Journal of Computational and Applied Mathematics 235 (2011) 2392–2404

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On the expected discounted penalty function for the compound Poisson risk model with delayed claims

Jie-hua Xie, Wei Zou *

Department of Science, NanChang Institute of Technology, NanChang 330099, PR China

ARTICLE INFO

Article history:
Received 10 April 2010
Received in revised form 18 August 2010

Keywords:
Compound Poisson risk model
Expected discounted penalty function
Delayed claim
Laplace transform
Defective renewal equation

ABSTRACT

In this paper, we consider an extension to the compound Poisson risk model for which the occurrence of the claim may be delayed. Two kinds of dependent claims, main claims and by-claims, are defined, where every by-claim is induced by the main claim and may be delayed with a certain probability. Both the expected discounted penalty functions with zero initial surplus and the Laplace transforms of the expected discounted penalty functions are obtained from an integro-differential equations system. We prove that the expected discounted penalty function satisfies a defective renewal equation. An exact representation for the solution of this equation is derived through an associated compound geometric distribution, and an analytic expression for this quantity is given for when the claim amounts from both classes are exponentially distributed. Moreover, the closed form expressions for the ruin probability and the distribution function of the surplus before ruin are obtained. We prove that the ruin probability for this risk model decreases as the probability of the delay of by-claims increases. Finally, numerical results are also provided to illustrate the applicability of our main result and the impact of the delay of by-claims on the expected discounted penalty functions.

1. Introduction

It is well known that the classical risk model, in which claims occur according to a Poisson process, has been extensively analyzed; see Bowers et al. [1, Chapter 13], [2,3] and references therein. Ruin probabilities and many ruin related quantities such as the marginal and the joint defective distributions of the time to ruin, the deficit at ruin and the surplus prior to ruin have been analytically studied.

A unified approach to studying together the above fundamental risk quantities in just one function has been proposed in the seminal paper [4], by introducing the expected discounted penalty function for the classical risk model. Soon after, much of the literature on the expected discounted penalty function for the compound Poisson risk model was extensively developed, for instance in [5–7].

In reality, insurance claims may be delayed for various reasons. Since the work in [8], risk models with this special feature have been discussed by many authors in the literature. For example, Yuen and Guo [9] studied a compound binomial model with delayed claims and obtained recursive formulas for the finite time ruin probabilities. Xiao and Guo [10] obtained the recursive formula for the joint distribution of the surplus immediately prior to ruin and the deficit at ruin in this model. Xie and Zou [11] studied an extension to the risk model proposed in Yuen and Guo [9]. Xie and Zou [12] also studied the expected present value of total dividends in a risk model with delayed claims under stochastic interest rates.

All risk models described in the paragraph above were discrete time risk models. Motivated by these papers, we explore analogous problems, but in the compound Poisson risk model with delayed claims. In our risk model, two kinds of dependent

* Corresponding address: Department of Science, NanChang Institute of Technology, TianXiang Road No. 289, NanChang, PR China.
E-mail address: zouwei@nit.edu.cn (W. Zou).

0377-0427/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2010.10.039

claims, main claims and by-claims, are defined, where every by-claim is induced by the main claim and may be delayed with a certain probability. This kind of specific dependent risk model may be of practical use: for instance, a serious motor accident causes different kinds of claims, such as ones for car damage, injury, and death; some can be dealt with immediately while others need a period of time to be settled. We study the expected discounted penalty function for this risk model and obtain many ruin related quantities through the expected discounted penalty function.

The model proposed in this paper is a generalization of the compound Poisson risk model. Hence our results in this paper include the corresponding results for the compound Poisson risk model obtained in [4]. The work of this paper can also be seen as a complement to the works of Yuen and Guo [9] and Xiao and Guo [10].

It is obvious that the incorporation of the delayed claims makes the problem more interesting. It also complicates the derivation of the expected discounted penalty function. Our aim is to give an exact representation for the expected discounted penalty function in the risk model with delayed claims. The paper is structured as follows. A brief description of the delayed claims risk model is considered in Section 2. In Section 3, we derive an integro-differential equations system for the expected discounted penalty function. Both the expected discounted penalty functions with zero initial surplus and the Laplace transforms of the expected discounted penalty functions are obtained in Section 4. Then the defective renewal equation for the expected discounted penalty function is obtained and an exact representation for the solution of this equation is derived through an associated compound geometric distribution in Section 5. The explicit results for the expected discounted penalty functions with positive initial surplus are given when the claim amounts from both classes are exponentially distributed in Section 6. Moreover, the closed form expressions for ruin probability and the distribution function of the surplus before ruin are obtained. We also prove that the ruin probability for this risk model decreases as the probability of the delay of by-claims increases in this section. Finally, in Section 7, numerical results are also provided to illustrate the applicability of our main result and the impact of the delay of by-claims on the expected discounted penalty functions.

2. Model description and notation

Here, we consider a continuous time model which involves two kinds of insurance claims, namely the main claims and the by-claims. Let the aggregate main claims process be a compound Poisson process and $\{N(t);\, t \geq 0\}$ be the corresponding Poisson claim number process, with intensity λ. Its jump times are denoted by $\{T_i\}_{i\geq 1}$ with $T_0 = 0$. The main claim amounts $\{Y_i\}_{i\geq 1}$ are assumed to be independent and identically distributed (i.i.d.) positive random variables with common distribution F. Let $\{X_i\}_{i\geq 1}$ be the by-claim amounts, assumed to be i.i.d. positive random variables with common distribution G. The main claim amounts and by-claim amounts are independent and their means are denoted by μ_F and μ_G, respectively.

In this risk model, we assume the claim occurrence process to be of the following type: there will be a main claim Y_i in every epoch T_i of the Poisson process and the main claim Y_i will induce a by-claim X_i. Moreover, the by-claim X_i and its associated main claim Y_i may occur simultaneously with probability θ, or the occurrence of the by-claim X_i may be delayed to T_{i+1} with probability $1-\theta$. If the occurrence of the by-claim X_i is delayed to T_{i+1}, we assume that the occurrence of the delayed by-claim X_i is independent of the occurrence of next main claim Y_{i+1}. When $\theta = 1$, that means that the main claim and its associated by-claim occur simultaneously in every epoch. Actually, this case is very similar to the classic compound Poisson risk model. In our set-up, there is a by-claim, X, occurring simultaneously with the main claim Y. Hence, the only difference is that we use $Y + X$ as our claim amount random variable while the compound Poisson risk model simply considers Y.

In this set-up, the surplus process $U(t)$ of this risk model is defined as

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i - R(t), \tag{2.1}$$

where u is the initial capital, c the constant rate of the premium, and $R(t)$ is the sum of all by-claims X_i that occurred before time t.

Now, we consider the number of claims that occurred before time t. From the definition of the aggregate main claims process, the number of main claims that occurred before time t is $N(t)$. The last main claim that occurred before time t is $Y_{N(t)}$. The main claim $Y_{N(t)}$ will induce a by-claim $X_{N(t)}$. If by-claim $X_{N(t)}$ and its associated main claim $Y_{N(t)}$ occur simultaneously, the number of by-claims that occur before time t is also $N(t)$. The probability of this event is θ. If the occurrence of by-claim $X_{N(t)}$ is delayed, the number of by-claims that occur before time t is $N(t) - 1$. The probability of this event is $1-\theta$. From these discussions, it follows that

$$\begin{aligned}
\mathrm{E}\left[\sum_{i=1}^{N(t)} Y_i + R(t)\right] &= \mathrm{E}\left[\sum_{i=1}^{N(t)} Y_i\right] + \mathrm{E}[R(t)] = \lambda t\mu_F + \theta\lambda t\mu_G + (1-\theta)e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}(n-1)\mu_G \\
&= \lambda t\mu_F + \theta\lambda t\mu_G + (1-\theta)e^{-\lambda t}\left(\lambda t e^{\lambda t} - \sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\right)\mu_G \\
&= \lambda t\mu_F + \theta\lambda t\mu_G + (1-\theta)e^{-\lambda t}(\lambda t e^{\lambda t} - (e^{\lambda t} - 1))\mu_G \\
&= \lambda t\mu_F + \lambda t\mu_G - (1-\theta)\mu_G(1 - e^{-\lambda t}).
\end{aligned}$$

Thus in order to guarantee the positivity of the security loading, we assume that

$$\lambda(\mu_F + \mu_G) < c. \tag{2.2}$$

We define the time of ruin by $T = \inf\{t \geq 0 : U(t) < 0\}$ ($T = \infty$ if the set is empty). Let $|U(T)|$ and $U(T-)$ be the deficit at ruin and the surplus immediately before ruin, respectively. The expected discounted penalty function $\Phi(u)$ is defined as

$$\Phi(u) = E[e^{-\delta T} w(U(T-), |U(T)|) I(T < \infty) | U(0) = u], \quad u \geq 0, \tag{2.3}$$

where $I(A)$ is the indicator function of a set A, $w(x_1, x_2)$, $0 \leq x_1, x_2 < \infty$, is the penalty function, and δ is a non-negative-valued parameter. We remark that choosing different forms of the penalty function $w(x_1, x_2)$ in Eq. (2.3) gives rise to different information relating to the deficit at ruin and the surplus before ruin. A special case of the expected discounted penalty function (with $\delta = 0$ and $w(x_1, x_2) = 1$) is the well-known ultimate ruin probability $\phi(u) = P(T < \infty | U(0) = u),\ u \geq 0$. The financial explanations of $w(x_1, x_2)$ can be found in [4].

3. The system of integro-differential equations

In order to derive the system of integro-differential equations for the expected discounted penalty functions, we need to consider an auxiliary risk model. With all else being the same, we consider a slight change in the risk model. Instead of having one main claim and a by-claim with probability θ in the first epoch T_1, another by-claim is added in the first epoch. We denote the corresponding expected discounted penalty function for this auxiliary model by $\Phi_1(u)$ which is very useful in the derivation of $\Phi(u)$.

