observe that the major information of the image is kept in low-frequency subband (LL), while the texture information of the image is kept in high-frequency subbands, respectively, $(LH_1, LH_2, H_1L, H_1H_1, H_1H_2, H_2L, H_2H_1, H_2H_2)$. Since wavelet frames possess 2 or more mother functions, they bear relatively high flexibility and a greater capability to handle high-frequency information.

For image compression, the masks of tight wavelet frames should be characterized in the following two aspects: on one hand, in the frequency area, the energy of the entire image should center in the low-frequency subband; on the other hand, the energy of the high-frequency subbands should center in a small portion of the coefficients. The energy in frequency area

(a) Conclusion 4.1. (b) Conclusion 4.2. (c) Conclusion 4.3. (d) Conclusion 4.4.

Fig. 6. Reconstruction of the image City based on the wavelet frames transform (Bpp = 0.5).

can be presented by the sum of the square power of all subbands of tight wavelet frames coefficients. We present an optimal model of tight wavelet frames aiming at image compression.

Model 4.1 (*The Model of the Optimal Filters*). An objective function is introduced as follows:

$$f(t_0, t_1, \dots, t_n) = \frac{\sum_{j,k,m}\left(\sum_{i=1}^{7}(D^i_{j,k,m})^r\right)}{\sum_{j,k,m}\left((C_{j,k,m})^r + \sum_{i=1}^{7}(D^i_{j,k,m})^r\right)}, \quad r \geq 2 \tag{4.8}$$

$$\min f(t_0, t_1, \dots, t_n) \tag{4.9}$$

and constraint conduction

$$h(0) = 1, \qquad h^{(j)}(-1) = 0, \quad 0 \leq j \leq J, \tag{4.10}$$

$$g^{(i)}(1) = 0, \qquad f^{(i)}(1) = 0, \quad 0 \leq i \leq K \tag{4.11}$$

where $C_{j,k,m}$ are coefficients of low-frequency subband (LL), $D^i_{j,k,m}$, $i = 1, 2, \dots, 8$ are coefficients of high-frequency subbands ($LH_1, LH_2, H_1L, H_1H_1, H_1H_2, H_2L, H_2H_1, H_2H_2$), given the constraint condition that if mask h of scale function has the sum rules of J, and if masks g, h of mother functions have the vanishing moments of K. If the objective function can reach its minimum value, the preponderant amount of energy is concentrated in the low-frequency part.

Model 4.1 is constrained on nonlinear optimization problem. The parameters of masks of tight wavelet frames in this paper provide high degrees of freedom to optimize the masks with respect to Model 4.1. We will present algorithm of Model 4.1.

Algorithm 4.3 (*Algorithm of* Model 4.1).

Step 4.1. Utilizing the filters obtained from Theorems 3.2, 3.5 and 3.6, a coefficients matrix can be got after tight wavelet frame transformations.

Step 4.2. Solving the constrained nonlinear optimization models (4.8), (4.10) and (4.11) by Sequential Quadratic Programming (SQP) [33], we get the value of parameterizations of filters.

Step 4.3. Utilize the filters obtained from the above step to compress wavelet frames coefficients through the bit-plane prediction coding method. Then, calculate the PSNR value of the reconstructed image.

Step 4.4. Set the group of solutions got by Step 2 as the original population. Then, the fitness function is defined by the PSNR of reconstruction of the image, we use the genetic algorithm [34], and get the optimal solutions of filters.

Note: In Algorithm 4.3, parameterizations of filters may be selected more than 1, thus it may be applicable in more than 2 dimensions.

The following results of experiments are acquired by Algorithm 4.3.

Conclusion 4.1. If mask h of scale function satisfies $h(1) = 1$ in Step 4.4 of Algorithm 4.3, an optimal filter among the filters in Example 3.1 can be acquired by Algorithm 4.3 as follows.

$$h_{-3} = 0.3012351745597467, \quad h_{-2} = 0.2510783697534620, \quad h_{-1} = -0.05231821603352702, \quad h_0 = 0,$$
$$g_{-3} = 0, \quad g_{-2} = 0, \quad g_{-1} = -0.2510783697534620, \quad g_0 = 0.4978339170524394,$$
$$f_{-3} = -0.3012351745597467, \quad f_{-2} = -0.2510783697534620, \quad f_{-1} = 0.05231821603352702, \quad f_0 = 0.$$

Table 4
Comparisons of the PSNR of compressed images.

