



On the expected discounted penalty function and optimal dividend strategy for a risk model with random incomes and interclaim-dependent claim sizes

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ABSTRACT

In this paper, we consider a risk model with dependence between claim sizes and interclaim arrivals. In contrast with the classical risk model where the premium process is a linear function of time, we consider a dependent risk model where the aggregate premium process is a compound Poisson process, moreover, there is a constant barrier strategy in this model. The integral equations for the expected discounted penalty function and the expected discounted dividend payments until ruin are obtained. In particular, when the individual stochastic premium amount is exponentially distributed, it is proved that both the expected discounted penalty function and the expected discounted dividend payments until ruin satisfy the Volterra integral equations. Furthermore, the representations of the solutions are derived, respectively. In addition, when the individual stochastic premium amount and claim amount are exponentially distributed, we can get the explicit expressions for the Laplace transform of the ruin time and the expected discounted dividend payments until ruin. Finally, the optimal barrier is presented under the condition of maximizing the expectation of the difference between discounted dividends until ruin and the deficit at ruin.

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1. Introduction

The classical compound Poisson risk model, in which claims occur according to a Poisson process has been extensively analyzed. Ruin probabilities and many ruin related quantities have been analytically studied. However, as is known, for the classical compound Poisson risk model, it is explicitly assumed that the interarrival time between two successive claims and the claim amounts are independent, which is extremely restrictive and sometimes unreal. To avoid this restriction, some papers considered the dependent risk models. For example, Albrecher and Boxma [1] have proposed an extension to the classical compound Poisson risk model, in which the distribution of a claim interval is controlled by the previous claim size. If the claim size exceeds a random level the next claim interval will follow one type of distribution, if not, it will follow another type. Boudreault et al. [2] have considered a risk model with the reverse dependence structure (i.e., the distribution of the next claim size depends on the last interarrival time). Landriault [3] have studied the risk model with interclaim-dependent claim sizes proposed by Boudreault et al. [2] in the presence of a constant dividend barrier. Xie and Zou [4] have proposed a risk model with a dependence setting where there exists a specific structure among the time between two claim occurrences, premium sizes and claim sizes.

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Recall that the barrier strategy has been initially proposed by De Finetti [5] for a binomial model. Historically, many papers have dealt with the finding of an optimal dividend strategy based on various criteria (see e.g., Dickson and Waters [6], Albrecher and Hartinger [7], Gerber and Shiu [8,9] and references therein).

Note that all risk models described in the paragraph above rely on the assumption that the premium process is a linear function of time. In fact, the structure of the premium process should be more complex owing to various factors of financial and insurance markets (see, for example, Melnikov [10]). Boikov [11] generalized the classical risk model to the case where the premium process was modeled by a compound Poisson process, and Temnov [12] also focused on the risk model with random premium process. Assuming that the premium process is a Poisson process, Bao [13] studied the expected discounted penalty function in the compound Poisson risk model. Zhang and Yang [14] considered the risk model with dependence between income and loss. This paper extends the risk model by adding stochastic premiums income under the work of Landriault [3]. Assume that the aggregate premium process is modeled as a compound Poisson process, we study the expected discounted penalty function and the expected discounted dividend payments until ruin for this risk model. When the individual stochastic premium amount is exponentially distributed, we prove that the expected discounted penalty function and the expected discounted dividend payments until ruin satisfy the Volterra integral equations, furthermore, the representations of the solutions are derived, respectively. In addition, when the individual stochastic premium amount and claim amount are exponentially distributed, we can get the explicit expressions for the Laplace transform of the ruin time and the expected discounted dividend payments until ruin. At last, we derive the optimal barrier under the condition of maximizing the expectation of the difference between discounted dividends until ruin and the deficit at ruin.

The rest of the paper is organized as follows. In Section 2, we introduce a brief description of the risk model with random incomes and dependence between claim sizes and claim intervals. In Section 3, we obtain that integral equations for the expected discounted penalty function and the expected discounted dividend payments until ruin. In particular, when the individual stochastic premium amount is exponentially distributed, we prove that both the expected discounted penalty function and the expected discounted dividend payments until ruin satisfy the Volterra integral equations. The representations of the solutions are also derived. In Section 4, we consider the special case where the individual stochastic premium amount and claim amount are exponential distributed. The explicit expression for the Laplace transform of the time of ruin is derived. In Section 5, we discuss the optimal dividend strategy, and derive the optimal barrier under the condition of maximizing the expectation of the difference between discounted dividends until ruin and the deficit at ruin. Finally, the concluding remarks are obtained in Section 6.

2. Model description and notation

Let $U_b(t)$ be the surplus of an insurance company at time t under a constant barrier strategy and assume that the initial surplus is $U_b(0) = u$. Then

$$dU_b(t) = \begin{cases} d \sum_{j=1}^{N_2(t)} X_j - d \sum_{i=1}^{N_1(t)} Y_i, & U_b(t) < b, \\ -d \sum_{i=1}^{N_1(t)} Y_i, & U_b(t) \geq b, \end{cases} \quad (2.1)$$

where $N_1(t)$ is a Poisson process with intensity $\lambda_1 > 0$ counting the number of claims up to time t , and Y_i is the i th claim size, $N_2(t)$ counting the number of individual premium income up to time t is a Poisson process with intensity $\lambda_2 > 0$, and $\{X_i\}_{i \geq 1}$ is a sequence of strictly positive random variables (r.v.'s) representing the individual premium amounts with common distribution G , probability density function (p.d.f.) g , and mean μ_G . Denote by $\{V_i\}_{i \geq 1}$ the sequence of claim inter-occurrence times and $\{W_i\}_{i \geq 1}$ the sequence of premium income inter-occurrence times. We suppose the bivariate random vectors (V_j, Y_j) for $j \in \mathbb{N}^+$ mutually independent whereas the r.v.'s V_j and Y_j are dependent (compared to the classical compound Poisson risk model). More precisely, the claim size r.v. Y_j is defined conditionally on the interarrival time r.v. V_j . In Boudreault et al. [2], a tractable dependence structure between the r.v.'s V_j and Y_j is proposed. Namely, we suppose that the conditional p.d.f. of the claim size r.v. Y_j (given that $V_j = v$), denoted $f_{Y_j|V_j}(v)$, to be defined as a special mixture of the two arbitrary p.d.f.'s f_1 and f_2 , i.e.,

$$f_{Y_j|V_j}(y|v) = e^{-\beta v} f_1(y) + (1 - e^{-\beta v}) f_2(y), \quad y \geq 0, \quad (2.2)$$

for $j = 1, 2, \dots$. As discussed in Boudreault et al. [2], the dependent structure (2.2) has a practical interpretation in a context of earthquake insurance for instance. Indeed, suppose V_j is the waiting time between the $(j-1)$ th and j th earthquakes and such an event has two possible intensities, say $I_j = 1$ (usual), 2 (severe). The result can be expressed as

$$\Pr(I_j = 1|V_j = v) = e^{-\beta v} = 1 - \Pr(I_j = 2|V_j = v),$$

which implies that the probability of a usual earthquake is an exponentially decreasing function of the time separating this event from the last one. By choosing the p.d.f. f_2 (associated to a severe earthquake) with a heavier tail than the p.d.f. f_1 (associated to a usual earthquake), we definitely have a more realistic/appropriate model (than the independent one) to approximate the behavior of the aggregate claim amount process in the context of earthquake insurance.