We are interested in the expected discounted penalty function $\Phi(u)$. Consider what will happen in the first epoch T_1. Obviously there will be a main claim Y_1. The main claim Y_1 will induce a by-claim X_1. If the by-claim X_1 also occurs in the first epoch T_1, the surplus process $U(t)$ will renew itself with a different initial reserve. The probability of this event is θ. If the occurrence of the by-claim X_1 is delayed to T_2, $U(t)$ will not renew itself in this case but will transfer to the auxiliary model described in the paragraph above. The probability of this event is $1 - \theta$. Remember that the expected discounted penalty function for the auxiliary model is $\Phi_1(u)$. Taking what happened at T_1 into account, we can set up the following equation for $\Phi(u)$ and $\Phi_1(u)$:

$$\begin{aligned}\Phi(u) &= \theta \int_0^\infty \lambda e^{-(\lambda+\delta)t} \left(\int_0^{u+ct} \Phi(u+ct-y) \mathrm{d}F * G(y) + \int_{u+ct}^\infty w(u+ct, y-u-ct) \mathrm{d}F * G(y) \right) \mathrm{d}t \\ &\quad + (1-\theta) \int_0^\infty \lambda e^{-(\lambda+\delta)t} \left(\int_0^{u+ct} \Phi_1(u+ct-y) \mathrm{d}F(y) + \int_{u+ct}^\infty w(u+ct, y-u-ct) \mathrm{d}F(y) \right) \mathrm{d}t,\end{aligned} \tag{3.1}$$

where $*$ denotes the distribution functions convolution. With the auxiliary model, similar analysis gives

$$\begin{aligned}\Phi_1(u) &= \theta \int_0^\infty \lambda e^{-(\lambda+\delta)t} \left(\int_0^{u+ct} \Phi(u+ct-y) \mathrm{d}F * G * G(y) + \int_{u+ct}^\infty w(u+ct, y-u-ct) \mathrm{d}F * G * G(y) \right) \mathrm{d}t \\ &\quad + (1-\theta) \int_0^\infty \lambda e^{-(\lambda+\delta)t} \left(\int_0^{u+ct} \Phi_1(u+ct-y) \mathrm{d}F * G(y) + \int_{u+ct}^\infty w(u+ct, y-u-ct) \mathrm{d}F * G(y) \right) \mathrm{d}t.\end{aligned} \tag{3.2}$$

Setting $s = u + ct$ in (3.1), (3.2) and differentiating with respect to u, we get the following system of integro-differential equations:

$$c\Phi'(u) = (\lambda+\delta)\Phi(u) - \lambda\theta \left(\int_0^u \Phi(u-y) \mathrm{d}F * G(y) + w_2(u) \right) - \lambda(1-\theta) \left(\int_0^u \Phi_1(u-y) \mathrm{d}F(y) + w_1(u) \right), \tag{3.3}$$

$$\begin{aligned}c\Phi_1'(u) &= (\lambda+\delta)\Phi_1(u) - \lambda\theta \left(\int_0^u \Phi(u-y) \mathrm{d}F * G * G(y) + w_3(u) \right) \\ &\quad - \lambda(1-\theta) \left(\int_0^u \Phi_1(u-y) \mathrm{d}F * G(y) + w_2(u) \right)\end{aligned} \tag{3.4}$$

where $w_1(u) = \int_u^\infty w(u, y-u) \mathrm{d}F(y)$, $w_2(u) = \int_u^\infty w(u, y-u) \mathrm{d}F * G(y)$, and $w_3(u) = \int_u^\infty w(u, y-u) \mathrm{d}F * G * G(y)$.

4. The Laplace transform

Henceforth, we focus our interest on the expected discounted penalty functions $\Phi(u)$ and $\Phi_1(u)$. Their Laplace transforms can be derived as follows.

As in [13], we define an operator Γ_r of a real-valued function f, with respect to a complex number r, to be

$$\Gamma_r f(x) = \int_x^\infty e^{-r(y-x)} f(y) \mathrm{d}y, \quad x \geq 0.$$

It is clear that the Laplace transform of f, $\tilde{f}(s)$, can be expressed as $\Gamma_s f(0)$, and that for distinct r_1 and r_2,

$$\Gamma_{r_1}\Gamma_{r_2}f(x) = \Gamma_{r_2}\Gamma_{r_1}f(x) = \frac{\Gamma_{r_1}f(x) - \Gamma_{r_2}f(x)}{r_2 - r_1}, \quad x \geq 0.$$

If $r_1 = r_2 = r$,

$$\Gamma_{r_1}\Gamma_{r_2}f(x) = \int_x^{\infty} (y-x)e^{-r(y-x)}f(y)\mathrm{d}y, \quad x \geq 0.$$

The properties for this operator can be found in [13–16].

For **Re** $s \geq 0$, we define

$$\tilde{b}_F(s) = \int_0^{\infty} \exp(-sy)\mathrm{d}F(y), \qquad \tilde{b}_G(s) = \int_0^{\infty} \exp(-sy)\mathrm{d}G(y),$$

$$\tilde{b}_{F*G}(s) = \int_0^{\infty} \exp(-sy)\mathrm{d}F*G(y), \qquad \tilde{b}_{F*G*G}(s) = \int_0^{\infty} \exp(-sy)\mathrm{d}F*G*G(y).$$

Note that $\tilde{b}_{F*G}(s) = \tilde{b}_F(s)\cdot\tilde{b}_G(s)$ and $\tilde{b}_{F*G*G}(s) = \tilde{b}_{F*G}(s)\cdot\tilde{b}_G(s)$.

We also define the Laplace transforms of $\Phi(u)$ and $\Phi_1(u)$ as

$$\tilde{\Phi}(s) = \int_0^{\infty} \exp(-su)\Phi(u)\mathrm{d}u, \qquad \tilde{\Phi}_1(s) = \int_0^{\infty} \exp(-su)\Phi_1(u)\mathrm{d}u.$$

Define $\tilde{w}_i(s)$ to be the Laplace transforms of $w_i(u)$ for $i = 1, 2, 3$. Taking Laplace transforms of (3.3) and (3.4) and making some simplifications, we obtain

$$c(-\Phi(0) + s\tilde{\Phi}(s)) = (\lambda+\delta)\tilde{\Phi}(s) - \lambda\theta(\tilde{\Phi}(s)\tilde{b}_{F*G}(s) + \tilde{w}_2(s)) - \lambda(1-\theta)(\tilde{\Phi}_1(s)\tilde{b}_F(s) + \tilde{w}_1(s)),$$
$$c(-\Phi_1(0) + s\tilde{\Phi}_1(s)) = (\lambda+\delta)\tilde{\Phi}_1(s) - \lambda\theta(\tilde{\Phi}(s)\tilde{b}_{F*G*G}(s) + \tilde{w}_3(s)) - \lambda(1-\theta)(\tilde{\Phi}_1(s)\tilde{b}_{F*G}(s) + \tilde{w}_2(s)),$$

which can further be simplified to

$$\tilde{\Phi}(s) = \frac{(cs-\delta-\lambda+\lambda(1-\theta)\tilde{b}_{F*G}(s))(\tilde{w}(s) - c\Phi(0)) - \lambda(1-\theta)\tilde{b}_F(s)(\tilde{w}^*(s) - c\Phi_1(0))}{-(cs-\delta-\lambda)^2 - \lambda\tilde{b}_{F*G}(s)(cs-\delta-\lambda)}, \tag{4.1}$$

$$\tilde{\Phi}_1(s) = \frac{(cs-\delta-\lambda+\lambda\theta\tilde{b}_{F*G}(s))(\tilde{w}^*(s) - c\Phi_1(0)) - \lambda\theta\tilde{b}_{F*G*G}(s)(\tilde{w}(s) - c\Phi(0))}{-(cs-\delta-\lambda)^2 - \lambda\tilde{b}_{F*G}(s)(cs-\delta-\lambda)}, \tag{4.2}$$

where

$$\tilde{w}(s) = \lambda(\theta\tilde{w}_2(s) + (1-\theta)\tilde{w}_1(s)), \qquad \tilde{w}^*(s) = \lambda(\theta\tilde{w}_3(s) + (1-\theta)\tilde{w}_2(s)).$$

In order to obtain $\tilde{\Phi}(s)$ and $\tilde{\Phi}_1(s)$, for the further sake of deriving $\Phi(u)$ and $\Phi_1(u)$, we only need to find $\Phi(0)$ and $\Phi_1(0)$.

Note that the denominators on the right-hand side of (4.1) and (4.2) coincide. Now we discuss analytically the roots of the equation

$$(cs-\delta-\lambda)^2 + \lambda\tilde{b}_{F*G}(s)(cs-\delta-\lambda) = 0. \tag{4.3}$$

Proposition 4.1. *Let δ be strictly positive; then Eq. (4.3) has exactly two distinct positive real roots, say, $\rho_1(\delta)$, and $\rho_2(\delta) = (\lambda+\delta)/c$. Further, $\rho_1(\delta)$ and $\rho_2(\delta)$ are the only roots on the right half of the complex plane.*

Proof. Noting that Eq. (4.3) can be rewritten as $(cs-\delta-\lambda)^2 + \lambda\tilde{b}_{F*G}(s)(cs-\delta-\lambda) = (cs-\delta-\lambda)l(s) = 0$, where $l(s) = cs-\delta-\lambda+\lambda\tilde{b}_{F*G}(s)$, it is easy to check that $l(0) = -\delta < 0$ and $\lim_{s\to+\infty} l(s) = +\infty$. Also,

$$l'(s) = c + \lambda\tilde{b}'_{F*G}(s) > c - \lambda(\mu_F + \mu_G) > 0,$$

and then $l(s)$ is an increasing function of s. Hence, $l(s) = 0$ has exactly one positive real root, say, $\rho_1(\delta)$. Then $\rho_1(\delta)$ is also one positive real root of Eq. (4.3). Note that $(\lambda+\delta)/c$ is another positive real root of Eq. (4.3), say, $\rho_2(\delta)$. Moreover, it is easy to see that $\rho_1(\delta) \neq \rho_2(\delta)$. We conclude that Eq. (4.3) has exactly two distinct positive real roots, say, $\rho_1(\delta)$ and $\rho_2(\delta)$.

