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Discrete C^1 convergence of linear multistep methods

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ABSTRACT

We prove that strongly stable linear multistep methods are convergent in the discrete C^1 norm.

1. Stability and convergence of linear multistep methods

Linear multistep methods are one of the most frequent choices to approximate the solution of initial value problems. Their convergence is investigated almost only with respect to the infinity norm (or some variants of it), see e.g. the "bible" of numerical solution of initial value problems [1]. Our focus is on a different type of convergence. We use the kC1 norm which is defined as

$$\|\mathbf{u}_N\|_{kC1} = \max_{0\le i<k} |u_i| + \max_{k\le i<N} \frac{1}{h} |u_i - u_{i-1}| \ .$$

This norm can capture the possible spurious oscillations of the numerical approximation which are typical e.g. for the weakly stable linear multistep methods. The name refers to its continuous counterpart $\|u\|_{C^1} = |u(0)| + \max |\dot{u}(t)|$.

We organized the paper as follows. After this introductory part in Section 2 we state and prove our main result about the kC1 convergence of LMMs. Finally, in Section 3 we list several remarks.

Without loss of generality we consider the scalar autonomous *initial value problem* (IVP)

$$\begin{cases} u(0) = u^0 \,, \\ \dot{u}(t) = f(u(t)) \,, \end{cases} \tag{1}$$

where $t \in [0, T]$, $u^0 \in \mathbb{R}$ is the initial value, $u : [0, T] \to \mathbb{R}$ is the unknown function and we assume that f is Lipschitz continuous.

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In practice we have to use a numerical method to approximate the solution of (1) since finding the solution analytically is impossible in most of the cases. There are many possible choices, one is the application of a linear multistep method (LMM).

Linear multistep methods can be given in the following way:

$$\begin{cases} u_i = c^i\,, & i = 0, \dots, k-1 \\ \dfrac{1}{h}\sum_{j=0}^{k} \alpha_j u_{i-j} = \sum_{j=0}^{k} \beta_j f(u_{i-j})\,, & i = k, \dots, n+k-1 = N\,, \end{cases} \tag{2}$$

where $h = T/N$ is the step size, α_j, $\beta_j \in \mathbb{R}$, $\alpha_0 \neq 0$ are the coefficients of the method and the constants c^i are some approximations of the solution on the first k time levels. When the latter ones are known (usually calculated by a one-step method or recursively by lower order members from the same family of the LMM we want to use) the method can "run", we can calculate the next approximation and so on. To get u_i which approximates the solution at the ith time level $u(i \cdot h)$, we only need to know the previous k approximations. Thus the formula represents a k-step method. Note that while k is fixed for the method, n, $N = k + n - 1$ and h can vary as the grid gets finer.

The *first characteristic polynomial* associated to (2) is defined as

$$\varrho(z) = \sum_{j=0}^{k} \alpha_j z^{k-j}\,.$$

Usually, two types of root-conditions are defined.

Definition 1.1. The method is said to be *strongly stable* if for every root $\xi_i \in \mathbb{C}$ of the first characteristic polynomial $|\xi_i| < 1$ holds except for $\xi_1 = 1$, which is a simple root.

A not strongly stable method is said to be *weakly stable* if for every root $\xi_i \in \mathbb{C}$ of the first characteristic polynomial $|\xi_i| \leq 1$ holds and if $|\xi_i| = 1$, then it is a simple root, moreover $\xi_1 = 1$.

In the following we rewrite LMMs (2) into the form for which we can define stability in the general sense [2,3]. A method can be represented with a sequence of operators $F_N : \mathcal{X}_N \to \mathcal{Y}_N$, where $\mathcal{X}_N$, $\mathcal{Y}_N$ are $k+n$ dimensional normed spaces with norms $\|\cdot\|_{\mathcal{X}_N}$, $\|\cdot\|_{\mathcal{Y}_N}$ respectively and

$$(F_N(\mathbf{u}_N))_i = \begin{cases} u_i - c^i\,, & i = 0, \dots, k-1 \\ \dfrac{1}{h}\sum_{j=0}^{k} \alpha_j u_{i-j} - \sum_{j=0}^{k} \beta_j f(u_{i-j})\,, & i = k, \dots, n+k-1 = N\,. \end{cases}$$

