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# Periodic boundary value problems for delay differential equations with impulses☆

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**Abstract**

Existence and approximation of solutions for an impulsive delay differential equation with periodic boundary value conditions are presented. We use comparison principles and monotone iterative techniques.


*Keywords:* Impulsive functional differential equation; Periodic boundary value problem; Lower and upper method; Extremal solutions

## 1. Introduction

Impulsive differential equations are now recognized as an excellent source of models to simulate processes and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnology, industrial robotic, optimal control, etc [13,1,30,2,29,33,15]. In recent years, impulsive differential equations have become a very active area of research and we refer the reader to the monographs [1,5,13] and the articles [3,4,7,9,16–18,20,23–25,31], where properties of their solutions are studied and extensive bibliographies are given. In consequence, it is very important to develop a complete basic theory of impulsive differential equations. This was initiated in [23].

In this paper, we study the periodic boundary value problem with impulses (PBVP)

$$\begin{aligned}
&u'(t) = g(t, u(t), u(\theta(t))), \quad t \in J = [0, T],\ t \neq t_k,\\
&\Delta u(t_k) = I_k(u(t_k)), \quad k = 1, \ldots, p,\\
&u(0) = u(T),
\end{aligned} \tag{1.1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, $J_0 = J\backslash\{t_1, \ldots, t_p\}$, $g : J \times R^2 \to R$ continuous, and $\theta : J \to R$ continuous verifying $0 \leqslant \theta(t) \leqslant t$, $t \in J$, $\Delta u(t_k) = u(t_k^+) - u(t_k)$. Denote by $PC(X, Y)$, where $X \subset R$, $Y \subset R$, the set of all functions $u : X \to Y$ which are piecewise continuous in $X$ with points of discontinuity of the first kind at the points

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$t_k \in X$, i.e., there exist the limits $\lim\{u(t) : t \to t_k, t > t_k\} = u(t_k^+) < \infty$ and $\lim\{u(t) : t \to t_k, t < t_k\} = u(t_k^-) = u(t_k) < \infty$. We denote by $PC^1(X, Y)$ the set of all functions $u \in PC(X, Y)$, that are continuously differentiate for $t \in X, t \neq t_k$. Let $\Omega = PC([0, T], R) \cap PC^1([0, T], R)$.

**Definition 1.** We say that the function $\alpha, \beta \in \Omega$ are lower and upper solution of the PBVP (1.1) if there exist $M, N \geqslant 0$ and $0 \leqslant L_k < 1$ such that

$$\begin{aligned} &\alpha'(t) \leqslant g(t, \alpha(t), \alpha(\theta(t))) - a(t), \quad t \in J_0, \\ &\Delta\alpha(t_k) \leqslant I_k(\alpha(t_k)) - L_k a_k, \quad k = 1, \ldots, p, \end{aligned}$$

where

$$a(t) = \begin{cases} 0 & \text{if } \alpha(0) \leqslant \alpha(T), \\ \dfrac{Mt + N\theta(t) + 1}{T}(\alpha(0) - \alpha(T)) & \text{if } \alpha(0) > \alpha(T), \end{cases} \tag{1.2}$$

$$a_k = \begin{cases} 0 & \text{if } \alpha(0) \leqslant \alpha(T), \\ \dfrac{t_k}{T}(\alpha(0) - \alpha(T)) & \text{if } \alpha(0) > \alpha(T), \end{cases} \tag{1.3}$$

and

$$\begin{aligned} &\beta'(t) \geqslant g(t, \beta(t), \beta(\theta(t))) + b(t), \quad t \in J_0, \\ &\Delta\beta(t_k) \geqslant I_k(\beta(t_k)) + L_k b_k, \quad k = 1, \ldots, p, \end{aligned}$$

where

$$b(t) = \begin{cases} 0 & \text{if } \beta(0) \geqslant \beta(T), \\ \dfrac{Mt + N\theta(t) + 1}{T}(\beta(T) - \beta(0)) & \text{if } \beta(0) < \beta(T), \end{cases} \tag{1.4}$$

$$b_k = \begin{cases} 0 & \text{if } \beta(0) \geqslant \beta(T), \\ \dfrac{t_k}{T}(\beta(T) - \beta(0)) & \text{if } \beta(0) < \beta(T). \end{cases} \tag{1.5}$$

The definition of classical lower and upper solutions makes reference to the case $\alpha(0) \leqslant \alpha(T)$ and $\beta(0) \geqslant \beta(T)$.

It is known that the method of upper and lower solutions coupled with the monotone iterative technique is an important method in studying PBVP for functional differential, see [5,12,22,26,27,6,34,19]. We also note it is used in [7–11,14,16,17,20,21,28,32] to approximate periodic solutions of impulsive equations.