Now, we prove that $\rho_1(\delta)$ is the exactly one positive real root of equation $l(s) = 0$ on the right half of the complex plane. When s is on the half-circle: $|z| = r (r > 0)$ and **Re**$(z) \geq 0$ on the complex plane, $|cs-\lambda-\delta| > \lambda = \lambda\tilde{b}_{F*G}(0) > |\lambda\tilde{b}_{F*G}(s)|$ for r sufficiently large; while for s on the imaginary axis, **Re**$(z) = 0$, $|cs-\lambda-\delta| > \lambda \geq |\lambda\tilde{b}_{F*G}(s)|$. That is to say, on the boundary of the contour enclosed by the half-circle and the imaginary axis, $|cs-\lambda-\delta| > |\lambda\tilde{b}_{F*G}(s)|$. Then we conclude, by Rouché's theorem, that on the right half of the complex plane, the number of roots of the equation $l(s) = 0$ equals the number of roots of the equation $cs-\lambda-\delta = 0$. Furthermore, the latter has exactly one root on the right half of the complex plane. It follows that $l(s) = 0$ has exactly one positive real root, say, $\rho_1(\delta)$, on the right half of the complex plane. It is easy to see that $\rho_2(\delta) = (\lambda+\delta)/c$ is the exactly one positive real root of equation $cs-\lambda-\delta = 0$ on the right half of the complex plane.

It follows from all of the above that Eq. (4.3) has exactly two distinct positive real roots, say, $\rho_1(\delta)$ and $\rho_2(\delta)$, on the right half of the complex plane. This completes the proof. □

Note that $\rho_1(\delta) \to 0^+$ as $\delta \to 0^+$. In the rest of the paper, the $\rho_i(\delta)$ are simply denoted by ρ_i, for $i = 1, 2$ and $\delta > 0$. Since $\tilde{\Phi}(s)$ is finite for **Re** $s \geq 0$, its numerator is zero if $s = \rho_1$ and ρ_2, i.e.,

$$(c\rho_i - \delta - \lambda + \lambda(1-\theta)\tilde{b}_{F*G}(\rho_i))(\tilde{w}(\rho_i) - c\Phi(0)) - \lambda(1-\theta)\tilde{b}_F(\rho_i)(\tilde{w}^*(\rho_i) - c\Phi_1(0)) = 0, \quad i = 1, 2.$$

By solving this linear equation system for $\Phi(0)$ and $\Phi_1(0)$, we get

$$\Phi(0) = \frac{\tilde{b}_F(\rho_1)\{-\tilde{w}(\rho_2)g(\rho_2) + \lambda(-1+\theta)\tilde{b}_F(\rho_2)\tilde{w^*}(\rho_2)\} - \tilde{b}_F(\rho_2)\{-\tilde{w}(\rho_1)g(\rho_1) + \lambda(-1+\theta)\tilde{b}_F(\rho_1)\tilde{w^*}(\rho_1)\}}{c(\tilde{b}_F(\rho_2)g(\rho_1) - \tilde{b}_F(\rho_1)g(\rho_2))}, \quad (4.4)$$

$$\Phi_1(0) = \frac{\tilde{w^*}(\rho_2)}{c} + \frac{g(\rho_2)(\tilde{w}(\rho_2) - c\Phi(0))}{\lambda c(1-\theta)\tilde{b}_F(\rho_2)} \quad (4.5)$$

where $g(s) = \lambda + \delta - \lambda\tilde{b}_{F*G}(s) + \lambda\theta\tilde{b}_{F*G}(s) - cs$.

5. The defective renewal equation for the expected discounted penalty function

In this section, our goal is to show that the expected discounted penalty function also satisfies a defective renewal equation in the compound Poisson risk model with delayed claims. To identify the form of this defective renewal equation, we first analyse the Laplace transform of $\Phi(u)$.

After some calculations, (4.1) can be rewritten as

$$\tilde{\Phi}(s) = \frac{\tilde{f}_1(s) + \tilde{f}_2(s)}{-(\tilde{h}_1(s) - \tilde{h}_2(s))}, \quad (5.1)$$

where $\tilde{f}_1(s) = -c\Phi(0)(cs-\lambda-\delta)$, $\tilde{f}_2(s) = (cs-\lambda-\delta)\tilde{w}(s) + \lambda(1-\theta)(\tilde{b}_{F*G}(s)\tilde{w}(s) - \tilde{b}_F(s)\tilde{w^*}(s)) + \lambda c(1-\theta)(\Phi_1(0)\tilde{b}_F(s) - \Phi(0)\tilde{b}_{F*G}(s))$, $\tilde{h}_1(s) = (cs-\delta-\lambda)^2$, $\tilde{h}_2(s) = -\lambda\tilde{b}_{F*G}(s)(cs-\delta-\lambda)$. Define the functions $f_1(u)$, $f_2(u)$, $h_1(u)$ and $h_2(u)$ to be the inverse image functions of $\tilde{f}_1(s)$, $\tilde{f}_1(s)$, $\tilde{h}_1(s)$, and $\tilde{h}_2(s)$, i.e., $\Gamma_s f_1(0) = \tilde{f}_1(s)$, $\Gamma_s f_2(0) = \tilde{f}_2(s)$, $\Gamma_s h_1(0) = \tilde{h}_1(s)$, and $\Gamma_s h_2(0) = \tilde{h}_2(s)$. We use the Lagrange interpolating theorem to rewrite (5.1), which will eventually lead to the defective renewal function for the expected discounted penalty function.

Lemma 5.1. *The Laplace transform $\tilde{\Phi}(s)$ of the expected discounted penalty function satisfies*

$$\tilde{\Phi}(s) = \frac{\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}h_2(0)}{c^2}\tilde{\Phi}(s) - \frac{\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}f_2(0)}{c^2}. \quad (5.2)$$

Proof. With $\tilde{\Phi}(s)$ analytic for **Re** $s \geq 0$, the numerator of (5.1) is zero if $s = \rho_1$ and ρ_2. Therefore, it follows that $\tilde{f}_1(\rho_i) = -\tilde{f}_2(\rho_i)$ for $i = 1, 2$. It is easy to see that $\tilde{f}_1(s)$ is a polynomial of degree 1 in s. Using the Lagrange interpolating theorem, one deduces

$$\tilde{f}_1(s) = \tilde{f}_1(\rho_1)\left(\frac{s-\rho_2}{\rho_1-\rho_2}\right) + \tilde{f}_1(\rho_2)\left(\frac{s-\rho_1}{\rho_2-\rho_1}\right) = -\frac{\tilde{f}_2(\rho_1)(s-\rho_2) - \tilde{f}_2(\rho_2)(s-\rho_1)}{\rho_1-\rho_2},$$

which implies

$$\begin{aligned}\tilde{f}_1(s) + \tilde{f}_2(s) &= \frac{(s-\rho_2)(\tilde{f}_2(s) - \tilde{f}_2(\rho_1)) - (s-\rho_1)(\tilde{f}_2(s) - \tilde{f}_2(\rho_2))}{\rho_1-\rho_2} \\ &= (s-\rho_1)(s-\rho_2)\frac{\Gamma_s\Gamma_{\rho_2}f_2(0) - \Gamma_s\Gamma_{\rho_1}f_2(0)}{\rho_1-\rho_2} \\ &= (s-\rho_1)(s-\rho_2)\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}f_2(0). \end{aligned} \quad (5.3)$$

A similar procedure is used to find an alternative expression for the denominator $-(\tilde{h}_1(s) - \tilde{h}_1(s))$ of $\tilde{\Phi}(s)$. From Proposition 4.1, we know that $\tilde{h}_1(\rho_i) = \tilde{h}_2(\rho_i)$ for $i = 1, 2$. Also, it is easy to see that $\tilde{h}_1(s)$ is a polynomial of degree 2 in s. Using the Lagrange interpolating theorem, one knows that

$$\begin{aligned}\tilde{h}_1(s) &= \tilde{h}_1(0)\frac{(s-\rho_1)(s-\rho_2)}{\rho_1\rho_2} + s\left(\frac{\tilde{h}_1(\rho_1)}{\rho_1}\frac{s-\rho_2}{\rho_1-\rho_2} + \frac{\tilde{h}_1(\rho_2)}{\rho_2}\frac{s-\rho_1}{\rho_2-\rho_1}\right) \\ &= \tilde{h}_1(0)\frac{(s-\rho_1)(s-\rho_2)}{\rho_1\rho_2} + s\left(\frac{\tilde{h}_2(\rho_1)}{\rho_1}\frac{s-\rho_2}{\rho_1-\rho_2} + \frac{\tilde{h}_2(\rho_2)}{\rho_2}\frac{s-\rho_1}{\rho_2-\rho_1}\right)\end{aligned}$$

$$= \tilde{h}_1(0)\frac{(s-\rho_1)(s-\rho_2)}{\rho_1\rho_2} + (s-\rho_1)(s-\rho_2)\left(\frac{\tilde{h}_2(\rho_1)}{\rho_1}\frac{1}{\rho_1-\rho_2} + \frac{\tilde{h}_2(\rho_2)}{\rho_2}\frac{1}{\rho_2-\rho_1}\right)$$
$$+ \tilde{h}_2(\rho_1)\frac{s-\rho_2}{\rho_1-\rho_2} + \tilde{h}_2(\rho_2)\frac{s-\rho_1}{\rho_2-\rho_1}.$$