We define the time of ruin by $T_b = \inf\{t \geq 0 : U_b(t) < 0\}$ ($T_b = \infty$ if the set is empty) and the ultimate ruin probability by $\phi_b(u) = \Pr(T_b < \infty | U_b(0) = u)$, $u \geq 0$. Let $w(x_1, x_2)$, $0 \leq x_1, x_2 < \infty$ be the penalty function and $I(A)$ be the indicator function of a set A . We also assume that $|U_b(T)|$ and $U_b(T-)$ are the deficit at ruin and the surplus immediately before ruin, respectively. For $\delta > 0$, define

$$\Phi_b(u) = E[e^{-\delta T_b} w(U(T_b-), |U(T_b)|) I(T_b < \infty) | U_b(0) = u], \quad u \geq 0, \quad (2.3)$$

as the expected discounted penalty (Gerber–Shiu) function at ruin. While δ may be interpreted as a force of interest, the function (2.3) may also be viewed in terms of a Laplace transform with δ serving as the argument. We remark that choosing different forms of the penalty function $w(x_1, x_2)$ in Eq. (2.3) gives rise to different information relating to the deficit at ruin and the surplus before ruin. In particular, if we let $w(x_1, x_2) = 1$, (2.3) is the Laplace transform of the time of ruin T_b . If we let $\delta = 0$ and $w(x_1, x_2) = I(x_1 \leq x)I(x_2 \leq y)$, (2.3) becomes the joint distribution function of the surplus before ruin and the deficit at ruin. Furthermore, if $\delta = 0$ and $w(x_1, x_2) = x_1^n$, we obtain the n th moment of the surplus before ruin. Likewise, if $\delta = 0$ and $w(x_1, x_2) = x_2^n$, we obtain the n th moment of the deficit at ruin. If $w(x_1, x_2)$ is interpreted as the benefit amount of an insurance (or reinsurance) payable at the time of ruin, then $\Phi_b(u)$ is the single premium of the insurance. The other financial explanations on $w(x_1, x_2)$ can be found in Gerber and Shiu [15].

Dividends are paid to the shareholders according to a barrier strategy. Let $D(t)$ denote the aggregate dividends paid by the time t . Then

$$U_b(t) = u + \sum_{j=1}^{N_2(t)} X_j - \sum_{i=1}^{N_1(t)} Y_i - D(t). \quad (2.4)$$

If the barrier strategy with parameter b is applied, no dividends are paid whenever $U_b(t) < b$, and the excess is paid out immediately as a dividend whenever the surplus exceeds the level b . Thus the surplus will never go above b .

Let $\delta \geq 0$ be the discounted factor, and define

$$D_{u,b} = \int_0^{T_b} e^{-\delta t} dD(t), \quad 0 \leq u \leq b,$$

as the present value of all dividends until the time of ruin T_b . The expected discounted dividend payments until ruin is defined as

$$V(u, b) = E[D_{u,b} | U_b(0) = u].$$

3. Integral equations and exact representations

In this section, we assume $U_b(0) = u \leq b$. Our aim is to give exact representations for the expected discounted dividend payments until ruin $V(u, b)$ and the expected discounted penalty function $\Phi_b(u)$. For this, we shall prove that these functions satisfy some integral equations.

3.1. Integral equations

First of all, we derive an integral equation satisfied by the expected discounted dividend payments until ruin $V(u, b)$.

Theorem 3.1. For $0 \leq u \leq b$, the expected discounted dividend payments until ruin $V(u, b)$ satisfies the following integral equation

$$\begin{aligned} V(u, b) = & \frac{\lambda_2}{\lambda_1 + \lambda_2 + \delta} \left(\int_0^{b-u} V(u+x, b) dG(x) + \int_{b-u}^{\infty} (x+u-b+V(b, b)) dG(x) \right) \\ & + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta + \beta} \left(\int_0^u V(u-y, b) f_1(y) dy - \int_0^u V(u-y, b) f_2(y) dy \right) \\ & + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta} \int_0^u V(u-y, b) f_2(y) dy. \end{aligned} \quad (3.1)$$

Proof. Firstly, we start from $0 \leq u < b$. The first claim can be or cannot be earlier than the first premium. Let $L = \min\{V_1, W_1\}$, then for $u \geq 0$, conditioning on the time of the first event (premium or claim), we have

$$\begin{aligned} V(u, b) = & \int_0^{\infty} \Pr(L = t, L = W_1) e^{-\delta t} \left(\int_0^{b-u} V(u+x, b) dG(x) + \int_{b-u}^{\infty} (x+u-b+V(b, b)) dG(x) \right) dt \\ & + \int_0^{\infty} \Pr(L = t, L = V_1) e^{-\delta t} \int_0^u V(u-y, b) [e^{-\beta t} f_1(y) + (1 - e^{-\beta t}) f_2(y)] dy dt. \end{aligned} \quad (3.2)$$

Note that

$$\begin{aligned}\Pr(L = W_1) &= \frac{\lambda_2}{\lambda_1 + \lambda_2}, & \Pr(L = V_1) &= \frac{\lambda_1}{\lambda_1 + \lambda_2}, \\ \Pr(L > t | L = W_1) &= \Pr(L > t | L = V_1) = \exp(-(\lambda_1 + \lambda_2)t).\end{aligned}\quad (3.3)$$

Setting these probabilities in (3.2) and by a bit of algebra, we can get the integral equation (3.1).

Now we consider $V(b, b)$. By similar arguments, we obtain

$$\begin{aligned}V(b, b) &= \int_0^\infty \Pr(L = t, L = W_1) e^{-\delta t} \int_0^\infty (x + V(b, b)) dG(x) dt \\ &\quad + \int_0^\infty \Pr(L = t, L = V_1) e^{-\delta t} \int_0^b V(b - y, b) [e^{-\beta t} f_1(y) + (1 - e^{-\beta t}) f_2(y)] dy dt \\ &= \frac{\lambda_2}{\lambda_1 + \lambda_2 + \delta} \int_0^\infty (x + V(b, b)) dG(x) + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta} \int_0^b V(b - y, b) f_2(y) dy \\ &\quad + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta + \beta} \left(\int_0^b V(b - y, b) f_1(y) dy - \int_0^b V(b - y, b) f_2(y) dy \right).\end{aligned}$$

Hence, (3.1) still holds for $u = b$. This completes the proof of this theorem. \square

Next we will consider the integral equation for the expected discounted penalty function $\Phi_b(u)$.

Theorem 3.2. For $0 \leq u \leq b$, the expected discounted penalty function $\Phi_b(u)$ satisfies the following integral equation

$$\begin{aligned}\Phi_b(u) &= \frac{\lambda_2}{\lambda_1 + \lambda_2 + \delta} \left(\int_0^{b-u} \Phi_b(u + x) dG(x) + \Phi_b(b) \bar{G}(b - u) \right) \\ &\quad + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta + \beta} \left(\int_0^u \Phi_b(u - y) (f_1(y) - f_2(y)) dy + \omega_1(u) - \omega_2(u) \right) \\ &\quad + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta} \left(\int_0^u \Phi_b(u - y) f_2(y) dy + \omega_2(u) \right),\end{aligned}\quad (3.4)$$

where $\omega_i(u) = \int_u^\infty w(u, y - u) f_i(y) dy$, $i = 1, 2$.

Proof. Similar to the proof of Theorem 3.1, we have

$$\begin{aligned}\Phi_b(u) &= \int_0^\infty \Pr(L = t, L = W_1) e^{-\delta t} \left(\int_0^{b-u} \Phi_b(u + x) dG(x) + \int_{b-u}^\infty \Phi_b(b) dG(x) \right) dt \\ &\quad + \int_0^\infty \Pr(L = t, L = V_1) e^{-\delta t} \int_0^u \Phi_b(u - y) [e^{-\beta t} f_1(y) + (1 - e^{-\beta t}) f_2(y)] dy dt \\ &\quad + \int_0^\infty \Pr(L = t, L = V_1) e^{-\delta t} \int_u^\infty w(u, y - u) [e^{-\beta t} f_1(y) + (1 - e^{-\beta t}) f_2(y)] dy dt, \quad 0 \leq u \leq b.\end{aligned}\quad (3.5)$$

Setting (3.3) in (3.5) and by a bit of algebra, we can get the integral equation (3.4). This completes the proof of this theorem. \square

As an application of Eq. (3.4), we will derive the integral equation for the discounted deficit at ruin.