Image	Bpp	Conclusion 4.1	Conclusion 4.2	Conclusion 4.3	Conclusion 4.4	Example 5.3	Example 5.4
	0.5	23.5660	26.8618	26.5886	**27.2626**	27.1421	26.0136
Lena	0.25	21.5470	**24.6486**	23.7462	23.8702	23.8367	24.4742
	0.125	19.6944	**23.0994**	16.5751	16.0117	15.9520	22.6960
	0.5	21.4846	23.1386	23.0504	23.3257	**23.3423**	23.1806
Barbara	0.25	19.5023	**21.9454**	21.0979	21.0585	20.0388	21.4397
	0.125	18.0856	**20.6072**	15.7232	14.9092	14.6454	20.3385
	0.5	18.9610	20.1095	21.2854	21.8824	**21.9495**	19.5639
City	0.25	17.2868	**18.8362**	18.1760	18.3710	18.3380	18.2775
	0.125	16.5887	**17.9895**	12.2047	12.1877	12.1498	17.3963

Conclusion 4.2. If mask h of scale function satisfies $h(1) = 1$ in Step 4.4 of Algorithm 4.3, an optimal filter among the filters in Example 3.2 can be acquired by Algorithm 4.3 as follows.

$$h_{-2} = -0.0075582613974531, \quad h_{-1} = 0.0080082255739456, \quad h_0 = 0.4995487948021224,$$
$$g_{-2} = 0.2426558270751309, \quad g_{-1} = -0.2571017986629554, \quad g_0 = 0.0155599822901763,$$
$$f_{-2} = -0.2427735111762980, \quad f_{-1} = 0.2572264888237020, \quad f_0 = 0.$$

Conclusion 4.3. An optimal filter among the filters in Example 3.1 can be acquired by Algorithm 4.3.

$$h_{-3} = 0.3028971268912868, \quad h_{-2} = -0.2477388684431995, \quad h_{-1} = -0.0506562637019869, \quad h_0 = 0,$$
$$g_{-3} = 0, \quad g_{-2} = 0, \quad g_{-1} = 0.2477388684431995, \quad g_0 = 0.5044817263785997,$$
$$f_{-3} = -0.3028971268912868, \quad f_{-2} = 0.2477388684431995, \quad f_{-1} = 0.0506562637019869, \quad f_0 = 0.$$

Conclusion 4.4. An optimal filter among the filters in Example 3.2 can be acquired by Algorithm 4.3.

$$h_{-2} = 0.1504810560179085, \quad h_{-1} = -0.3191739410721348, \quad h_0 = 0.1600876715921230,$$
$$g_{-2} = 0.05496105927950028, \quad g_{-1} = -0.1165737293447069, \quad g_0 = -0.4383132747518949,$$
$$f_{-2} = -0.1602038272245381, \quad f_{-1} = 0.3397961727754619, \quad f_0 = 0.$$

The Conclusions 4.1–4.4 and Examples 5.3 and 5.4 from [27] are applied to achieve compression. The method of coding is based on the bit-plane prediction algorithm. In Table 4, we provide the performance of Conclusions 4.1–4.4 and Examples 5.3 and 5.4 [27]. The best result of each transform has been blackened. From Table 4, we can see that Conclusion 4.2 has the six best results. In other words, the transforms proposed in this paper are comparable to Examples 5.3 and 5.4.

Fig. 6 are the corresponding reconstructed images. The compression effect of each conclusion can be seen visually. As can be seen from the reconstructed images, the presented transforms keep the pattern of texture of remote sensing images. However, there still exists an issue. In this paper, we use the coding method of wavelet frame coefficients the same as biorthogonal wavelet transforms. Based on both inter-band and intra-band correlation of wavelet frame coefficients, we may study a highly efficient prediction model with bi-plane methods. We will continue working on this study in the future.

5. Conclusion and future research

In this paper, we discuss tight wavelet frames constructed via multiresolution analysis (MRA). The mask parameterizations for tight wavelet frames with two symmetric/antisymmetric generators are constructed, which are of arbitrary lengths and of arbitrary symmetric/antisymmetric centers. Additionally, we explore the applicability of newly designed tight wavelet frames to image compression and denoising. Experimental results show that the presented transforms have a better performance. The paper focuses only on the construction of masks, while the properties of the corresponding wavelet and scaling function (e.g., the smoothness and approximation ability etc.) are significant as well. We will give a more in-depth study in our future work.

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