Therefore, using Property 6 of the Dickson–Hipp operator of Li and Garrido [4], $\tilde{h}_1(s) - \tilde{h}_2(s)$ becomes

$$\begin{aligned}
\tilde{h}_1(s) - \tilde{h}_2(s) &= \tilde{h}_1(0)\frac{(s-\rho_1)(s-\rho_2)}{\rho_1\rho_2} + (s-\rho_1)(s-\rho_2)\left(\frac{\tilde{h}_2(\rho_1)}{\rho_1(\rho_1-\rho_2)} + \frac{\tilde{h}_2(\rho_2)}{\rho_2(\rho_2-\rho_1)}\right) \\
&\quad - \left(\tilde{h}_2(s) - \tilde{h}_2(\rho_1)\frac{s-\rho_2}{\rho_1-\rho_2} - \tilde{h}_2(\rho_2)\frac{s-\rho_1}{\rho_2-\rho_1}\right) \\
&= (s-\rho_1)(s-\rho_2)\left(\mathit{\Gamma}_0\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}h_1(0) - \left(\frac{\tilde{h}_2(s)}{(s-\rho_1)(s-\rho_2)}\right.\right. \\
&\quad \left.\left. - \frac{\tilde{h}_2(\rho_1)}{(s-\rho_1)(\rho_1-\rho_2)} - \frac{\tilde{h}_2(\rho_2)}{(s-\rho_2)(\rho_2-\rho_1)}\right)\right) \\
&= (s-\rho_1)(s-\rho_2)(\mathit{\Gamma}_0\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}h_1(0) - \mathit{\Gamma}_s\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}h_2(0)).
\end{aligned} \tag{5.4}$$

It is easy to prove that $\mathit{\Gamma}_0\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}h_1(0) = c^2$ which implies that (5.4) becomes

$$\tilde{h}_1(s) - \tilde{h}_2(s) = (s-\rho_1)(s-\rho_2)(c^2 - \mathit{\Gamma}_s\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}h_2(0)). \tag{5.5}$$

Combining (5.3) and (5.5) with $\tilde{\Phi}(s) = \frac{\tilde{f}_1(s)+\tilde{f}_2(s)}{-(\tilde{h}_1(s)-\tilde{h}_2(s))}$, one deduces $\tilde{\Phi}(s) = -\frac{\mathit{\Gamma}_s\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}f_2(0)}{c^2-\mathit{\Gamma}_s\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}h_2(0)}$ which leads to (5.2). This completes the proof. □

Using Lemma 5.1, we are now in a position to derive the defective renewal equation for $\Phi(u)$.

Theorem 5.1. *$\Phi(u)$ satisfies the following defective renewal equation:*

$$\Phi(u) = \kappa_\delta \int_0^u \Phi(u-y)\zeta(y)\mathrm{d}y + \vartheta(u), \tag{5.6}$$

where

$$\kappa_\delta = \frac{\lambda}{c}\mathit{\Gamma}_0\mathit{\Gamma}_{\rho_1}b_{F*G}(0), \qquad \zeta(y) = \frac{\mathit{\Gamma}_{\rho_1}b_{F*G}(y)}{\mathit{\Gamma}_0\mathit{\Gamma}_{\rho_1}b_{F*G}(0)},$$

and

$$\vartheta(u) = -\frac{[\lambda(1-\theta)(\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}A_2(u) - \mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}A_1(u)) + \lambda c(1-\theta)(\Phi_1(0)\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}b_F(u) - \Phi(0)\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}b_{F*G}(u)) - c\mathit{\Gamma}_{\rho_1}w(u)]}{c^2},$$

$A_1(u)$ and $A_2(u)$ are the inverse image functions of $\tilde{b}_F(s)\tilde{\omega}^*(s)$ and $\tilde{b}_{F*G}(s)\tilde{\omega}(s)$, i.e., $\mathit{\Gamma}_sA_1(0) = \tilde{b}_F(s)\tilde{\omega}^*(s)$ and $\mathit{\Gamma}_sA_2(0) = \tilde{b}_{F*G}(s)\tilde{\omega}(s)$.

Proof. From the definition of the Dickson–Hipp operator $\mathit{\Gamma}$, one deduces

$$\begin{aligned}
\mathit{\Gamma}_s\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}h_2(0) &= \lambda(\lambda+\delta)\mathit{\Gamma}_s\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}b_{F*G}(0) + \frac{\lambda c\left(\frac{s\tilde{b}_{F*G}(s)-\rho_2\tilde{b}_{F*G}(\rho_2)}{s-\rho_2} - \frac{s\tilde{b}_{F*G}(s)-\rho_1\tilde{b}_{F*G}(\rho_1)}{s-\rho_1}\right)}{\rho_1-\rho_2} \\
&= \lambda(\lambda+\delta)\mathit{\Gamma}_s\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}b_{F*G}(0) \\
&\quad + \lambda c\left(\frac{\tilde{b}_{F*G}(s) - \rho_2\mathit{\Gamma}_s\mathit{\Gamma}_{\rho_2}b_{F*G}(0)}{\rho_1-\rho_2} - \frac{\tilde{b}_{F*G}(s) - \rho_1\mathit{\Gamma}_s\mathit{\Gamma}_{\rho_1}b_{F*G}(0)}{\rho_1-\rho_2}\right) \\
&= \lambda(\lambda+\delta)\mathit{\Gamma}_s\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}b_{F*G}(0) + \lambda c\left(\frac{\rho_1\mathit{\Gamma}_s\mathit{\Gamma}_{\rho_1}b_{F*G}(0) - \rho_2\mathit{\Gamma}_s\mathit{\Gamma}_{\rho_2}b_{F*G}(0)}{\rho_1-\rho_2}\right) \\
&= \lambda(\lambda+\delta)\mathit{\Gamma}_s\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}b_{F*G}(0) + \lambda c(\mathit{\Gamma}_s\mathit{\Gamma}_{\rho_1}b_{F*G}(0) - \rho_2\mathit{\Gamma}_s\mathit{\Gamma}_{\rho_2}\mathit{\Gamma}_{\rho_1}b_{F*G}(0)) \\
&= \lambda c\mathit{\Gamma}_s\mathit{\Gamma}_{\rho_1}b_{F*G}(0).
\end{aligned} \tag{5.7}$$

Let $\tilde{A}_1(s) = \tilde{b}_F(s)\tilde{w^*}(s)$ and $\tilde{A}_2(s) = \tilde{b}_{F*G}(s)\tilde{w}(s)$. From the definition of the Dickson–Hipp operator Γ, we can deduce

$$\begin{aligned}
\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}f_2(0) &= \lambda(1-\theta)(\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}A_2(0) - \Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}A_1(0)) + \lambda c(1-\theta)(\Phi_1(0)\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}b_F(0) \\
&\quad - \Phi(0)\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}b_{F*G}(0)) - (\lambda+\delta)\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}w(0) + \frac{c\left(\frac{s\tilde{w}(s)-\rho_2\tilde{w}(\rho_2)}{s-\rho_2} - \frac{s\tilde{w}(s)-\rho_1\tilde{w}(\rho_1)}{s-\rho_1}\right)}{\rho_2-\rho_1} \\
&= \lambda(1-\theta)(\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}A_2(0) - \Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}A_1(0)) + \lambda c(1-\theta)(\Phi_1(0)\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}b_F(0) \\
&\quad - \Phi(0)\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}b_{F*G}(0)) - (\lambda+\delta)\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}w(0) + \frac{c(\rho_1\Gamma_s\Gamma_{\rho_1}w(0) - \rho_2\Gamma_s\Gamma_{\rho_2}w(0))}{\rho_2-\rho_1} \\
&= \lambda(1-\theta)(\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}A_2(0) - \Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}A_1(0)) + \lambda c(1-\theta)(\Phi_1(0)\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}b_F(0) \\
&\quad - \Phi(0)\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}b_{F*G}(0)) - c\Gamma_s\Gamma_{\rho_1}w(0) \\
&= -c^2\Gamma_s\vartheta(0).
\end{aligned} \tag{5.8}$$

Therefore, substituting (5.7) and (5.8) into (5.2), one deduces

$$\tilde{\Phi}(s) = \frac{\lambda}{c}\tilde{\Phi}(s)\Gamma_s\Gamma_{\rho_1}b_{F*G}(0) + \Gamma_s\vartheta(0). \tag{5.9}$$

Inverting the Laplace transform in (5.9), one finds

$$\begin{aligned}
\Phi(u) &= \frac{\lambda}{c}\int_0^u \Phi(u-y)\Gamma_{\rho_1}b_{F*G}(y)\mathrm{d}y + \vartheta(u) \\
&= \frac{\lambda}{c}\Gamma_0\Gamma_{\rho_1}b_{F*G}(0)\int_0^u \Phi(u-y)\frac{\Gamma_{\rho_1}b_{F*G}(y)}{\Gamma_0\Gamma_{\rho_1}b_{F*G}(0)}\mathrm{d}y + \vartheta(u)
\end{aligned}$$

which corresponds to (5.6).

For (5.6) to be a defective renewal equation, it remains to show that $\kappa_\delta < 1$. Let us first assume that $\delta > 0$. By comparing (5.7) at $s = 0$ to the expression for κ_δ, it follows that $\kappa_\delta = \frac{\Gamma_0\Gamma_{\rho_2}\Gamma_{\rho_1}h_2(0)}{c^2}$. From (5.5) at $s = 0$,

$$\kappa_\delta = \frac{\Gamma_0\Gamma_{\rho_2}\Gamma_{\rho_1}h_2(0)}{c^2} = 1 - \frac{\tilde{h}_1(0) - \tilde{h}_2(0)}{c^2\rho_1\rho_2} = 1 - \frac{\delta(\delta+\lambda)}{c^2\rho_1\rho_2} < 1,$$

given that $\rho_1(\delta) > 0$ and $\rho_2(\delta) > 0$. For $\delta = 0$, we know that

$$\kappa_0 = \frac{\lambda}{c}\Gamma_0\Gamma_0 b_{F*G}(0) = \frac{\lambda}{c}(\mu_F + \mu_G) < 1,$$

where the inequality is derived via (2.2). □

Now, we define an associated compound geometric distribution function $K(u) = 1 - \overline{K}(u)$ as follows:

$$\overline{K}(u) = \frac{\epsilon}{1+\epsilon}\sum_{n=1}^{\infty}\left(\frac{1}{1+\epsilon}\right)^n \overline{Z}^{*n}(u), \quad u \geq 0,$$

where $\epsilon = (1-\kappa_\delta)/\kappa_\delta$, $\overline{Z}^{*n}(u)$ is the tail of the n-fold convolution of $Z(u) = 1 - \overline{Z}(u) = \int_0^u \zeta(y)\mathrm{d}y$. Explicit solutions of the defective renewal equation (5.6) can be derived directly by applying Theorem 2.1 of [5].