Finding the approximating solution means that we have to solve the non-linear system of equations $F_N(\mathbf{u}_N) = \mathbf{0}$. F_N can be represented in the following way:

$$F_N(\mathbf{u}_N) = \mathbf{A}_N \mathbf{u}_N - \mathbf{B}_N f(\mathbf{u}_N) - \mathbf{c}_N\,,$$

where $\mathbf{u}_N = (\mathbf{u}_k, \mathbf{u}_n)^T = (u_0, \dots, u_{k-1}, u_k, \dots, u_{n+k-1})^T \in \mathbb{R}^{k+n}$, $\mathbf{u}_k \in \mathbb{R}^k$, $\mathbf{u}_n \in \mathbb{R}^n$,
$f(\mathbf{u}_N) = (f(u_0), f(u_1), \dots, f(u_{n+k-1}))^T \in \mathbb{R}^{k+n}$, $\mathbf{c}_n = (c^0, c^1, \dots, c^{k-1}, 0, \dots, 0)^T \in \mathbb{R}^{k+n}$,

$$\mathbf{A}_N = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}_k & \mathbf{A}_n \end{pmatrix}, \qquad \mathbf{B}_N = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_k & \mathbf{B}_n \end{pmatrix},$$

where $\mathbf{I} \in \mathbb{R}^{k\times k}$ is the identity matrix, $\mathbf{A}_k, \mathbf{B}_k \in \mathbb{R}^{n\times k}$, $\mathbf{A}_n, \mathbf{B}_n \in \mathbb{R}^{n\times n}$,

$$\mathbf{A}_k = \frac{1}{h}\begin{pmatrix} \alpha_k & \dots & \alpha_2 & \alpha_1 \\ 0 & \alpha_k & \dots & \alpha_2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \alpha_k \\ 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \quad \mathbf{A}_n = \frac{1}{h}\begin{pmatrix} \alpha_0 & 0 & \dots & \dots & \dots & 0 \\ \alpha_1 & \alpha_0 & 0 & \dots & \dots & 0 \\ \alpha_2 & \alpha_1 & \alpha_0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \alpha_k & \dots & \alpha_0 \end{pmatrix}$$

and $\mathbf{B}_k$, $\mathbf{B}_n$ are the same as $\mathbf{A}_k$, $\mathbf{A}_n$, except that we have to omit the factor $\frac{1}{h}$ and the α-s have to be changed to β-s. Formally, for $k = n$ the definition of these matrices are contradicting to each other however, k is always a fixed number and with that $\mathbf{A}_k$ too, while n is a running index which means that $\mathbf{A}_n$ denotes a family of matrices not a single one. If this is still not enough for the Reader to dissolve the contradiction then let us assume that $k < n$.

Definition 1.2. We call a method *stable in the norm pair* $\left(\|\cdot\|_{\mathcal{X}_n}, \|\cdot\|_{\mathcal{Y}_n}\right)$ if for all IVPs (1) $\exists S \in \mathbb{R}_0^+$ and $\exists N_0 \in \mathbb{N}$ such that $\forall N \geq N_0$, $\forall \mathbf{u}_N, \mathbf{v}_N \in \mathbb{R}^{k+n}$ the estimate

$$\|\mathbf{u}_N - \mathbf{v}_N\|_{\mathcal{X}_N} \leq S\,\|F_N(\mathbf{u}_N) - F_N(\mathbf{v}_N)\|_{\mathcal{Y}_N} \tag{3}$$

holds.

To define stability in this way has a definite profit. It is general in the sense that it works for almost every type of numerical method approximating the solution of ODEs and PDEs as well. Convergence can be proved by the popular recipe "consistency + stability = convergence"

$$\|\varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}}) - \bar{\mathbf{u}}_N\|_{\mathcal{X}_N} \le S\|F_N(\varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})) - F_N(\bar{\mathbf{u}}_N)\|_{\mathcal{Y}_N} = S\|F_N(\varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}}))\|_{\mathcal{Y}_N} \to 0\,,$$

where $\bar{u}$, $\bar{\mathbf{u}}_N$ denote the solution of the original problem (1) and the approximating problem $F_N(\mathbf{u}_N) = \mathbf{0}$ respectively, $\varphi_N : \mathcal{X} \to \mathcal{X}_N$ are projections from the normed space where the original problem is set, thus $\varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}}) - \bar{\mathbf{u}}_N$ represents the error (measured in $\mathcal{X}_N$). Finally, $\|F_N(\varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}}))\|_{\mathcal{Y}_N} \to 0$ is exactly the definition of consistency in this framework. We note that the existence of $\bar{\mathbf{u}}_N$ (from some index) is also the consequence of stability, see [2, Lemmas 24. and 25.], cf. [3, Lemma 1.2.1]. There are many versions of Definition 1.2 which require the stability estimate only in some neighborhood, see [2]. As we defined it is satisfactory for the IVP (1) since we assumed global Lipschitz continuity, for detailed explanation see [3, first Example in Section 1.1.4].