Corollary 3.1. For $0 \leq u \leq b$, the discounted deficit at ruin $\Psi_b(u)$ satisfies the following integral equation

$$\begin{aligned}\Psi_b(u) &= \frac{\lambda_2}{\lambda_1 + \lambda_2 + \delta} \left(\int_0^{b-u} \Psi_b(u + x) dG(x) + \Psi_b(b) \bar{G}(b - u) \right) \\ &\quad + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta + \beta} \left(\int_0^u \Psi_b(u - y) (f_1(y) - f_2(y)) dy + \int_u^\infty (y - u) (f_1(y) - f_2(y)) dy \right) \\ &\quad + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta} \left(\int_0^u \Psi_b(u - y) f_2(y) dy + \int_u^\infty (y - u) f_2(y) dy \right).\end{aligned}\quad (3.6)$$

Proof. Assume that $w(x_1, x_2) = x_2$, then the expected discounted penalty function $\Phi_b(u)$ is the discounted deficit at ruin $\Psi_b(u) = E[e^{-\delta T_b} | U(T_b) | I(T_b < \infty) | U_b(0) = u]$. Moreover, in this case, $\omega_i(u)$ can be simplified as

$$\omega_i(u) = \int_u^\infty (y - u) f_i(y) dy.$$

Substituting this result into Eq. (3.4), we can obtain the integral equation for $\Psi_b(u)$. \square

3.2. Analytic expressions for $u = b$

Now, we define an associated compound geometric distribution function $K(u) = 1 - \bar{K}(u)$ by

$$\bar{K}(u) = \frac{\epsilon}{1 + \epsilon} \sum_{n=1}^{\infty} \left(\frac{1}{1 + \epsilon} \right)^n \bar{H}^{*n}(u), \quad u \geq 0, \quad i = 1, 2,$$

where $\epsilon = \delta/\lambda_1$, $\bar{H}^{*n}(u)$ is the tail of the n -fold convolution of $H(u) = 1 - \bar{H}(u) = \int_0^u f^*(y) dy$.

Using Theorem 3.2, we are now in a position to derive the defective renewal equation for $\Phi_b(b)$. Based on the defective renewal equation, we will give analytic expressions for the solution of this equation through the associated compound geometric distribution.

Theorem 3.3. For $u = b$ and $\delta > 0$, the expected discounted penalty function $\Phi_b(b)$ can be expressed as

$$\Phi_b(b) = \frac{1}{\epsilon} \int_0^b [1 - \bar{K}(b - y)] dW(y) + \frac{W(0)}{\epsilon} [1 - \bar{K}(b)], \quad (3.7)$$

where $W(b) = [\omega^*(b)(\lambda_1 + \delta)]/\lambda_1$.

Proof. Assume that $u = b$, then Eq. (3.4) is simplified to

$$\Phi_b(b) = \frac{\lambda_1}{\lambda_1 + \delta} \int_0^b \Phi_b(b - y) f^*(y) dy + \omega^*(b), \quad (3.8)$$

where

$$f^*(y) = \frac{\lambda_1 + \lambda_2 + \delta}{\lambda_1 + \lambda_2 + \delta + \beta} \left(f_1(y) + \frac{\beta f_2(y)}{\lambda_1 + \lambda_2 + \delta} \right),$$

$$\omega^*(b) = \frac{\lambda_1(\lambda_1 + \lambda_2 + \delta)}{(\lambda_1 + \lambda_2 + \delta + \beta)(\lambda_1 + \delta)} \left(\omega_1(b) + \frac{\beta \omega_2(b)}{\lambda_1 + \lambda_2 + \delta} \right).$$

Since $\lambda_1/(\lambda_1 + \delta) < 1$, then Eq. (3.8) is an defective renewal equations. Applying Theorem 2.1 of Lin and Willmot [16], we can derive the analytic expression for the solution of Eq. (3.8). This completes the proof of this theorem. \square

Similarly, assume that $u = b$, then Eq. (3.1) is simplified to the defective renewal equation

$$V(b; b) = \frac{\lambda_1}{\lambda_1 + \delta} \int_0^b V(b - y; b) f^*(y) dy + \frac{\lambda_2 \mu_G}{\lambda_1 + \delta}. \quad (3.9)$$

Using similar arguments as in Theorem 3.3, we can derive the analytic expressions for the solution of Eq. (3.9).

Theorem 3.4. The expected discounted dividend payments until ruin $V(b; b)$ satisfying the defective renewal equation (3.9) can be expressed as

$$V(b, b) = \frac{\lambda_2 \mu_G}{\epsilon \lambda_1} (K(b) - K(0)) + \frac{\lambda_2 \mu_G}{(1 + \epsilon) \lambda_1}. \quad (3.10)$$

3.3. Volterra integral equations and exact representations for exponential premium

Now, we will find the exact representations for the solutions of integral equations (3.1) and (3.4). We pay attention to the situation in which the premium sizes are exponentially distributed with probability density function $g(x) = \alpha e^{-\alpha x}$, $x \geq 0$.

In this case, Eq. (3.4) is simplified to

$$\Phi_b(u) = \frac{\lambda_2}{\lambda_1 + \lambda_2 + \delta} \left(\alpha e^{\alpha u} \int_b^u \Phi_b(x) e^{-\alpha x} dx + \Phi_b(b) e^{-\alpha(b-u)} \right)$$

$$+ \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta + \beta} \int_0^u \Phi_b(u - y) f(y) dy + \omega(u), \quad (3.11)$$

where

$$f(y) = f_1(y) + \frac{\beta f_2(y)}{\lambda_1 + \lambda_2 + \delta}, \quad \omega^*(b) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta + \beta} \left(\omega_1(b) + \frac{\beta \omega_2(b)}{\lambda_1 + \lambda_2 + \delta} \right).$$

Differentiating the above equation with respect to u , we obtain for $0 \leq u \leq b$,

$$\Phi_b^{(1)}(u) = \frac{(\lambda_1 + \delta)\alpha}{\lambda_1 + \lambda_2 + \delta} \Phi_b(u) + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta + \beta} \left(\frac{d}{du} - \alpha \right) \left(\int_0^u \Phi_b(u-y)f(y)dy \right) + (\omega'(u) - \alpha\omega(u)). \quad (3.12)$$

Replacing u by x in (3.12) and then integrating both sides of the equation from 0 to u with respect to x , we obtain for $0 \leq u \leq b$,

$$\begin{aligned} \Phi_b(u) - \Phi_b(0) &= \frac{(\lambda_1 + \delta)\alpha}{\lambda_1 + \lambda_2 + \delta} \int_0^u \Phi_b(x)dx - \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta + \beta} \int_0^u \Phi_b(x)[\alpha Z(u-x) - f(u-x)]dx \\ &\quad + \int_0^u (\omega'(x) - \alpha\omega(x))dx, \end{aligned}$$

where $Z(u) = \int_0^u f(y)dy$.

Rearranging this equation, we have the following theorem.

Theorem 3.5. *If the premium size distribution G is an exponential distribution with mean $1/\alpha$, $\alpha > 0$. Then the integral equation (3.4) can be represented as the Volterra integral equation*

$$\Phi_b(u) = \int_0^u \Phi_b(x)p(u, x)dx + l(u), \quad 0 \leq x \leq u \leq b, \quad (3.13)$$

where

$$p(u, x) = \frac{(\lambda_1 + \delta)\alpha}{\lambda_1 + \lambda_2 + \delta} - \frac{\lambda_1(\alpha Z(u-x) - f(u-x))}{\lambda_1 + \lambda_2 + \delta + \beta}, \quad l(u) = \Phi_b(0) + \int_0^u (\omega'(x) - \alpha\omega(x))dx.$$

If $\Phi_b(0)$ is available, then the solution for $\Phi_b(u)$ is available. Therefore, we have to determine $\Phi_b(0)$. It is easy to verify that $l(u)$ is continuous in $0 \leq u \leq b$ since $w(x_1, x_2)$ is bounded and $\omega(u)$ is continuous. Obviously, $p(u, x)$ is continuous in $0 \leq x \leq u$ in that both $Z(x)$ and $f(x)$ are continuous functions. Then, according to Cai and Dickson [17], the unique solution for $\Phi_b(u)$ has the following representation, for $0 \leq u \leq b$,

$$\Phi_b(u) = l(u) + \sum_{m=1}^{\infty} \int_0^u p_m(u, x)l(x)dx, \quad (3.14)$$

where $p_m(u, x) = \int_x^u p(u, t)p_{m-1}(t, x)dt$, $m = 2, 3, \dots$, $0 \leq x \leq u$, with $p_1(u, x) = p(u, x)$. Setting $u = b$ in (3.14) and combining with (3.13), we see that

$$\Phi_b(0) = \frac{\Phi_b(b) - \int_0^b (\omega'(x) - \alpha\omega(x))dx - \int_0^b P(b, x) \int_0^x (\omega'(y) - \alpha\omega(y))dydx}{1 + \int_0^b P(b, x)dx}, \quad (3.15)$$

where $P(b, x) = \sum_{m=1}^{\infty} p_m(b, x)$, $0 \leq x \leq b$ and $\Phi_b(b)$ is given in (3.9).