Proposition 5.1. *The expected discounted penalty function $\Phi(u)$ satisfying the defective renewal equation (5.6) can be expressed as*

$$\Phi(u) = \frac{1}{\epsilon}\int_0^u [1 - \overline{K}(u-y)]\mathrm{d}B(y) + \frac{B(0)}{\epsilon}[1 - \overline{K}(u)], \tag{5.10}$$

or

$$\Phi(u) = \frac{1}{\epsilon}\int_0^u B(u-y)\mathrm{d}K(y) + \frac{1}{1+\epsilon}B(u), \tag{5.11}$$

where $B(u) = \vartheta(u)/\kappa_\delta$.

Proof. The proof is straightforward using Theorem 2.1 of [5] and Eq. (5.6). □

Remarks. When $\theta = 1$, that is to say, in any time period, the main claim and its associated by-claim occur simultaneously. Actually, this risk model is the compound Poisson risk model and the claim amounts are $\{Y_i + X_i\}_{i\geq 1}$ with common distribution function $F * G(x)$. In this case, Eq. (5.6) may be simplified as

$$\Phi(u) = \frac{\lambda}{c}\int_0^u \Phi(u-x)\int_x^\infty e^{-\rho_1(y-x)}\mathrm{d}F*G(y)\mathrm{d}x + \frac{\lambda}{c}\int_u^\infty e^{-\rho_1(x-u)}\int_x^\infty w(x,y-x)\mathrm{d}F*G(y)\mathrm{d}x.$$

This equation is consistent with Eq. (2.34) in [4]; the only difference is that we use $X + Y$ as our claim amount random variable while Gerber and Shiu [4] consider X.

6. Explicit results for exponential claim size distributions

We now consider the case where the claim amounts from both classes are exponentially distributed, with distribution functions $F \sim \mathrm{Exp}(\nu)$ and $G \sim \mathrm{Exp}(\omega)$, respectively, where $\nu = 1/\mu_F$ and $\omega = 1/\mu_G$; then $\tilde{b}_F(s) = \nu/(\nu+s)$ and $\tilde{b}_G(s) = \omega/(\omega+s)$. Moreover, if $\nu \neq \omega$, then

$$F*G(x) = 1 - \frac{\nu e^{-\omega x} - \omega e^{-\nu x}}{\nu-\omega}, \qquad F*G*G(x) = 1 - \frac{\omega^2 e^{-\nu x} + \nu e^{-\omega x}(\nu + \nu\omega x - \omega(2 + x\omega))}{(\nu-\omega)^2},$$

and if $\nu = \omega$, then

$$F*G(x) = e^{-x\nu}(e^{x\nu} - 1 - x\nu), \qquad F*G*G(x) = \frac{1}{2}e^{-x\nu}(2e^{x\nu} - 2 - 2x\nu - x^2\nu^2).$$

From (5.2), we know that

$$\tilde{\Phi}(s) = \frac{(s-\rho_1)(s-\rho_2)\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}f_2(0)}{-(cs-\delta-\lambda)^2 - \lambda\tilde{b}_{F*G}(s)(cs-\delta-\lambda)}. \tag{6.1}$$

It turns out that (6.1) can be transformed to another expression by multiplying both the denominator and numerator by $(s+\nu)(s+\omega)$:

$$\tilde{\Phi}(s) = \frac{(s+\nu)(s+\omega)(s-\rho_1)(s-\rho_2)\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}f_2(0)}{(s+\nu)(s+\omega)\{-(cs-\delta-\lambda)^2 - \lambda\tilde{b}_{F*G}(s)(cs-\delta-\lambda)\}}, \tag{6.2}$$

where in the numerator of (6.2)

$$\begin{aligned}(s+\nu)(s+\omega)\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}f_2(0) &= (s+\nu)(s+\omega)[\lambda(1-\theta)(\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}A_2(0) - \Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}A_1(0)) - c\Gamma_s\Gamma_{\rho_1}w(0)] \\ &\quad + \lambda c(1-\theta)\left\{(s+\omega)\times\left[\frac{\nu\Phi_1(0)}{(\rho_1+\nu)(\rho_2+\nu)} - \frac{\nu\omega\Phi(0)}{(\rho_2+\nu)(\rho_1+\omega)(\rho_2+\omega)}\right.\right. \\ &\quad \left.\left. - \frac{\nu\omega\Phi(0)}{(\rho_1+\nu)(\rho_2+\nu)(\rho_1+\omega)}\right] + \frac{\nu\omega\Phi(0)}{(\rho_1+\omega)(\rho_2+\omega)}\right\}.\end{aligned}$$

The common denominator of (6.2), denoted by $D_4(s)$, is a polynomial of degree 4 with the leading coefficient $-c^2$, given by

$$D_4(s) = -(cs-\delta-\lambda)^2(s+\nu)(s+\omega) - \lambda\nu\omega(cs-\delta-\lambda),$$

which has four roots on the complex plane and all the complex roots are in conjugate pairs. Noting that $s = \rho_1$ and $s = \rho_2$ are two roots, we have

$$D_4(s) = -c^2(s-\rho_1)(s-\rho_2)\prod_{i=1}^{2}(s+R_i).$$

Note also that all R_i's have positive real parts, since, otherwise, they would also be roots of Eq. (4.3), which is a contradiction to the conclusion of Proposition 4.1.

Furthermore, if R_1, R_2 are distinct, we obtain, by partial fractions, that

$$\frac{1}{(s+R_1)(s+R_2)} = \frac{a_1}{s+R_1} + \frac{a_2}{s+R_2}, \qquad \frac{s+\omega}{(s+R_1)(s+R_2)} = \frac{b_1}{s+R_1} + \frac{b_2}{s+R_2}$$
$$\frac{(s+\nu)(s+\omega)}{(s+R_1)(s+R_2)} = 1 + \frac{c_1}{s+R_1} + \frac{c_2}{s+R_2},$$

where $a_1 = 1/(R_2 - R_1)$, $a_2 = 1/(R_1 - R_2)$, $b_i = (\omega - R_i)a_i$, and $c_i = (\nu - R_i)(\omega - R_i)a_i$, for $i = 1, 2$. Then (6.2) can be simplified to

$$\begin{aligned}\tilde{\Phi}(s) = {} & \lambda c(1-\theta)\left\{\frac{\nu\omega\Phi(0)}{(\rho_1+\omega)(\rho_2+\omega)}\sum_{i=1}^{2}\frac{a_i}{s+R_i} + \sum_{i=1}^{2}\frac{b_i}{s+R_i}\left[\frac{\nu\Phi_1(0)}{(\rho_1+\nu)(\rho_2+\nu)}\right.\right.\\ & \left.\left. - \frac{\nu\omega\Phi(0)}{(\rho_2+\nu)(\rho_1+\omega)(\rho_2+\omega)} - \frac{\nu\omega\Phi(0)}{(\rho_1+\nu)(\rho_2+\nu)(\rho_1+\omega)}\right]\right\}\\ & + \left(1+\sum_{i=1}^{2}\frac{c_i}{s+R_i}\right)[\lambda(1-\theta)(\Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}A_2(0) - \Gamma_s\Gamma_{\rho_2}\Gamma_{\rho_1}A_1(0)) - c\Gamma_s\Gamma_{\rho_1}w(0)],\end{aligned}$$

where $\Phi(0)$, $\Phi_1(0)$ can be derived from (4.4) and (4.5).

Accordingly, explicit expressions for $\Phi(u)$, when the claim sizes from both of the classes are exponentially distributed, are given by

$$\begin{aligned}\Phi(u) = {} & \lambda c(1-\theta)\left\{\frac{\nu\omega\Phi(0)}{(\rho_1+\omega)(\rho_2+\omega)}\sum_{i=1}^{2}a_i e^{-R_i u} + \sum_{i=1}^{2}b_i e^{-R_i u}\left[\frac{\nu\Phi_1(0)}{(\rho_1+\nu)(\rho_2+\nu)}\right.\right.\\ & \left.\left. - \frac{\nu\omega\Phi(0)}{(\rho_2+\nu)(\rho_1+\omega)(\rho_2+\omega)} - \frac{\nu\omega\Phi(0)}{(\rho_1+\nu)(\rho_2+\nu)(\rho_1+\omega)}\right]\right\}\\ & + \lambda(1-\theta)(\Gamma_{\rho_2}\Gamma_{\rho_1}A_2(u) - \Gamma_{\rho_2}\Gamma_{\rho_1}A_1(u)) - c\Gamma_{\rho_1}w(u)\\ & + \sum_{i=1}^{2}c_i e^{-R_i u}\underline{*}\{\lambda(1-\theta)(\Gamma_{\rho_2}\Gamma_{\rho_1}A_2(u) - \Gamma_{\rho_2}\Gamma_{\rho_1}A_1(u)) - c\Gamma_{\rho_1}w(u)\},\end{aligned} \tag{6.3}$$

where $\underline{*}$ denotes the operation of convolution which is different from the distribution functions convolution.