In the following we introduce norm pairs which are interesting for us. We start with some norm notations: for $k \in \mathbb{N}$ fixed, $\mathbf{u}_N \in \mathbb{R}^{k+n}$, the $k\infty$ norm is defined as

$$\|\mathbf{u}_N\|_{k\infty} = \max_{0\le i\le k-1} |u_i| + \max_{k\le i\le N} |u_i|\,.$$

It is known that strongly and weakly stable LMMs are stable in the norm pair $(\|\cdot\|_{k\infty}\,,\|\cdot\|_{k\infty})$, cf. [4]. Stability in the norm pair $\left(\|\cdot\|_{k\infty}\,,\|\cdot\|_{k\$}\right)$ where $\|\cdot\|_{k\$}$ denotes a k-Spijker norm, was first investigated in [5], see [3, Example 2 in Section 1.1.4] for a more available reference and [6,7] for its extensions.

Here we focus on a different norm pair. First, we introduce $\mathbf{E}_N \in \mathbb{R}^{N\times N}$

$$\mathbf{E}_N = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{E}_k & \mathbf{E}_n \end{pmatrix}$$

where $\mathbf{E}_k \in \mathbb{R}^{n\times k}$, $\mathbf{E}_n \in \mathbb{R}^{n\times n}$ are

$$\mathbf{E}_k = \frac{1}{h}\begin{pmatrix} 0 & \dots & 0 & -1 \\ 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{E}_n = \frac{1}{h}\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix}.$$

Note that $\mathbf{E}_n$ represents the linear part of the explicit Euler method (without the initial step). Second, if $\mathbf{A}$ is a regular matrix and $\|\cdot\|_\star$ is a norm, then $\|\mathbf{u}\|_{\mathbf{A},\star} = \|\mathbf{A}\mathbf{u}\|_\star$ defines a norm. Then the kC1 norm can be expressed as

$$\|\mathbf{u}_N\|_{kC1} = \|\mathbf{E}_N\mathbf{u}_N\|_{k\infty} = \|\mathbf{u}_N\|_{\mathbf{E}_N,k\infty}\,.$$

In the next section we formulate and prove our statements about the convergence of LMMs with respect to this norm.

2. kC1 norm convergence of linear multistep methods

Our intention is to investigate the order of the kC1 convergence of LMMs.

Definition 2.1. An LMM is said to be kC1 convergent with order γ if

$$\left\|\varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}}) - \bar{\mathbf{u}}_N\right\|_{kC1} = \mathcal{O}\left(h^\gamma\right) \to 0\,.$$

Note that if we have $k\infty$ convergence with order γ, then

$$\left|\frac{\mathcal{O}\left(h^\gamma\right) - \mathcal{O}\left(h^\gamma\right)}{h}\right| \le \mathcal{O}\left(h^{\gamma-1}\right)$$

thus we may lose one order with respect to the $kC1$ norm. The question is whether we lose it indeed. Note that if $m \in \mathbb{N}$ is fixed (independent of n), then $|\varphi_N(\bar{u})_m - \bar{u}_m| = \mathcal{O}\left(h^{\gamma+1}\right)$ and $\left|\frac{\varphi_N(\bar{u})_{m+1} - \bar{u}_m}{h}\right| = \left|\frac{\varphi_N(\bar{u})_{m+1} - \varphi_N(\bar{u})_m}{h} + \mathcal{O}\left(h^\gamma\right)\right|$, which plays an important role in the proof of the main result where we need to modify the usual consistency proof.

The first step toward convergence is the appropriate choice of the stability. The natural choice is to pick the $(\|\cdot\|_{kC1}\,,\|\cdot\|_{k\infty})$ norm pair. But it will not work directly. First we show that each LMM is stable "in its own sense".