Using the same techniques, we can obtain the Volterra integral equation for the expected discounted dividend payments until ruin $V(u; b)$.

Theorem 3.6. *If the premium size distribution G is an exponential distribution with mean $1/\alpha$, $\alpha > 0$. Then the integral equation (3.1) can be represented as the Volterra integral equation*

$$V(u, b) = \int_0^u V(x, b)p(u, x)dx + V(0, b), \quad 0 \leq x \leq u \leq b. \quad (3.16)$$

Furthermore, for $0 \leq u < b$,

$$V(u, b) = \frac{V(b, b) \left(1 + \int_0^u P(u, x)dx \right)}{1 + \int_0^b P(b, x)dx}. \quad (3.17)$$

4. Laplace transform of the time of ruin

Assume that $w(x_1, x_2) = 1$ and $\delta > 0$, then the expected discounted penalty function $\Phi_b(u)$ is the Laplace transform of the time of ruin $\psi_b(u) = E[e^{-\delta T_b} I(T_b < \infty) | U_b(0) = u]$. Generally, it is difficult to find the Laplace transform of the time of ruin analytically for general premium size distributions and claim size distributions. In the case of exponential claims and exponential premium amounts, an explicit expression for $\psi_b(u)$ can be obtained. In the rest of this paper, we assume that the individual premium sizes and claim sizes are exponentially distributed with p.d.f.

$$g(x) = \alpha e^{-\alpha x}, \quad f_1(x) = \alpha_1 e^{-\alpha_1 x}, \quad f_2(x) = \alpha_2 e^{-\alpha_2 x}, \quad x \geq 0. \quad (4.1)$$

Letting $w(x_1, x_2) = 1$ and substituting $z = u + x$ (or $u - y$), we can rewrite Eq. (3.4) as

$$\begin{aligned} \psi_b(u) = & \frac{\lambda_2}{\lambda_1 + \lambda_2 + \delta} \left(\int_0^{b-u} \psi_b(u+x)g(x)dx + \psi_b(b)e^{-\alpha(b-u)} \right) \\ & + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \beta + \delta} \left(\int_0^u \psi_b(u-y)(f_1(y) - f_2(y))dy + e^{-\alpha_1 u} - e^{-\alpha_2 u} \right) \\ & + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta} \left(\int_0^u \psi_b(u-y)f_2(y)dy + e^{-\alpha_2 u} \right). \end{aligned} \quad (4.2)$$

Differentiating Eq. (4.2) with respect to u , we get the following integro-differential equation:

$$\begin{aligned} \psi_b^{(1)}(u) = & \left(\frac{\lambda_1(\alpha_1 - \alpha_2)}{\lambda_1 + \lambda_2 + \delta + \beta} + \frac{\lambda_1\alpha_2 - \lambda_2\alpha}{\lambda_1 + \lambda_2 + \delta} \right) \psi_b(u) + \frac{\lambda_2\alpha}{\lambda_1 + \lambda_2 + \delta} \left(\int_u^b \psi_b(z)g(z-u)dz + \psi_b(b)e^{-\alpha(b-u)} \right) \\ & + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \beta + \delta} \left(-\alpha_1 \int_0^u \psi_b(z)f_1(u-z)dz + \alpha_2 \int_0^u \psi_b(z)f_2(u-z)dz - \alpha_1 e^{-\alpha_1 u} + \alpha_2 e^{-\alpha_2 u} \right) \\ & - \frac{\lambda_1\alpha_2}{\lambda_1 + \lambda_2 + \delta} \left(\int_0^u \psi_b(z)f_2(u-z)dz + e^{-\alpha_2 u} \right). \end{aligned} \quad (4.3)$$

Differentiating once again yields

$$\begin{aligned} \psi_b^{(2)}(u) = & \left(\frac{\lambda_1(\alpha_1 - \alpha_2)}{\lambda_1 + \lambda_2 + \beta + \delta} + \frac{\lambda_1\alpha_2 - \lambda_2\alpha}{\lambda_1 + \lambda_2 + \delta} \right) \psi_b^{(1)}(u) - \left(\frac{\lambda_1(\alpha_1^2 - \alpha_2^2)}{\lambda_1 + \lambda_2 + \beta + \delta} + \frac{\lambda_1\alpha_2^2 + \lambda_2\alpha^2}{\lambda_1 + \lambda_2 + \delta} \right) \psi_b(u) \\ & + \frac{\lambda_2\alpha^2}{\lambda_1 + \lambda_2 + \delta} \left(\int_u^b \psi_b(z)g(z-u)dz + \psi_b(b)e^{-\alpha(b-u)} \right) \\ & + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \beta + \delta} \left(\alpha_1^2 \int_0^u \psi_b(z)f_1(u-z)dz - \alpha_2^2 \int_0^u \psi_b(z)f_2(u-z)dz + \alpha_1^2 e^{-\alpha_1 u} - \alpha_2^2 e^{-\alpha_2 u} \right) \\ & + \frac{\lambda_1\alpha_2^2}{\lambda_1 + \lambda_2 + \delta} \left(\int_0^u \psi_b(z)f_2(u-z)dz + e^{-\alpha_2 u} \right). \end{aligned} \quad (4.4)$$

Furthermore, differentiating Eq. (4.4) with respect to u and using (4.2)–(4.4), we obtain

$$\begin{aligned} \psi_b^{(3)}(u) = & \frac{A_1}{(\lambda_1 + \lambda_2 + \delta)(\lambda_1 + \lambda_2 + \beta + \delta)} \psi_b^{(2)}(u) \\ & + \frac{A_2}{(\lambda_1 + \lambda_2 + \delta)(\lambda_1 + \lambda_2 + \beta + \delta)} \psi_b^{(1)}(u) + \frac{\alpha\alpha_1\alpha_2\delta}{\lambda_1 + \lambda_2 + \delta} \psi_b(u), \end{aligned} \quad (4.5)$$

where $A_1 = \alpha(\lambda_1 + \delta)(\lambda_1 + \lambda_2 + \beta + \delta) - \alpha_1(\lambda_1 + \lambda_2 + \delta)(\lambda_2 + \beta + \delta) - \alpha_2[(\lambda_1 + \lambda_2 + \delta)^2 + \beta(\lambda_2 + \delta)]$, $A_2 = \alpha\alpha_1(\beta(\lambda_1 + \delta) + \delta(\lambda_1 + \lambda_2 + \delta)) + \alpha\alpha_2(\beta\delta + (\lambda_1 + \delta)(\lambda_1 + \lambda_2 + \delta)) - \alpha_1\alpha_2(\lambda_2 + \delta)(\lambda_1 + \lambda_2 + \beta + \delta)$.