Now, we discuss the special case $\delta = 0$. In this situation,

$$D_4(s) = -(cs-\lambda)^2(s+\nu)(s+\omega) - \lambda\nu\omega(cs-\lambda) = 0$$

has four roots, namely

$$s_1 = \rho_1 = 0, \qquad s_2 = \rho_2 = \frac{\lambda}{c}, \qquad s_3 = -R_1 = \frac{\lambda - c\nu - c\omega - \Lambda}{2c}, \qquad s_4 = -R_2 = \frac{\lambda - c\nu - c\omega + \Lambda}{2c},$$

where $\Lambda = \sqrt{(c\nu + c\omega - \lambda)^2 - 4c(c\nu\omega - \lambda\nu - \lambda\omega)}$. The positive relative security loading condition, $c > \lambda(1/\omega + 1/\nu)$, implies that only s_2 is positive. This result also confirms the conclusion of Proposition 4.1.

Example 6.1. Assume that $\delta = 0$, $w(x_1, x_2) = 1$; then (6.3) is the ruin probability $\phi(u)$. Accordingly, we have $w_1(u) = \bar{F}(u)$, $w_2(u) = \overline{F * G}(u)$, and $w_3(u) = \overline{F * G * G}(u)$. From (6.3), we obtain the ruin probability

$$\begin{aligned}\phi(u) = {} & e^{\frac{(\lambda - c\nu - c\omega - \Lambda)u}{2c}}\frac{\lambda(\lambda(\omega+\nu) - c\omega\nu)\{\lambda^2 + c\omega(c(\omega + 2\theta\nu - \nu) - \Lambda) + \lambda(c(\nu + 2\omega) - \Lambda)\}}{c\omega\nu\Lambda(\lambda\theta + c\omega)(c(\omega+\nu) - \lambda + \Lambda)}\\ & + e^{\frac{(\lambda - c\nu - c\omega + \Lambda)u}{2c}}\frac{\lambda(\lambda(\omega+\nu) - c\omega\nu)\{\lambda^2 + c\omega(c(\omega + 2\theta\nu - \nu) + \Lambda) + \lambda(c(\nu + 2\omega) + \Lambda)\}}{c\omega\nu\Lambda(\lambda\theta + c\omega)(\lambda - c(\omega+\nu) + \Lambda)}, \quad u \geq 0.\end{aligned} \tag{6.4}$$

Two extreme cases of (6.4) are

$$\begin{aligned}\phi(u) = {} & e^{\frac{(\lambda - c\nu - c\omega - \Lambda)u}{2c}}\frac{\lambda(\lambda(\omega+\nu) - c\omega\nu)\{\lambda^2 + c\omega(c(\omega - \nu) - \Lambda) + \lambda(c(\nu + 2\omega) - \Lambda)\}}{c^2\omega^2\nu\Lambda(c(\omega+\nu) - \lambda + \Lambda)}\\ & + e^{\frac{(\lambda - c\nu - c\omega + \Lambda)u}{2c}}\frac{\lambda(\lambda(\omega+\nu) - c\omega\nu)\{\lambda^2 + c\omega(c(\omega - \nu) + \Lambda) + \lambda(c(\nu + 2\omega) + \Lambda)\}}{c^2\omega^2\nu\Lambda(\lambda - c(\omega+\nu) + \Lambda)}, \quad \text{for } \theta = 0,\end{aligned}$$

$$\phi(u) = \frac{e^{\frac{(\lambda - c\nu - c\omega - \Lambda)u}{2c}}\lambda[(\omega+\nu)(\Lambda - \lambda) - c(\nu^2 + \omega^2)] + e^{\frac{(\lambda - c\nu - c\omega + \Lambda)u}{2c}}\lambda[(\omega+\nu)(\Lambda + \lambda) + c(\nu^2 + \omega^2)]}{2c\omega\nu\Lambda},$$

for $\theta = 1$.

Another value of interest in Example 6.1 is the impact of the delay of by-claims on the ruin probability. We can prove the following result.

Theorem 6.1. *For the risk model considered in Example 6.1, the ruin probability, $\phi(u)$, decreases as the probability of the delay of the by-claims increases, i.e., ruin probability is an increasing function of θ.*

Proof. Differentiating $\phi(u)$ with respect to θ, we can get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\theta}\left[e^{\frac{(\lambda-c\nu-c\omega-\Lambda)u}{2c}}\frac{\lambda(\lambda(\omega+\nu)-c\omega\nu)\{\lambda^2+c\omega(c(\omega+2\theta\nu-\nu)-\Lambda)+\lambda(c(\nu+2\omega)-\Lambda)\}}{c\omega\nu\Lambda(\lambda\theta+c\omega)(c(\omega+\nu)-\lambda+\Lambda)}\right.\\&\quad\left.+\;e^{\frac{(\lambda-c\nu-c\omega+\Lambda)u}{2c}}\frac{\lambda(\lambda(\omega+\nu)-c\omega\nu)\{\lambda^2+c\omega(c(\omega+2\theta\nu-\nu)+\Lambda)+\lambda(c(\nu+2\omega)+\Lambda)\}}{c\omega\nu\Lambda(\lambda\theta+c\omega)(\lambda-c(\omega+\nu)+\Lambda)}\right]\\&=e^{\frac{(\lambda-c\nu-c\omega-\Lambda)u}{2c}}\frac{\lambda(\lambda(\omega+\nu)-c\omega\nu)(\lambda+c\omega)\{2c^2\omega\nu-c\lambda(\omega+\nu)+\lambda(\Lambda-\lambda)\}}{c\omega\nu(c(\omega+\nu)-\lambda+\Lambda)(\lambda\theta+c\omega)^2}\\&\quad+e^{\frac{(\lambda-c\nu-c\omega+\Lambda)u}{2c}}\frac{\lambda(\lambda(\omega+\nu)-c\omega\nu)(\lambda+c\omega)\{2c^2\omega\nu-c\lambda(\omega+\nu)-\lambda(\Lambda+\lambda)\}}{c\omega\nu(\lambda-c(\omega+\nu)+\Lambda)(\lambda\theta+c\omega)^2}.\end{aligned}$$

Assume

$$\begin{aligned}\Lambda_1&=\frac{\lambda(\lambda(\omega+\nu)-c\omega\nu)(\lambda+c\omega)\{2c^2\omega\nu-c\lambda(\omega+\nu)+\lambda(\Lambda-\lambda)\}}{c\omega\nu(c(\omega+\nu)-\lambda+\Lambda)(\lambda\theta+c\omega)^2},\\\Lambda_2&=\frac{\lambda(\lambda(\omega+\nu)-c\omega\nu)(\lambda+c\omega)\{2c^2\omega\nu-c\lambda(\omega+\nu)-\lambda(\Lambda+\lambda)\}}{c\omega\nu(\lambda-c(\omega+\nu)+\Lambda)(\lambda\theta+c\omega)^2}.\end{aligned}$$

The positive relative security loading condition, $c>\lambda(1/\omega+1/\nu)$, implies that $\lambda(\omega+\nu)-c\omega\nu<0$, $c(\omega+\nu)-\lambda+\Lambda>0$, and $\lambda-c(\omega+\nu)+\Lambda<0$. Moreover,

$$\begin{aligned}&2c^2\omega\nu-c\lambda(\omega+\nu)+\lambda(\Lambda-\lambda)=2c(c\omega\nu-\lambda(\omega+\nu))+\lambda(c(\omega+\nu)-\lambda+\Lambda)>0,\\&2c^2\omega\nu-c\lambda(\omega+\nu)-\lambda(\Lambda+\lambda)=2c(c\omega\nu-\lambda(\omega+\nu))+\lambda(c(\omega+\nu)-\lambda-\Lambda)>0.\end{aligned}$$

From these discussions, it follows that $\Lambda_1<0$ and $\Lambda_2>0$. According to the definitions of Λ_1 and Λ_2, we know that

$$\frac{|\Lambda_1|}{|\Lambda_2|}=\frac{(2c^2\omega\nu-c\lambda(\omega+\nu)+\lambda(\Lambda-\lambda))(c(\omega+\nu)-\lambda-\Lambda)}{(2c^2\omega\nu-c\lambda(\omega+\nu)-\lambda(\Lambda+\lambda))(c(\omega+\nu)-\lambda+\Lambda)},$$

and moreover,

$$\begin{aligned}&(2c^2\omega\nu-c\lambda(\omega+\nu)+\lambda(\Lambda-\lambda))(c(\omega+\nu)-\lambda-\Lambda)-(2c^2\omega\nu-c\lambda(\omega+\nu)-\lambda(\Lambda+\lambda))(c(\omega+\nu)-\lambda+\Lambda)\\&\quad=4c\Lambda(\lambda(\omega+\nu)-c\omega\nu)<0,\end{aligned}$$

and then $\frac{|\Lambda_1|}{|\Lambda_2|}<1$. Also, it is easy to see that $\lambda-c\nu-c\omega-\Lambda<\lambda-c\nu-c\omega+\Lambda<0$. Hence

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\theta}\left[e^{\frac{(\lambda-c\nu-c\omega-\Lambda)u}{2c}}\frac{\lambda(\lambda(\omega+\nu)-c\omega\nu)\{\lambda^2+c\omega(c(\omega+2\theta\nu-\nu)-\Lambda)+\lambda(c(\nu+2\omega)-\Lambda)\}}{c\omega\nu\Lambda(\lambda\theta+c\omega)(c(\omega+\nu)-\lambda+\Lambda)}\right.\\&\quad\left.+\;e^{\frac{(\lambda-c\nu-c\omega+\Lambda)u}{2c}}\frac{\lambda(\lambda(\omega+\nu)-c\omega\nu)\{\lambda^2+c\omega(c(\omega+2\theta\nu-\nu)+\Lambda)+\lambda(c(\nu+2\omega)+\Lambda)\}}{c\omega\nu\Lambda(\lambda\theta+c\omega)(\lambda-c(\omega+\nu)+\Lambda)}\right]>0,\end{aligned}$$

and then the ruin probability is an increasing function of θ.