Lemma 2.1. *Consider a weakly or strongly stable LMM with its associated matrix $\mathbf{A}_N$. Then it is stable in the norm pair $\left(\|\cdot\|_{\mathbf{A}_N,k\infty}\,,\|\cdot\|_{k\infty}\right)$.*

Proof. Our starting point is the identity

$$F_N(\mathbf{u}_N) - F_N(\mathbf{v}_N) = \mathbf{A}_N(\mathbf{u}_N - \mathbf{v}_N) - \mathbf{B}_N(f(\mathbf{u}_N) - f(\mathbf{v}_N)). \tag{4}$$

Taking (elementwise) absolute value of (4) and using the Lipschitz continuity we can estimate the right side. Note that $\leq$ is also meant elementwise.

$$|F_N(\mathbf{u}_N) - F_N(\mathbf{v}_N)| = |\mathbf{A}_N(\mathbf{u}_N - \mathbf{v}_N) - \mathbf{B}_N(f(\mathbf{u}_N) - f(\mathbf{v}_N))| \geq$$
$$|\mathbf{A}_N(\mathbf{u}_N - \mathbf{v}_N)| - |\mathbf{B}_N||f(\mathbf{u}_N) - f(\mathbf{v}_N)| \geq |\mathbf{A}_N(\mathbf{u}_N - \mathbf{v}_N)| - L|\mathbf{B}_N||\mathbf{u}_N - \mathbf{v}_N|$$

Thus

$$|F_N(\mathbf{u}_N) - F_N(\mathbf{v}_N)| + L|\mathbf{B}_N||\mathbf{u}_N - \mathbf{v}_N| \geq |\mathbf{A}_N(\mathbf{u}_N - \mathbf{v}_N)|.$$

Taking $k\infty$ norms and using the triangle inequality we have

$$\|\mathbf{u}_N - \mathbf{v}_N\|_{\mathbf{A}_N,k\infty} = \|\mathbf{A}_N(\mathbf{u}_N - \mathbf{v}_N)\|_{k\infty} \leq \|F_N(\mathbf{u}_N) - F_N(\mathbf{v}_N)\|_{k\infty} + L\,\|\mathbf{B}_N\|_{k\infty}\,\|\mathbf{u}_N - \mathbf{v}_N\|_{k\infty}$$

where we can use the known stability estimate

$$\|\mathbf{u}_N - \mathbf{v}_N\|_{k\infty} \leq S\,\|F_N(\mathbf{u}_N) - F_N(\mathbf{v}_N)\|_{k\infty}$$

for the right side to arrive at the required stability estimate

$$\|\mathbf{u}_N - \mathbf{v}_N\|_{\mathbf{A}_N,k\infty} \leq C\,\|F_N(\mathbf{u}_N) - F_N(\mathbf{v}_N)\|_{k\infty}. \quad \square \tag{5}$$

Note that $\mathbf{A}_N$ is similar to $\mathbf{E}_N$ since both represent a numerical differentiation. In particular, for one-step and Adams methods $\mathbf{A}_N = \mathbf{E}_N$. In these cases we already obtained kC1 convergence with the same order as in the $k\infty$ case since we rely on the same type of consistency.

Lemma 2.2. *Consider a strongly stable LMM with its associated matrix $\mathbf{A}_N$. Then $\exists C > 0$ such that $\forall \mathbf{u}_n \in \mathbb{R}^n$ the following inequality is valid:*

$$\|\mathbf{u}_n\|_{\mathbf{E}_n,\infty} \leq C\,\|\mathbf{u}_n\|_{\mathbf{A}_n,\infty}. \tag{6}$$

Proof. The statement is equivalent to

$$\left\|\mathbf{E}_n\mathbf{A}_n^{-1}\right\|_\infty \leq C.$$

Note that

$$\mathbf{E}_n\mathbf{A}_n^{-1} = \frac{1}{\alpha_0}\prod_{i=2}^{k}(\mathbf{I}_n - \xi_i\mathbf{H}_n)^{-1}$$

where $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ is the identity matrix and $\mathbf{H}_n \in \mathbb{R}^{n\times n}$ is defined as

$$\mathbf{H}_n = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix},$$

and ξ_i, $i = 1,\dots,k$ are the roots of the first characteristic polynomial, see [4, formula 3.5 in the proof of Lemma 3.1.] or [7]. Finally, we can use that