Its characteristic equation

$$z^3 = \frac{A_1}{(\lambda_1 + \lambda_2 + \delta)(\lambda_1 + \lambda_2 + \beta + \delta)} z^2 + \frac{A_2}{(\lambda_1 + \lambda_2 + \delta)(\lambda_1 + \lambda_2 + \beta + \delta)} z + \frac{\alpha\alpha_1\alpha_2\delta}{\lambda_1 + \lambda_2 + \delta}, \quad (4.6)$$

has three roots, namely z_1, z_2, z_3 . It is easy to see that $z_1 \rightarrow 0$ as $\delta \rightarrow 0$. Therefore, the general solution for $\psi_b(u)$ is

$$\psi_b(u) = C_1 e^{z_1 u} + C_2 e^{z_2 u} + C_3 e^{z_3 u}, \quad 0 \leq u \leq b. \quad (4.7)$$

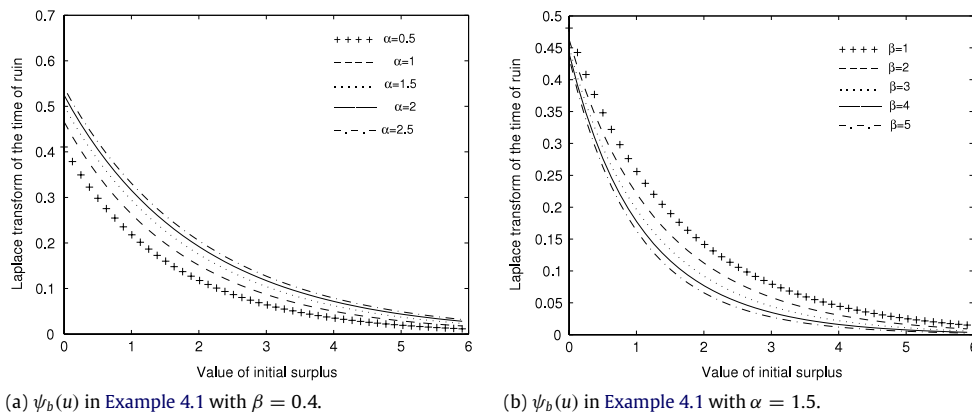


Fig. 1. Laplace transform of the time of ruin in Example 4.1.

Next we want to determine the coefficients C_1 , C_2 and C_3 . Substituting (4.7) into (4.2), we have

$$\begin{aligned}
 & e^{uz_1} C_1 \left(\frac{\lambda_2 \alpha}{(\lambda_1 + \lambda_2 + \delta)(\alpha - z_1)} + \frac{\lambda_1 \left(\Delta_1 + \frac{\beta \Lambda_1}{\lambda_1 + \lambda_2 + \delta} \right)}{\lambda_1 + \lambda_2 + \beta + \delta} - 1 \right) + \frac{e^{-u\alpha_1} \lambda_1 (1 - (C_1 \Delta_1 + C_2 \Delta_2 + C_3 \Delta_3))}{\lambda_1 + \lambda_2 + \delta + \beta} \\
 & + e^{uz_2} C_2 \left(\frac{\lambda_2 \alpha}{(\lambda_1 + \lambda_2 + \delta)(\alpha - z_2)} + \frac{\lambda_1 \left(\Delta_2 + \frac{\beta \Lambda_2}{\lambda_1 + \lambda_2 + \delta} \right)}{\lambda_1 + \lambda_2 + \beta + \delta} - 1 \right) + \frac{e^{-u\alpha_2} \lambda_1 \beta (1 - (C_1 \Lambda_1 + C_2 \Lambda_2 + C_3 \Lambda_3))}{(\lambda_1 + \lambda_2 + \delta)(\lambda_1 + \lambda_2 + \delta + \beta)} \\
 & + e^{uz_3} C_3 \left(\frac{\lambda_2 \alpha}{(\lambda_1 + \lambda_2 + \delta)(\alpha - z_3)} + \frac{\lambda_1 \left(\Delta_3 + \frac{\beta \Lambda_3}{\lambda_1 + \lambda_2 + \delta} \right)}{\lambda_1 + \lambda_2 + \beta + \delta} - 1 \right) + \frac{\lambda_2 e^{-(b-u)\alpha}}{\lambda_1 + \lambda_2 + \delta} (C_1 \Gamma_1 + C_2 \Gamma_2 + C_3 \Gamma_3) = 0,
 \end{aligned}$$

where

$$\Delta_i = \frac{\alpha_1}{z_i + \alpha_1}, \quad \Lambda_i = \frac{\alpha_2}{z_i + \alpha_2}, \quad \Gamma_i = \frac{e^{bz_i} z_i}{z_i - \alpha}, \quad i = 1, 2, 3.$$

The above equation holds for all $0 \leq u \leq b$. Hence, comparing the coefficients of $e^{-u\alpha_1}$, $e^{-u\alpha_2}$ and $e^{-(b-u)\alpha}$, we see that

$$\begin{cases} 1 - (C_1 \Delta_1 + C_2 \Delta_2 + C_3 \Delta_3) = 0, \\ 1 - (C_1 \Lambda_1 + C_2 \Lambda_2 + C_3 \Lambda_3) = 0, \\ C_1 \Gamma_1 + C_2 \Gamma_2 + C_3 \Gamma_3 = 0. \end{cases} \quad (4.8)$$

Solving the system of equations (4.8) gives

$$\begin{cases} C_1 = -\frac{(\Gamma_2 \Delta_3 - \Gamma_3 \Delta_2)(\Lambda_2 - \Delta_2) + \Gamma_2(\Delta_2 \Lambda_3 - \Delta_3 \Lambda_2)}{(\Gamma_2 \Delta_3 - \Gamma_3 \Delta_2)(\Delta_2 \Lambda_1 - \Delta_1 \Lambda_2) - (\Gamma_2 \Delta_1 - \Gamma_1 \Delta_2)(\Delta_2 \Lambda_3 - \Delta_3 \Lambda_2)}, \\ C_2 = -\frac{(\Gamma_1 \Delta_3 - \Gamma_3 \Delta_1)(\Lambda_1 - \Delta_1) + \Gamma_1(\Delta_1 \Lambda_3 - \Delta_3 \Lambda_1)}{(\Gamma_1 \Delta_3 - \Gamma_3 \Delta_1)(\Delta_1 \Lambda_2 - \Delta_2 \Lambda_1) - (\Gamma_1 \Delta_2 - \Gamma_2 \Delta_1)(\Delta_1 \Lambda_3 - \Delta_3 \Lambda_1)}, \\ C_3 = -\frac{(\Gamma_1 \Delta_2 - \Gamma_2 \Delta_1)(\Lambda_1 - \Delta_1) + \Gamma_1(\Delta_1 \Lambda_2 - \Delta_2 \Lambda_1)}{(\Gamma_1 \Delta_2 - \Gamma_2 \Delta_1)(\Delta_1 \Lambda_3 - \Delta_3 \Lambda_1) - (\Gamma_1 \Delta_3 - \Gamma_3 \Delta_1)(\Delta_1 \Lambda_2 - \Delta_2 \Lambda_1)}. \end{cases} \quad (4.9)$$

Example 4.1. Assume that $\delta = 0.6$, $\lambda_1 = 1$, $\lambda_2 = 1.5$, $\alpha_1 = 1$, $\alpha_2 = 2$, $b = 10$. From (4.7) and (4.9), we can calculate numerical values of $\psi_b(u)$. Fig. 1(a) shows the Laplace transforms of the time of ruin $\psi_b(u)$ with $\beta = 0.4$, for different values of $u \in [0, 6]$ and $\alpha = 0.5, 1, 1.5, 2, 2.5$. From this graph, we can see that, in this case, these Laplace transforms of the time of ruin decrease as the initial surplus u increases. Moreover, with fixed u , Laplace transforms of the time of ruin increase as α increases. Fig. 1(b) shows the Laplace transforms of the time of ruin $\psi_b(u)$ with $\alpha = 1.5$, for different values of $u \in [0, 6]$ and $\beta = 1, 2, 3, 4, 5$. In this case, we can see that, with fixed u , Laplace transforms of the time of ruin decrease as β increases.