Since the probability of the delay of the by-claim is $1-\theta$, the ruin probability decreases as the probability of the delay of the by-claims increases. This completes the proof. □

Example 6.2. Assume that $\delta=0$, $w(x_1,x_2)=I(x_1\le x)$; then (6.3) is the distribution function of the surplus before ruin, denoted by $F(u,x)$. Accordingly, we have

$$w_1(u)=\int_u^\infty w(u,s-u)\mathrm{d}F(s)=\int_u^\infty I(u\le x)\mathrm{d}F(s)=I(u\le x)\int_u^\infty \mathrm{d}F(s)=I(u\le x)\bar{F}(u),$$

$w_2(u)=I(u\le x)\overline{F*G}(u)$, $w_3(u)=I(u\le x)\overline{F*G*G}(u)$. Hence, when $\nu=\omega$, by (4.4) and (4.5), we have

$$\begin{aligned}&F(0,x)\\&\quad=\frac{e^{-x\nu}\lambda(\lambda+c\nu)(x^2(\theta-1)\theta\nu^2-2x\nu-4)+e^{-x(\frac{\lambda}{c}+\nu)}\lambda c\nu(1-\theta)(2+2x\nu+x^2\theta\nu^2)+2\lambda(2\lambda+c(1+\theta)\nu)}{2c\nu(\lambda\theta+c\nu)},\end{aligned}\tag{6.5}$$

and

$$F_1(0,x)=\frac{e^{-x\nu}\lambda(x^2(\theta-1)\theta\nu^2-2x\nu-4)-e^{-x(\frac{\lambda}{c}+\nu)}\lambda\theta(2+2x\nu+x^2\theta\nu^2)+2\lambda(2+\theta)}{2(\lambda\theta+c\nu)}.\tag{6.6}$$

Then, by (6.3), $F(u,x)$ can be derived for when $0 \le u < x$ as

$$F(u,x) = \frac{c\lambda\{2\lambda(\theta-1)(\lambda-c\theta\nu)(\nu^2-R_1R_2)+e^{-\frac{\lambda}{c}(x-u)-x\nu}c^2\nu^2R_1R_2(\theta-1)(2+2x\nu+\theta x^2\nu^2)+2e^{-u\nu}\chi_1\}}{2R_1R_2\nu(\lambda+c\nu)^2}$$
$$+\frac{c\lambda\{e^{-uR_1-x\nu}R_2(R_1-\nu)^2-e^{-uR_2-x\nu}R_1(R_2-\nu)^2-e^{-x\nu}(R_1-R_2)(R_1R_2-\nu^2)\}\chi_2}{2R_1R_2\nu(R_1-R_2)(\lambda+c\nu)^2}$$
$$+\frac{\lambda^2c(\theta-1)\{\xi_1(R_1)-\xi_1(R_2)\}}{2\nu(R_1-R_2)(\lambda+c\nu)^2(\lambda\theta+c\nu)}+\frac{c\lambda\{e^{-uR_2}R_1\xi_2(R_2)-e^{-uR_1}R_2\xi_2(R_1)\}}{R_1R_2\nu(R_1-R_2)(\lambda+c\nu)^2}$$
$$-\frac{e^{-u\nu}c\lambda(1+\theta+u\theta\nu)}{\nu}+\frac{e^{-x\nu}c\lambda\{4+2x\nu+x^2(1-\theta)\theta\nu^2\}}{2\nu}, \quad (6.7)$$

where $\xi_1(s)=2e^{-us}(c\nu^2(\lambda\theta+c\nu)+s(2\lambda^2+3c\lambda\nu+c^2\theta\nu^2))-e^{-us-x(\frac{\lambda}{c}+\nu)}c\nu(2+2x\nu+x^2\theta\nu^2)\{\nu(\theta\lambda+c\nu)-s(\lambda+c(2-\theta)\nu)\}-e^{-us-x\nu}(\lambda+c\nu)^2(4+2x\nu+x^2(1-\theta)\theta\nu^2)s$, $\xi_2(s)=\lambda\nu^2(\theta-1)(c\theta\nu-\lambda)+s^2(2\lambda^2+c\lambda\nu(2+\theta+\theta^2)+c^2\nu^2(1+\theta))-s\nu(3\lambda^2+2c\lambda\nu(2+\theta+\theta^2)+c^2\nu^2(1+2\theta))$, $\chi_1=R_1R_2\left\{\theta((2\lambda^2+c^2\nu^2)(1+u\nu)+c\lambda\nu(4+3u\nu))-\nu(u\lambda(\lambda+c\nu)-c^2\nu)\right\}$, $\chi_2=2c^2\nu^2(1+\theta+\theta x\nu)+\lambda^2(4+2x\nu+x^2(1-\theta)\theta\nu^2)+c\lambda\nu(4+2x\nu+\theta^2(2+2x\nu-x^2\nu^2)+\theta(2+x^2\nu^2))$, and for when $u \ge x$ as

$$F(u,x) = \frac{c\lambda^2(1-\theta)(\lambda-c\theta\nu)(R_1R_2-\nu^2)}{R_1R_2\nu(\lambda+c\nu)^2}+\frac{c\lambda(e^{-uR_1-x\nu}R_2(R_1-\nu)^2-e^{-uR_2-x\nu}R_1(R_2-\nu)^2)\chi_2}{2R_1R_2\nu(R_1-R_2)(\lambda+c\nu)^2}$$
$$+\frac{c\lambda^2(\theta-1)(c\theta\nu-\lambda)[e^{-(u-x)R_1}R_2(R_1-\nu)^2-e^{-(u-x)R_2}R_1(R_2-\nu)^2]}{R_1R_2\nu(R_1-R_2)(\lambda+c\nu)^2}+\frac{e^{-u\nu}cx\lambda^2\nu(\theta-1)\chi_3}{2(\lambda+c\nu)^2}$$
$$+\frac{c\lambda\{2e^{-(u-x)R_1}\gamma_1(R_1)-e^{-uR_1-x\nu}\gamma_2(R_1)+e^{-uR_1-x(\frac{\lambda}{c}+\nu)}\gamma_3(R_1)-2e^{-uR_1}\gamma_4(R_1)\}}{2R_1\nu(R_2-R_1)(\lambda+c\nu)^2(\lambda\theta+c\nu)}$$
$$+\frac{c\lambda\{2e^{-(u-x)R_2}\gamma_1(R_2)-e^{-uR_2-x\nu}\gamma_2(R_2)+e^{-uR_2-x(\frac{\lambda}{c}+\nu)}\gamma_3(R_2)-2e^{-uR_2}\gamma_4(R_2)\}}{2R_2\nu(R_1-R_2)(\lambda+c\nu)^2(\lambda\theta+c\nu)}$$
$$+\frac{c\lambda\{2\nu(\lambda+c\nu)^2(e^{-uR_1-x(\nu-R_1)}R_2\gamma_5(R_1)-e^{-uR_2-x(\nu-R_2)}R_1\gamma_5(R_2))-e^{-x\nu}\chi_4\}}{2R_1R_2\nu(R_1-R_2)(\lambda+c\nu)^2}, \quad (6.8)$$

where $\gamma_1(s)=\lambda(\theta-1)(\nu-s)^2(c\theta\nu-\lambda)(\lambda\theta+c\nu)$, $\gamma_2(s)=\lambda(1-\theta)s^2(\lambda+c\nu)^2[4+2x\nu+\theta(1-\theta)x^2\nu^2]$, $\gamma_3(s)=s[\nu(\lambda\theta+c\nu)-s(\lambda+c(2-\theta)\nu)]c\nu\lambda(\theta-1)(2+2x\nu+x^2\theta\nu^2)$, $\gamma_4(s)=\lambda(1-\theta)\nu^2(c\theta\nu-\lambda)(\lambda\theta+c\nu)-s^2[2\lambda^3+c(5-\theta(1-\theta-\theta^2))\lambda^2\nu+c^2(2+3\theta+\theta^2)\lambda\nu^2+c^3\nu^3(1+\theta)]+s\nu[2c\theta^3\lambda^2\nu+c\lambda\nu\theta^2(3\lambda+4c\nu)+c\nu(3\lambda^2+c\lambda\nu+c^2\nu^2)+\theta(3\lambda^3+c\lambda^2\nu+4c^2\lambda\nu^2+2c^3\nu^3)]$, $\gamma_5(s)=s(1+x\theta\nu)-\nu(1+\theta+x\theta\nu)$, $\chi_3=c[x^2\theta\nu^2+2x(1-\theta)\nu-2-u\nu(2+x\theta\nu)]-\lambda[x(\theta-2-x\theta\nu)+u(2+x\theta\nu)]$, $\chi_4=\lambda(1-\theta)(R_1-R_2)(R_1R_2-\nu^2)[\lambda(2+2x\nu+x^2\theta\nu^2)+c\nu(2x\nu-\theta(2+2x\nu-x^2\nu^2))]$.