$$(\mathbf{I}_n - \xi_i\mathbf{H}_n)^{-1} = \mathbf{I}_n + \xi_i\mathbf{H}_n + \dots + (\xi_i\mathbf{H}_n)^{n-1} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \xi_i & 1 & 0 & \dots & 0 \\ \xi_i^2 & \xi_i & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \xi_i^{n-1} & \dots & \xi_i^2 & \xi_i & 1 \end{pmatrix}$$

resulting in

$$\left\|(\mathbf{I}_n - \xi_i\mathbf{H}_n)^{-1}\right\|_\infty \leq \frac{1-|\xi_i|^n}{1-|\xi_i|} < C_i$$

since $|\xi_i| < 1$. $\square$

Theorem 2.1. *Consider a strongly stable k-step LMM whose order of convergence is γ with respect to the $k\infty$ norm. Assume that the starting values are calculated with a one-step method with the same order. Then it is convergent with respect to the kC1 norm and its order is also γ.*

Proof. The main idea is to split the error into two parts and apply Lemmas 2.1 and 2.2. We assume that the one-step method is determined by G_N and $G_N(\mathbf{w}_N) = \mathbf{0}$. Thus $c^i = w_i = \bar{u}_i$, $0 \le i < k$.

$$\|\varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}}) - \bar{\mathbf{u}}_N\|_{kC1} = \|\varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}}) - \bar{\mathbf{u}}_N\|_{\mathbf{E}_N,k\infty} \le \left\| \begin{pmatrix} \varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_k \\ \mathbf{w}_n \end{pmatrix} - \begin{pmatrix} \mathbf{w}_k \\ \mathbf{w}_n \end{pmatrix} \right\|_{\mathbf{E}_N,k\infty} + \left\| \begin{pmatrix} \mathbf{w}_k \\ \varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_n \end{pmatrix} - \begin{pmatrix} \mathbf{w}_k \\ \bar{\mathbf{u}}_n \end{pmatrix} \right\|_{\mathbf{E}_N,k\infty} \tag{7}$$

1. For the first term of the sum in (7) we can directly use Lemma 2.1.

$$\left\| \begin{pmatrix} \varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_k \\ \mathbf{w}_n \end{pmatrix} - \begin{pmatrix} \mathbf{w}_k \\ \mathbf{w}_n \end{pmatrix} \right\|_{\mathbf{E}_N,k\infty} \le C_1 \left\| G_N\left(\begin{pmatrix} \varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_k \\ \mathbf{w}_n \end{pmatrix}\right) - G_N\left(\begin{pmatrix} \mathbf{w}_k \\ \mathbf{w}_n \end{pmatrix}\right) \right\|_{k\infty} = C_1 \left\| G_N\left(\begin{pmatrix} \varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_k \\ \mathbf{w}_n \end{pmatrix}\right) \right\|_{k\infty} .$$

To estimate the latter expression we have to modify slightly the usual consistency argument. We assume that G_N can be written as

$$(G_N(\mathbf{u}_N))_i = \begin{cases} u_0 - c_0 \quad \text{if} \quad i = 0\,, \\ \dfrac{u_i - u_{i-1}}{h} - g(u_{i-1}, u_i) \quad \text{if} \quad 1 \le i \le N \end{cases}$$

where g is Lipschitz continuous in both variables. Then

$$\left(G_N\left(\begin{pmatrix} \varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_k \\ \mathbf{w}_n \end{pmatrix}\right)\right)_i = \begin{cases} \varphi_N(\bar{u})_0 - c_0 = 0 \quad \text{if} \quad i = 0\,, \\ \dfrac{\varphi_N(\bar{u})_i - \varphi_N(\bar{u})_{i-1}}{h} - g(\varphi_N(\bar{u})_{i-1}, \varphi_N(\bar{u})_i) \quad \text{if} \quad 1 \le i < k \\ \dfrac{w_k - \varphi_N(\bar{u})_{k-1}}{h} - g(\varphi_N(\bar{u})_{k-1}, w_k) \quad \text{if} \quad i = k \\ \dfrac{w_i - w_{i-1}}{h} - g(w_{i-1}, w_i) = 0 \quad \text{if} \quad k < i \le N \end{cases}$$