Remark 4.1. When $\delta = 0$, then $z_1 = 0$ and the Laplace transform of the time of ruin $\psi_b(u)$ is the ruin probability $\phi_b(u)$. Moreover, solving the system of Eq. (4.8) in this case gives $C_1 = 1$, $C_2 = 0$, $C_3 = 0$, and $\phi_b(u) = 1$, $0 \leq u \leq b$. This illustrates that the ruin is certain when there is a horizontal barrier $b < \infty$. Then the survival probability $\varphi_b(u) = 1 - \phi_b(u) = 0$.

5. The optimal dividend barrier

Dickson and Waters [6] argued that the shareholders should be obliged to cover the deficit at ruin. Consequently, the shareholders will want to maximize the expectation of the difference of the discounted dividends until ruin and discounted deficit at ruin. In other words, b is chosen to maximize

$$M(u, b) = V(u, b) - \Psi_b(u), \quad (5.1)$$

for given $0 \leq u \leq b$. We denote the optimal value of b by b^* and assume that $b^* > 0$. It follows that

$$\frac{\partial}{\partial b} M(u, b) \Big|_{b=b^*} = 0. \quad (5.2)$$

We still assume that the individual premium sizes and claim sizes are exponentially distributed with p.d.f. $g(x) = \alpha e^{-\alpha x}$, $f_1(x) = \alpha_1 e^{-\alpha_1 x}$, $f_2(x) = \alpha_2 e^{-\alpha_2 x}$, $x \geq 0$.

Substituting $z = u + x$ (or $u - y$), we can rewrite Eq. (3.1) as

$$\begin{aligned} V(u, b) = & \frac{\lambda_2}{\lambda_1 + \lambda_2 + \delta} \left(\int_u^b V(z, b)g(z-u)dz + \int_b^\infty (z-b+V(b, b))g(z-u)dz \right) \\ & + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta + \beta} \left(\int_0^u V(z, b)f_1(u-z)dz - \int_0^u V(z, b)f_2(u-z)dz \right) \\ & + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta} \int_0^u V(z, b)f_2(u-z)dz. \end{aligned} \quad (5.3)$$

Differentiating Eq. (5.3) with respect to u , we get the following integro-differential equation:

$$\begin{aligned} V^{(1)}(u, b) = & \left(\frac{\lambda_1(\alpha_1 - \alpha_2)}{\lambda_1 + \lambda_2 + \delta + \beta} + \frac{\lambda_1\alpha_2 - \lambda_2\alpha}{\lambda_1 + \lambda_2 + \delta} \right) V(u, b) \\ & + \frac{\lambda_2\alpha}{\lambda_1 + \lambda_2 + \delta} \left(\int_u^b V(z, b)g(z-u)dz + \int_b^\infty (z-b+V(b, b))g(z-u)dz \right) \\ & + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \beta + \delta} \left(-\alpha_1 \int_0^u V(z, b)f_1(u-z)dz + \alpha_2 \int_0^u V(z, b)f_2(u-z)dz \right) \\ & - \frac{\lambda_1\alpha_2}{\lambda_1 + \lambda_2 + \delta} \int_0^u V(z, b)f_2(u-z)dz. \end{aligned} \quad (5.4)$$

Differentiating once again yields

$$\begin{aligned} V^{(2)}(u, b) = & \left(\frac{\lambda_1(\alpha_1 - \alpha_2)}{\lambda_1 + \lambda_2 + \delta + \beta} + \frac{\lambda_1\alpha_2 - \lambda_2\alpha}{\lambda_1 + \lambda_2 + \delta} \right) V^{(1)}(u, b) \\ & - \left(\frac{\lambda_1(\alpha_1^2 - \alpha_2^2)}{\lambda_1 + \lambda_2 + \delta + \beta} + \frac{\lambda_1\alpha_2^2 + \lambda_2\alpha^2}{\lambda_1 + \lambda_2 + \delta} \right) V(u, b) \\ & + \frac{\lambda_2\alpha^2}{\lambda_1 + \lambda_2 + \delta} \left(\int_u^b V(z, b)g(z-u)dz + \int_b^\infty (z-b+V(b, b))g(z-u)dz \right) \\ & + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \beta + \delta} \left(\alpha_1^2 \int_0^u V(z, b)f_1(u-z)dz - \alpha_2^2 \int_0^u V(z, b)f_2(u-z)dz \right) \\ & + \frac{\lambda_1\alpha_2^2}{\lambda_1 + \lambda_2 + \delta} \int_0^u V(z, b)f_2(u-z)dz. \end{aligned} \quad (5.5)$$

Furthermore, differentiating Eq. (5.5) with respect to u and using (5.3) and (5.4), we obtain

$$\begin{aligned} V^{(3)}(u, b) = & \frac{A_1}{(\lambda_1 + \lambda_2 + \delta)(\lambda_1 + \lambda_2 + \beta + \delta)} V^{(2)}(u, b) \\ & + \frac{A_2}{(\lambda_1 + \lambda_2 + \delta)(\lambda_1 + \lambda_2 + \beta + \delta)} V^{(1)}(u, b) + \frac{\alpha\alpha_1\alpha_2\delta}{\lambda_1 + \lambda_2 + \delta} V(u, b), \end{aligned} \quad (5.6)$$

where A_1 and A_2 are defined in Eq. (4.5). Its characteristic equation is Eq. (4.6). Then the general solution for $V(u, b)$ is

$$V(u, b) = D_1(b)e^{z_1 u} + D_2(b)e^{z_2 u} + D_3(b)e^{z_3 u}, \quad 0 \leq u \leq b. \quad (5.7)$$

Substituting (5.7) into (5.3), we have

$$\begin{aligned}
 & e^{uz_1} D_1(b) \left(\frac{\lambda_2 \alpha}{(\lambda_1 + \lambda_2 + \delta)(\alpha - z_1)} + \frac{\lambda_1 \left(\Delta_1 + \frac{\beta \Lambda_1}{\lambda_1 + \lambda_2 + \delta} \right)}{\lambda_1 + \lambda_2 + \beta + \delta} - 1 \right) \\
 & - \frac{e^{-u\alpha_1} \lambda_1 (D_1(b) \Delta_1 + D_2(b) \Delta_2 + D_3(b) \Delta_3)}{\lambda_1 + \lambda_2 + \delta + \beta} \\
 & + e^{uz_2} C_2 \left(\frac{\lambda_2 \alpha}{(\lambda_1 + \lambda_2 + \delta)(\alpha - z_2)} + \frac{\lambda_1 \left(\Delta_2 + \frac{\beta \Lambda_2}{\lambda_1 + \lambda_2 + \delta} \right)}{\lambda_1 + \lambda_2 + \beta + \delta} - 1 \right) \\
 & - \frac{e^{-u\alpha_2} \lambda_1 \beta (D_1(b) \Lambda_1 + D_2(b) \Lambda_2 + D_3(b) \Lambda_3)}{(\lambda_1 + \lambda_2 + \delta)(\lambda_1 + \lambda_2 + \delta + \beta)} \\
 & + e^{uz_3} C_3 \left(\frac{\lambda_2 \alpha}{(\lambda_1 + \lambda_2 + \delta)(\alpha - z_3)} + \frac{\lambda_1 \left(\Delta_3 + \frac{\beta \Lambda_2}{\lambda_1 + \lambda_2 + \delta} \right)}{\lambda_1 + \lambda_2 + \beta + \delta} - 1 \right) \\
 & + \frac{\lambda_2 e^{-(b-u)\alpha}}{\lambda_1 + \lambda_2 + \delta} \left(\frac{1}{\alpha} + D_1(b) \Gamma_1 + D_2(b) \Gamma_2 + D_3(b) \Gamma_3 \right) = 0.
 \end{aligned}$$

Comparing the coefficients of $e^{-u\alpha_1}$, $e^{-u\alpha_2}$ and $e^{-(b+u)\alpha}$, yields

$$\begin{cases} D_1(b) \Delta_1 + D_2(b) \Delta_2 + D_3(b) \Delta_3 = 0, \\ D_1(b) \Lambda_1 + D_2(b) \Lambda_2 + D_3(b) \Lambda_3 = 0, \\ \frac{1}{\alpha} + D_1(b) \Gamma_1 + D_2(b) \Gamma_2 + D_3(b) \Gamma_3 = 0. \end{cases} \quad (5.8)$$