On the other hand, when $\nu \ne \omega$, by (4.4) and (4.5), we have

$$F(0,x) = \frac{1}{c\nu(\lambda+c\nu)(\lambda\theta+c\omega)}\left(e^{-x\nu}\frac{\lambda(\lambda+c\nu)(\lambda+c\omega)(\nu^2\theta(\theta-1)+\omega(\nu-\omega))}{(\nu-\omega)^2}\right.$$
$$+e^{-x\omega}\frac{\lambda\nu^2(\lambda+c\nu)(\lambda+c\omega)(\omega-\nu+\omega\theta(1-\theta)(1+x(\omega-\nu)))}{(\nu-\omega)^2\omega}$$
$$+\frac{\lambda(\lambda+c\nu)(\lambda(\nu+\omega)+c\omega(\theta\nu+\omega))}{\omega}+e^{-x(\frac{\lambda}{c}+\nu)}\frac{c\lambda\nu\omega(1-\theta)(\lambda+c\nu)(\omega-(1-\theta)\nu)}{(\nu-\omega)^2}$$
$$\left.-e^{-x(\frac{\lambda}{c}+\omega)}\frac{\lambda c(\theta-1)\nu^2(\lambda+c\nu)(\nu+x\theta\nu\omega-\omega(1+\theta+x\theta\omega))}{(\nu-\omega)^2}\right), \quad (6.9)$$

and

$$F_1(0,x) = \frac{\lambda}{\nu(\nu-\omega)^2(\lambda\theta+c\omega)}((\nu-\omega)^2(\nu(1+\theta)+\omega)+e^{-x\nu}\omega(\omega(\nu-\omega)+\theta\nu^2(\theta-1))$$
$$+e^{-x\omega}\nu^2(\nu(x\omega\theta(\theta-1)-1)+\omega(1+\theta+x\theta\omega-\theta^2(1+x\omega)))-e^{-x(\frac{\lambda}{c}+\nu)}\theta\nu\omega(\omega-\nu(1-\theta))$$
$$-e^{-x(\frac{\lambda}{c}+\omega)}\theta\nu^2(\nu+x\theta\nu\omega-\omega(1+\theta+x\theta\omega))). \quad (6.10)$$

Hence, the expression for $F(u,x)$ can also be given by (6.3).

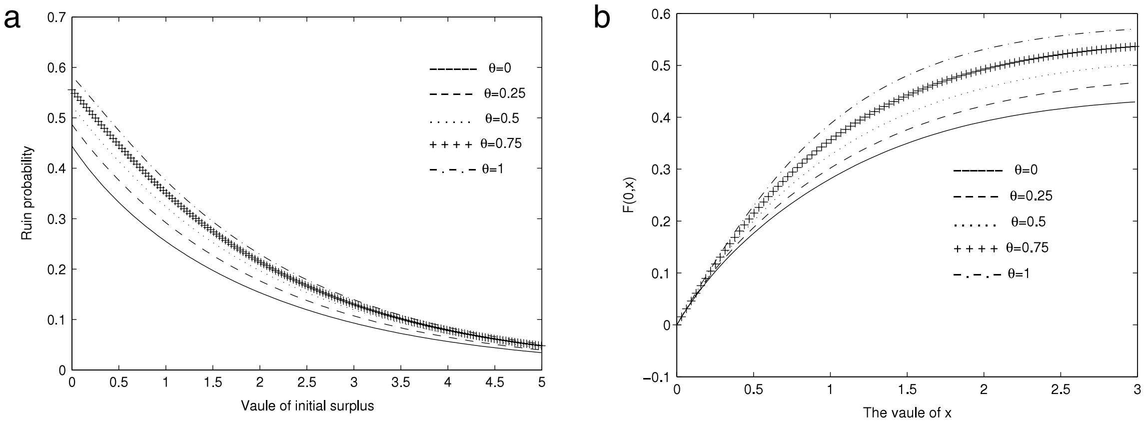

Fig. 1. (a) Ruin probabilities in Example 6.1. (b) The distribution functions, $F(0,x)$, in Example 6.2.

7. Numerical illustrations

Let $\lambda = 1, c = 2, F \sim \text{Exp}(2), G \sim \text{Exp}(1.5)$. The positive relative security loading condition (2.2) is obviously fulfilled. In this case, $R_1 = 2.5$ and $R_2 = 0.5$.

Fig. 1(a) shows the ruin probabilities $\Phi(u)$ in Example 6.1, for different values of $u \in [0, 5]$ and $\theta = 0, 0.25, 0.5, 0.75, 1$. From this graph, we can see that, as expected, these ruin probabilities decrease as the initial surplus u increases. Moreover, with fixed u, ruin probabilities increase as θ increases. This result confirms the conclusion of Theorem 6.1.

Fig. 1(b) shows the distribution functions of the surplus before ruin, $F(0, x)$ in Example 6.2, for different values of $x \in [0, 3]$ and $\theta = 0, 0.25, 0.5, 0.75, 1$. From this graph, we can see that, with fixed x, these distribution functions also increase as θ increases.

With fixed $\theta = 0.75$, the $F(u, x)$ in Example 6.2 can be derived for when $0 \le u < x$ as

$$\begin{aligned} F(u,x) = {} & -0.4e^{-2.5(u+x)} - 0.16667e^{-2.5u} - 2.05556e^{-0.5u} + 0.8e^{-0.5u-2.5x} - 2e^{0.5u-2.5x} \\ & + e^{-1.5x}(5.6 + 3.6x) - e^{-0.5u-1.5x}(3.57778 + 2.3x) + e^{0.5u-2x}(1.66667 - 1.5x) \\ & + e^{-2.5u-2x}(0.33333 - 0.3x) + e^{-2.5u-1.5x}(0.46667 + 0.3x) + e^{-0.5u-2x}(-0.66667 + 0.6x), \end{aligned}$$

and for when $u \ge x$ as

$$\begin{aligned} F(u,x) = {} & -0.3e^{-2.5(u+x)} - 0.1e^{-0.5u-2.5x} + 0.2e^{-2.5u+0.5x} + e^{-2.5u-2x}(0.275 - 0.225x) \\ & - e^{-0.5u-2x}(0.291667 + 0.075x) + e^{-0.5u-1.5x}(1.42222 - 0.05x) + e^{-2.5u-1.5x}(0.23333 + 0.15x) \\ & + e^{-0.5u-x}(-1 + 0.675x) - 0.075e^{-2.5u+x}x + 0.18e^{-2.5(u-x)}(0.346668 + 0.80000x - x^2) \\ & + e^{-0.5(u-x)}(49.5 - 27x + 6.75x^2) + e^{-0.5u}(-49.5306 + 2.25x - 1.125x^3) \\ & + e^{-2.5u}(-0.339533 - 0.15x + 0.075x^3). \end{aligned}$$

8. Concluding remarks

In this paper, we study the compound Poisson risk model with delayed claims. Two kinds of dependent claims: main claims and by-claims, are defined. In this risk model, there will be a main claim Y_i in every epoch T_i of the Poisson process and the main claim Y_i will induce a by-claim X_i. Moreover, the by-claim X_i and its associated main claim Y_i may occur simultaneously with probability θ, or the occurrence of the by-claim X_i may be delayed to T_{i+1} with probability $1-\theta$. If the occurrence of the by-claim X_i is delayed to T_{i+1}, we assume that the occurrence of the delayed by-claim X_i is independent of the occurrence of the next main claim Y_{i+1}. The results obtained in this paper show (although the risk process is neither a compound renewal nor a compound Poisson one) that the expected discounted penalty function satisfies the defective renewal equation. The results also illustrate the impact of the delay of by-claims on the expected discounted penalty function.

We also derive the explicit expressions for the expected discounted penalty functions when the claims from both classes are exponentially distributed. The results may be extended if, for example, the claim size distributions for both classes are Erlang(n), or more generally, from the $K_n (n \in \mathbf{N}^+)$ family.

Acknowledgements

The author thanks the referees for their valuable comments and suggestions which led to improvement of the paper. The research was fully supported by the Science and Technology Foundation of JiangXi Province (Project No. GJJ10267).

References

[1] N.L. Bowers, H.U. Gerber, J.C. Hickman, D.A. Jones, C.J. Nesbitt, Actuarial Mathematics, 2nd, Society of Actuaries, 1997.
[2] H.U. Gerber, An Introduction to Mathematical Risk Theory, S.S. Huebner Foundation, University of Pennsylvania, Philadelphia, 1979.
[3] H.H. Panjer, G.E. Willmot, Insurance Risk Models, Society of Actuaries, Schaumburg, 1992.
[4] H.U. Gerber, E.S.W. Shiu, On the time value of ruin, North American Actuarial Journal 2 (1998) 48–78.
[5] X.S. Lin, G.E. Willmot, Analysis of a defective renewal equation arising in ruin theory, Insurance: Mathematics and Economics 25 (1999) 63–84.
[6] X.S. Lin, G.E. Willmot, The moments of the time of ruin, the surplus before ruin, and the deficit at ruin, Insurance: Mathematics and Economics 27 (2000) 19–44.
[7] C.C.L. Tsai, On the discounted distribution functions of the surplus process, Insurance: Mathematics and Economics 28 (2001) 401–419.
[8] H.R. Waters, A. Papatriandafylou, Ruin probabilities allowing for delay in claims settlement, Insurance: Mathematics and Economics 4 (1985) 113–122.
[9] K.C. Yuen, J.Y. Guo, Ruin probabilities for time-correlated claims in the compound binomial model, Insurance: Mathematics and Economics 29 (2001) 47–57.
[10] Y.T. Xiao, J.Y. Guo, The compound binomial risk model with time-correlated claims, Insurance: Mathematics and Economics 41 (2007) 124–133.
[11] J.H. Xie, W. Zou, Ruin probabilities of a risk model with time-correlated claims, Journal of the Graduate School of the Chinese Academy of Sciences 3 (2008) 319–326.
[12] J.H. Xie, W. Zou, Expected present value of total dividends in a delayed claims risk model under stochastic interest rates, Insurance: Mathematics and Economics 46 (2010) 415–422.
[13] D.C.M. Dickson, C. Hipp, On the time to ruin for Erlang(2) risk processes, Insurance: Mathematics and Economics 29 (2001) 333–344.
[14] S. Li, J. Garrido, On ruin for Erlang(n) risk process, Insurance: Mathematics and Economics 34 (2004) 391–408.
[15] S. Li, Y. Lu, On the expected discounted penalty functions for two classes of risk processes, Insurance: Mathematics and Economics 36 (2005) 179–193.
[16] Z. Zhang, S. Li, H. Yang, The Gerber–Shiu discounted penalty functions for a risk model with two classes of claims, Journal of Computational and Applied Mathematics 230 (2009) 643–655.