Observe that $\left| \dfrac{\varphi_N(\bar{u})_i - \varphi_N(\bar{u})_{i-1}}{h} - g(\varphi_N(\bar{u})_{i-1}, \varphi_N(\bar{u})_i) \right| = \mathcal{O}(h^\gamma)$ if $1 \le i < k$, which follows from the fact that the one-step method is consistent with order γ. Furthermore,

$$\left| \frac{w_k - \varphi_N(\bar{u})_{k-1}}{h} - g(\varphi_N(\bar{u})_{k-1}, w_k) \right| = \left| \frac{w_k - w_{k-1} + \mathcal{O}\left(h^{\gamma+1}\right)}{h} - g(w_{k-1}, w_k) + \mathcal{O}\left(h^{\gamma+1}\right) \right| = \mathcal{O}(h^\gamma)$$

where we used the Lipschitz continuity of g. These two facts show that

$$\left\| G_N\left(\begin{pmatrix} \varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_k \\ \mathbf{w}_n \end{pmatrix}\right) \right\|_{k\infty} = \mathcal{O}(h^\gamma) \,.$$

2. For the second term of the sum in (7) first we use Lemma 2.2 to obtain

$$\left\| \begin{pmatrix} \mathbf{w}_k \\ \varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_n \end{pmatrix} - \begin{pmatrix} \mathbf{w}_k \\ \bar{\mathbf{u}}_n \end{pmatrix} \right\|_{\mathbf{E}_N,k\infty} = \|\varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_n - \bar{\mathbf{u}}_n\|_{\mathbf{E}_n,\infty} \le C_2 \|\varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_n - \bar{\mathbf{u}}_n\|_{\mathbf{A}_n,\infty}$$
$$= C_2 \left\| \begin{pmatrix} \mathbf{w}_k \\ \varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_n \end{pmatrix} - \begin{pmatrix} \mathbf{w}_k \\ \bar{\mathbf{u}}_n \end{pmatrix} \right\|_{\mathbf{A}_N,k\infty} ,$$

then we exploit that $\mathbf{w}_k = \bar{\mathbf{u}}_k$ and we use Lemma 2.1 arriving at

$$\left\| \begin{pmatrix} \mathbf{w}_k \\ \varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_n \end{pmatrix} - \begin{pmatrix} \mathbf{w}_k \\ \bar{\mathbf{u}}_n \end{pmatrix} \right\|_{\mathbf{A}_N,k\infty} \le C_3 \left\| F_N\left(\begin{pmatrix} \mathbf{w}_k \\ \varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_n \end{pmatrix}\right) - F_N\left(\begin{pmatrix} \bar{\mathbf{u}}_k \\ \bar{\mathbf{u}}_n \end{pmatrix}\right) \right\|_{k\infty} = C_3 \left\| F_N\left(\begin{pmatrix} \mathbf{w}_k \\ \varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_n \end{pmatrix}\right) \right\|_{k\infty}$$

Similarly to the previous case we have to modify the usual consistency proof.

$$\left(F_N\left(\begin{pmatrix} \mathbf{w}_k \\ \varphi_{\boldsymbol{N}}(\bar{\boldsymbol{u}})_n \end{pmatrix}\right)\right)_i = \begin{cases} w_i - c_i = 0 \quad \text{if} \quad 1 \le i < k\,, \\ \dfrac{1}{h}\left(\alpha_0\varphi_N(\bar{u})_i + \cdots + \alpha_{i-k}\varphi_N(\bar{u})_k + \alpha_{i-k+1}w_{k-1} + \cdots + \alpha_k w_{i-k}\right) - \\ \left(\beta_0 f\left(\varphi_N(\bar{u})_i\right) + \cdots + \beta_{i-k} f\left(\varphi_N(\bar{u})_k\right) + \beta_{i-k+1} f\left(w_{k-1}\right) + \cdots + \beta_k f\left(w_{i-k}\right)\right) \\ \qquad \text{if} \quad k \le i < 2k \\ \dfrac{1}{h}\sum_{j=0}^k \alpha_j\varphi_N(\bar{u})_{i-j} - \sum_{j=0}^k \beta_j f(\varphi_N(\bar{u})_{i-j}) \quad \text{if} \quad 2k \le i \le N \end{cases}$$