The solution to (5.8) is

$$\begin{cases} D_1(b) = \frac{\Delta_2(\Delta_2 \Lambda_3 - \Delta_3 \Lambda_2)}{\alpha \{(\Gamma_3 \Delta_2 - \Gamma_2 \Delta_3)(\Delta_2 \Lambda_1 - \Delta_1 \Lambda_2) - (\Gamma_1 \Delta_2 - \Gamma_2 \Delta_1)(\Delta_2 \Lambda_3 - \Delta_3 \Lambda_2)\}}, \\ D_2(b) = \frac{\Delta_1(\Delta_1 \Lambda_3 - \Delta_3 \Lambda_1)}{\alpha \{(\Gamma_3 \Delta_1 - \Gamma_1 \Delta_3)(\Delta_1 \Lambda_2 - \Delta_2 \Lambda_1) - (\Gamma_2 \Delta_1 - \Gamma_1 \Delta_2)(\Delta_1 \Lambda_3 - \Delta_3 \Lambda_1)\}}, \\ D_3(b) = \frac{\Delta_1(\Delta_1 \Lambda_2 - \Delta_2 \Lambda_1)}{\alpha \{(\Gamma_2 \Delta_1 - \Gamma_1 \Delta_2)(\Delta_1 \Lambda_3 - \Delta_3 \Lambda_1) - (\Gamma_3 \Delta_1 - \Gamma_1 \Delta_3)(\Delta_1 \Lambda_2 - \Delta_2 \Lambda_1)\}}. \end{cases} \quad (5.9)$$

Using the same techniques, we have

$$\begin{aligned}
 \Psi_b^{(3)}(u) &= \frac{A_1}{(\lambda_1 + \lambda_2 + \delta)(\lambda_1 + \lambda_2 + \beta + \delta)} \Psi_b^{(2)}(u) \\
 &+ \frac{A_2}{(\lambda_1 + \lambda_2 + \delta)(\lambda_1 + \lambda_2 + \beta + \delta)} \Psi_b^{(1)}(u) + \frac{\alpha \alpha_1 \alpha_2 \delta}{\lambda_1 + \lambda_2 + \delta} \Psi_b(u),
 \end{aligned} \quad (5.10)$$

and

$$\Psi_b(u) = E_1(b) e^{z_1 u} + E_2(b) e^{z_2 u} + E_3(b) e^{z_3 u}, \quad 0 \leq u \leq b. \quad (5.11)$$

Similarly, substituting (5.11) into (3.6), we have

$$\begin{aligned}
 & \frac{e^{-u\alpha_1} \lambda_1 (1 - \alpha_1 (E_1(b) \Delta_1 + E_2(b) \Delta_2 + E_3(b) \Delta_3))}{\alpha_1 (\lambda_1 + \lambda_2 + \delta + \beta)} + \frac{e^{-u\alpha_2} \lambda_1 \beta (1 - \alpha_2 (E_1(b) \Lambda_1 + E_2(b) \Lambda_2 + E_3(b) \Lambda_3))}{\alpha_2 (\lambda_1 + \lambda_2 + \delta)(\lambda_1 + \lambda_2 + \delta + \beta)} \\
 & + e^{uz_1} E_1(b) \left(\frac{\lambda_2 \alpha}{(\lambda_1 + \lambda_2 + \delta)(\alpha - z_1)} + \frac{\lambda_1 \left(\Delta_1 + \frac{\beta \Lambda_1}{\lambda_1 + \lambda_2 + \delta} \right)}{\lambda_1 + \lambda_2 + \beta + \delta} - 1 \right) \\
 & + e^{uz_2} E_2(b) \left(\frac{\lambda_2 \alpha}{(\lambda_1 + \lambda_2 + \delta)(\alpha - z_2)} + \frac{\lambda_1 \left(\Delta_2 + \frac{\beta \Lambda_2}{\lambda_1 + \lambda_2 + \delta} \right)}{\lambda_1 + \lambda_2 + \beta + \delta} - 1 \right) \\
 & + e^{uz_3} E_3(b) \left(\frac{\lambda_2 \alpha}{(\lambda_1 + \lambda_2 + \delta)(\alpha - z_3)} + \frac{\lambda_1 \left(\Delta_3 + \frac{\beta \Lambda_2}{\lambda_1 + \lambda_2 + \delta} \right)}{\lambda_1 + \lambda_2 + \beta + \delta} - 1 \right) \\
 & + \frac{\lambda_2 e^{-(b-u)\alpha}}{\lambda_1 + \lambda_2 + \delta} (E_1(b) \Gamma_1 + E_2(b) \Gamma_2 + E_3(b) \Gamma_3) = 0,
 \end{aligned}$$

Hence, comparing the coefficients of $e^{-u\alpha_1}$, $e^{-u\alpha_2}$ and $e^{(-b+u)\alpha}$, we see that

$$\begin{cases} 1 - \alpha_1(E_1(b)\Delta_1 + E_2(b)\Delta_2 + E_3(b)\Delta_3) = 0, \\ 1 - \alpha_2(E_1(b)\Delta_1 + E_2(b)\Delta_2 + E_3(b)\Delta_3) = 0, \\ E_1(b)\Gamma_1 + E_2(b)\Gamma_2 + E_3(b)\Gamma_3 = 0. \end{cases} \quad (5.12)$$

Solving the system of equations (5.12) gives

$$\begin{cases} E_1(b) = -\frac{(\Gamma_2\Delta_3 - \Gamma_3\Delta_2)(\Delta_2\alpha_2 - \Delta_2\alpha_1) + \Gamma_2\alpha_2(\Delta_2\Delta_3 - \Delta_3\Delta_2)}{(\Gamma_2\Delta_3 - \Gamma_3\Delta_2)(\Delta_2\Delta_1 - \Delta_1\Delta_2)\alpha_1\alpha_2 - (\Gamma_2\Delta_1 - \Gamma_1\Delta_2)(\Delta_2\Delta_3 - \Delta_3\Delta_2)\alpha_1\alpha_2}, \\ E_2(b) = -\frac{(\Gamma_1\Delta_3 - \Gamma_3\Delta_1)(\Delta_1\alpha_2 - \Delta_1\alpha_1) + \Gamma_1\alpha_2(\Delta_1\Delta_3 - \Delta_3\Delta_1)}{(\Gamma_1\Delta_3 - \Gamma_3\Delta_1)(\Delta_1\Delta_2 - \Delta_2\Delta_1)\alpha_1\alpha_2 - (\Gamma_1\Delta_2 - \Gamma_2\Delta_1)(\Delta_1\Delta_3 - \Delta_3\Delta_1)\alpha_1\alpha_2}, \\ E_3(b) = -\frac{(\Gamma_1\Delta_2 - \Gamma_2\Delta_1)(\Delta_1\alpha_2 - \Delta_1\alpha_1) + \Gamma_1\alpha_2(\Delta_1\Delta_2 - \Delta_2\Delta_1)}{(\Gamma_1\Delta_2 - \Gamma_2\Delta_1)(\Delta_1\Delta_3 - \Delta_3\Delta_1)\alpha_1\alpha_2 - (\Gamma_1\Delta_3 - \Gamma_3\Delta_1)(\Delta_1\Delta_2 - \Delta_2\Delta_1)\alpha_1\alpha_2}. \end{cases} \quad (5.13)$$