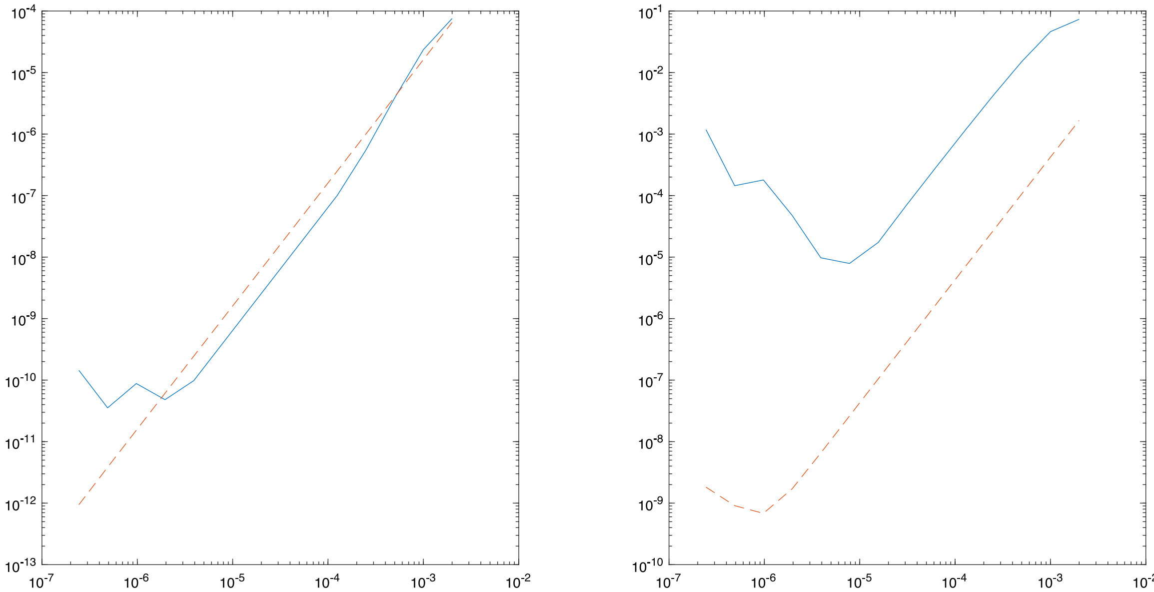

Fig. 1. $k\infty$ error vs. h (left) and $kC1$ error vs. h (right) on a log–log scale. The IVP is $y(0) = 1$, $\dot{y}(t) = \lambda\,(y(t) - \sin 3t) + 3\cos 3t$ with $T = 1$, $\lambda = -10$, the solution is $\sin 3t + e^{\lambda t}$. y_1 was determined by the classical Runge–Kutta second order method. Midpoint method (solid line), which is only weakly stable; Adams–Bashforth 2 (dashed line). The Midpoint method seems to be a second order method wrt. both norms, but it is more sensitive to roundoff errors especially wrt the kC1 norm.

Observe that $\left|\dfrac{1}{h}\sum_{j=0}^{k}\alpha_j\varphi_N(\bar{u})_{i-j} - \sum_{j=0}^{k}\beta_j f(\varphi_N(\bar{u})_{i-j})\right| = \mathcal{O}\left(h^{\gamma}\right)$ because the method is consistent of order γ. Furthermore,

$$\left|\frac{1}{h}\left(\alpha_0\varphi_N(\bar{u})_i + \cdots + \alpha_{i-k}\varphi_N(\bar{u})_k + \alpha_{i-k+1}w_{k-1} + \cdots + \alpha_k w_{i-k}\right) - \right.$$

$$\left.\left(\beta_0 f\left(\varphi_N(\bar{u})_i\right) + \cdots + \beta_{i-k} f\left(\varphi_N(\bar{u})_k\right) + \beta_{i-k+1} f\left(w_{k-1}\right) + \cdots + \beta_k f\left(w_{i-k}\right)\right)\right| =$$

$$\left|\frac{1}{h}\sum_{j=0}^{k}\alpha_j\varphi_N(\bar{u})_{i-j} - \sum_{j=0}^{k}\beta_j f(\varphi_N(\bar{u})_{i-j}) + \mathcal{O}\left(h^{\gamma}\right)\right| = \mathcal{O}\left(h^{\gamma}\right)\ .$$

These facts imply that

$$\left\| F_N\left(\begin{pmatrix} \mathbf{w}_k \\ \varphi_{\mathbf{N}}(\bar{\mathbf{u}})_n \end{pmatrix}\right)\right\|_{k\infty} = \mathcal{O}\left(h^{\gamma}\right)\ .$$

These estimates complete the proof. □

3. Conclusions

Our results which culminated in Theorem 2.1 are sharp from one point of view, but loose from another one. In this section we list the important points where the strengths and weaknesses of these results reveal themselves.