Theorem 5.1. The condition for the optimal value b^* satisfies

$$\begin{aligned} & -\alpha(D_1(b^*))^2 \frac{\alpha \{(\bar{\Gamma}_3\Delta_2\Delta_3 - \bar{\Gamma}_2\Delta_3\Delta_2)(\Delta_2\Delta_1 - \Delta_1\Delta_2) - (\bar{\Gamma}_1\Delta_2\Delta_1 - \bar{\Gamma}_2\Delta_1\Delta_2)(\Delta_2\Delta_3 - \Delta_3\Delta_2)\}}{\Delta_2(\Delta_2\Delta_3 - \Delta_3\Delta_2)} \\ & = E_1(b^*) \frac{(\bar{\Gamma}_2\Delta_3\Delta_2 - \bar{\Gamma}_3\Delta_2\Delta_3)(\Delta_2\alpha_2 - \Delta_2\alpha_1) + \bar{\Gamma}_2\alpha_2(\Delta_2\Delta_3 - \Delta_3\Delta_2)}{(\bar{\Gamma}_2\Delta_3 - \bar{\Gamma}_3\Delta_2)(\Delta_2\Delta_1 - \Delta_1\Delta_2)\alpha_1\alpha_2 - (\bar{\Gamma}_2\Delta_1 - \bar{\Gamma}_1\Delta_2)(\Delta_2\Delta_3 - \Delta_3\Delta_2)\alpha_1\alpha_2} \\ & - (E_1(b^*))^2 \frac{(\bar{\Gamma}_2\Delta_3\Delta_2 - \bar{\Gamma}_3\Delta_2\Delta_3)(\Delta_2\Delta_1 - \Delta_1\Delta_2) - (\bar{\Gamma}_2\Delta_1\Delta_2 - \bar{\Gamma}_1\Delta_2\Delta_1)(\Delta_2\Delta_3 - \Delta_3\Delta_2)}{(\bar{\Gamma}_2\Delta_3 - \bar{\Gamma}_3\Delta_2)(\Delta_2\Delta_1 - \Delta_1\Delta_2) - (\bar{\Gamma}_2\Delta_1 - \bar{\Gamma}_1\Delta_2)(\Delta_2\Delta_3 - \Delta_3\Delta_2)}, \end{aligned} \quad (5.14)$$

or

$$\begin{aligned} & -\alpha(D_2(b^*))^2 \frac{\alpha \{(\bar{\Gamma}_3\Delta_1\Delta_3 - \bar{\Gamma}_1\Delta_3\Delta_1)(\Delta_1\Delta_2 - \Delta_2\Delta_1) - (\bar{\Gamma}_2\Delta_1\Delta_2 - \bar{\Gamma}_1\Delta_2\Delta_1)(\Delta_1\Delta_3 - \Delta_3\Delta_1)\}}{\Delta_1(\Delta_1\Delta_3 - \Delta_3\Delta_1)} \\ & = E_2(b^*) \frac{(\bar{\Gamma}_1\Delta_3\Delta_1 - \bar{\Gamma}_3\Delta_1\Delta_3)(\Delta_1\alpha_2 - \Delta_1\alpha_1) + \bar{\Gamma}_1\alpha_2(\Delta_1\Delta_3 - \Delta_3\Delta_1)}{(\bar{\Gamma}_1\Delta_3 - \bar{\Gamma}_3\Delta_1)(\Delta_1\Delta_2 - \Delta_2\Delta_1)\alpha_1\alpha_2 - (\bar{\Gamma}_1\Delta_2 - \bar{\Gamma}_2\Delta_1)(\Delta_1\Delta_3 - \Delta_3\Delta_1)\alpha_1\alpha_2} \\ & - (E_2(b^*))^2 \frac{(\bar{\Gamma}_1\Delta_3\Delta_1 - \bar{\Gamma}_3\Delta_1\Delta_3)(\Delta_1\Delta_2 - \Delta_2\Delta_1) - (\bar{\Gamma}_1\Delta_2\Delta_1 - \bar{\Gamma}_2\Delta_1\Delta_2)(\Delta_1\Delta_3 - \Delta_3\Delta_1)}{(\bar{\Gamma}_1\Delta_3 - \bar{\Gamma}_3\Delta_1)(\Delta_1\Delta_2 - \Delta_2\Delta_1) - (\bar{\Gamma}_1\Delta_2 - \bar{\Gamma}_2\Delta_1)(\Delta_1\Delta_3 - \Delta_3\Delta_1)}, \end{aligned} \quad (5.15)$$

or

$$\begin{aligned} & -\alpha(D_3(b^*))^2 \frac{\alpha \{(\bar{\Gamma}_1\Delta_3\Delta_1 - \bar{\Gamma}_3\Delta_1\Delta_3)(\Delta_1\Delta_2 - \Delta_2\Delta_1) - (\bar{\Gamma}_2\Delta_1\Delta_2 - \bar{\Gamma}_1\Delta_2\Delta_1)(\Delta_1\Delta_3 - \Delta_3\Delta_1)\}}{\Delta_1(\Delta_1\Delta_2 - \Delta_2\Delta_1)} \\ & = E_3(b^*) \frac{(\bar{\Gamma}_1\Delta_2\Delta_1 - \bar{\Gamma}_2\Delta_1\Delta_2)(\Delta_1\alpha_2 - \Delta_1\alpha_1) + \bar{\Gamma}_1\alpha_2(\Delta_1\Delta_3 - \Delta_3\Delta_1)}{(\bar{\Gamma}_1\Delta_2 - \bar{\Gamma}_2\Delta_1)(\Delta_1\Delta_3 - \Delta_3\Delta_1)\alpha_1\alpha_2 - (\bar{\Gamma}_1\Delta_3 - \bar{\Gamma}_3\Delta_1)(\Delta_1\Delta_2 - \Delta_2\Delta_1)\alpha_1\alpha_2} \\ & - (E_3(b^*))^2 \frac{(\bar{\Gamma}_1\Delta_3\Delta_1 - \bar{\Gamma}_3\Delta_1\Delta_3)(\Delta_1\Delta_2 - \Delta_2\Delta_1) + (\bar{\Gamma}_1\Delta_2\Delta_1 - \bar{\Gamma}_2\Delta_1\Delta_2)(\Delta_1\Delta_3 - \Delta_3\Delta_1)}{(\bar{\Gamma}_1\Delta_2 - \bar{\Gamma}_2\Delta_1)(\Delta_1\Delta_3 - \Delta_3\Delta_1) - (\bar{\Gamma}_1\Delta_3 - \bar{\Gamma}_3\Delta_1)(\Delta_1\Delta_2 - \Delta_2\Delta_1)}, \end{aligned} \quad (5.16)$$

where

$$\bar{\Gamma}_i = \frac{e^{b^*z_i}z_i}{z_i - \alpha}, \quad i = 1, 2, 3.$$

Proof. From (5.2), we can know that the condition for b^* can be represented as

$$\begin{aligned} & \left(\frac{dD_1(b)}{db} \Big|_{b=b^*} - \frac{dE_1(b)}{db} \Big|_{b=b^*} \right) e^{z_1u} + \left(\frac{dD_2(b)}{db} \Big|_{b=b^*} - \frac{dE_2(b)}{db} \Big|_{b=b^*} \right) e^{z_2u} \\ & \left(+ \frac{dD_3(b)}{db} \Big|_{b=b^*} - \frac{dE_3(b)}{db} \Big|_{b=b^*} \right) e^{z_3u} = 0. \end{aligned} \quad (5.17)$$

Consequently, we have the following three equivalent conditions

$$\frac{dD_i(b)}{db} \Big|_{b=b^*} - \frac{dE_i(b)}{db} \Big|_{b=b^*} = 0, \quad i = 1, 2, 3. \quad (5.18)$$

Then, after some calculations we can get the results of Theorem 5.1. \square

6. Concluding remarks

In this paper, we have studied a risk model with interclaim-dependence claim sizes, where the distribution of the next claim size depends on the last interarrival time. Provided that the aggregate premium process driven by a compound Poisson process, we consider the constant barrier strategy under this dependent risk model with random incomes. The Volterra integral equations for the expected discounted penalty function and the expected discounted dividend payments until ruin are derived and solved when the individual stochastic premium amount is exponentially distributed. Furthermore, the condition of the optimal barrier is calculated when the individual stochastic premium amount and claim amount are exponentially distributed.

In future work, it can be shown that the individual stochastic premium amount and claim amount are Erlang(n), or more generally, from the $K_n(n \in \mathbf{N}^+)$ family.

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