- Note that inequality (6) is not valid for weakly stable LMMs because a counterexample can be constructed in exactly the same way as in [7].
- It is unfortunately not true that

  $$\|\mathbf{u}_N\|_{kC1} = \|\mathbf{u}_N\|_{\mathbf{E}_N,k\infty} \leq C\,\|\mathbf{u}_N\|_{\mathbf{A}_N,k\infty}$$

  since $\mathbf{E}_k - \mathbf{E}_n\mathbf{A}_n^{-1}\mathbf{A}_k$ the lower-left block of $\mathbf{E}_N\mathbf{A}_N^{-1}$ contains a factor $\dfrac{1}{h}$. This was the reason that we could not apply directly stability in the norm pair $(\|\cdot\|_{kC1}\ ,\ \|\cdot\|_{k\infty})$.
- It is clear that this line of proof cannot be used for weakly stable methods. However, the numerical experiments suggest that the statement of Theorem 2.1 is also valid for weakly stable methods. See Fig. 1.

- In practice it is usual to use a one-step method with order of only $\gamma - 1$ to produce the starting values (or recursively by lower order members from the same family of the LMM we want to use). It is well known that these choices of the starting values will result in the required order of convergence if we use the $k\infty$ norm. The question is what is the situation if we use the $kC1$ norm? The numerical results suggest that we can use them in this case also without losing an order.
- If we want to compare the $kC1$ norm to other norms then it is appropriate to compare the families of norms and not the individual members. We will call the $\|\cdot\|_a$ family of norms not weaker than the $\|\cdot\|_b$ family of norms if: $\exists c_1 \in \mathbb{R}^+$ such that $\forall N \in \mathbb{N}$, $\forall \mathbf{u}_N \in \mathbb{R}^{N+1}$

$$c_1 \|\mathbf{u}_N\|_a \geq \|\mathbf{u}_N\|_b \ .$$

We will call the $\|\cdot\|_a$ family of norms stronger than the $\|\cdot\|_b$ family of norms if it is not weaker and it is not true that the $\|\cdot\|_b$ family of norms is not weaker than the $\|\cdot\|_a$ family of norms. Comparing two families of norms the relation can be decided usually by taking $N \to \infty$.
It is easy to show that the $\|\cdot\|_{kC1}$ family is stronger than the $\|\cdot\|_{k\infty}$ family, since we can choose $c_1 = 2$, while taking $N \to \infty$ we can see that it means that the $\|\cdot\|_{C^1}$ norm cannot be overestimated by the $\|\cdot\|_\infty$ norm. Similarly the $\|\cdot\|_{kC1}$ family is stronger than the $\|\cdot\|_{kH1}$ family, where the latter norm is the discrete H^1 norm: $\max_{0\leq i<k} |u_i| + \sqrt{\frac{1}{h} \sum_{i=k}^{N} (u_i - u_{i-1})^2}$. Basically, this comes from the fact when $N \to \infty$, we have to compare the $L^2(0, 1)$ norm with the $L^\infty(0, 1)$ norm.
- Surprisingly, the $kC1$ norms are not equivalent for different k-s. Let us assume that $k_1 < k_2$. Consider the vector $\mathbf{u}_N$: $(\mathbf{u}_N)_i = 0$ if $0 \leq i < k_1$, and $(\mathbf{u}_N)_i = 1$ if $k_1 \leq i \leq N$. Then $\|\mathbf{u}_N\|_{k_1C1} = \frac{1}{h}$, while $\|\mathbf{u}_N\|_{k_2C1} = 1$.

To summarize, we showed that strongly stable LMMs are convergent in the kC1 norm under the assumption that the method is convergent in the $k\infty$ norm, moreover, the order is the same. In the last section we compared different types of norms to emphasize the strength of our result, moreover, we asked two open questions which can lead to the completion of this theory.

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