In the present paper, we study the existence of extremal solutions for PB VP (1.1) by using this method. This paper is organized as follows. In Section 2, some lemmas are given, which are important for proving our main result. The main results are contained in Sections 3 and 4.

## 2. Preliminaries

In this section, we prove some comparison results. To this end, we need the following lemma.

**Lemma 1** (*Bainov and Simeonov [1]*). *Let $s \in [0, T)$, $c_k \geqslant 0$, $\alpha_k$, $k = 1, \ldots, p$ are constants and let $p, q \in PC(J, R)$, $x \in PC^1(J, R)$. If*

$$\begin{cases} x'(t) \leqslant p(t)x(t) + q(t), & t \in [s, T), \ t \neq t_k, \\ x(t_k^+) \leqslant c_k x(t_k) + \alpha_k, & t_k \in [s, T), \end{cases}$$

*then for* $t \in [s, T]$

$$x(t) \leqslant x(s^+)\left(\prod_{s<t_k<t} c_k\right)\exp\left(\int_s^t p(u)\,\mathrm{d}u\right) + \int_s^t \left(\prod_{u<t_k<t} c_k\right)\exp\left(\int_u^t p(\tau)\,\mathrm{d}\tau\right) q(u)\,\mathrm{d}u$$
$$+ \sum_{s<t_k<t}\left(\prod_{t_k<t_i<t} c_i\right)\exp\left(\int_{t_k}^t p(\tau)\,\mathrm{d}\tau\right)\alpha_k.$$

We will establish two new comparison results, which plays an important role in monotone iterative technique.

**Lemma 2.** *Let function* $u \in \Omega$, $M > 0$, $N \geqslant 0$, $0 \leqslant L_k < 1$ *such that*

($A_1$) $u'(t) + Mu(t) + Nu(\theta(t)) \leqslant 0$, $t \in J_0$,
($A_2$) $\Delta u(t_k) \leqslant -L_k u(t_k)$, $k = 1, \ldots, p$,
($A_3$) $u(0) \leqslant 0$,
($A_4$) $N\int_0^T \prod_{t<t_k<T}(1 - L_k)\mathrm{e}^{M(t-\theta(t))}\,\mathrm{d}t \leqslant \prod_{k=1}^p (1 - L_k)$.

*Then* $u \leqslant 0$ *on* $J$.

**Proof.** Let $v(t) = \mathrm{e}^{Mt}u(t)$, $t \in [0, T]$. Then by ($A_1$),

$$v'(t) = M\mathrm{e}^{Mt}u(t) + \mathrm{e}^{Mt}u'(t) \leqslant -N\mathrm{e}^{Mt}u(\theta(t)), \quad t \in J_0,$$

or

$$v'(t) \leqslant -N\mathrm{e}^{M(t-\theta(t))}v(\theta(t)), \quad t \in J_0, \tag{2.1}$$

and

$$v(t_k^+) = \mathrm{e}^{Mt_k}u(t_k^+) \leqslant (1 - L_k)v(t_k), \quad k = 1, \ldots, p. \tag{2.2}$$

It is sufficient to show $v(t) \leqslant 0$ on $J$. If this no true, then there exists $t^* \in J$ such that $v(t^*) > 0$. Since $v(0) = u(0) \leqslant 0$, then $t^* \in (0, T]$. Let $\bar{t} \in [0, t^*)$ such that $v(\bar{t}) = \inf_{t\in[0,t^*)} v(t) = -\lambda \leqslant 0$ $(\lambda \geqslant 0)$. We assume that $\bar{t} \neq t_i^+$, if $\bar{t} = t_i^+$ for some $i \in \{1, \ldots, p\}$, the proof is similar. From (2.1) we have

$$v'(t) \leqslant \lambda N\mathrm{e}^{M(t-\theta(t))}, \quad t \in [0, t^*]. \tag{2.3}$$

By (2.3) and (2.2), using Lemma 1, we have for $t \in [\bar{t}, t^*]$

$$v(t) \leqslant v(\bar{t})\prod_{\bar{t}<t_k<t}(1 - L_k) + \int_{\bar{t}}^t \prod_{s<t_k<t}(1 - L_k)\lambda N\mathrm{e}^{M(s-\theta(s))}\,\mathrm{d}s$$
$$\leqslant -\lambda\prod_{k=1}^p(1 - L_k) + \lambda N\int_0^T \prod_{s<t_k<T}(1 - L_k)\mathrm{e}^{M(s-\theta(s))}\,\mathrm{d}s.$$

Let $t = t^*$, we obtain

$$v(t^*) \leqslant \lambda\left\{N\int_0^T \prod_{s<t_k<T}(1 - L_k)\mathrm{e}^{M(s-\theta(s))}\,\mathrm{d}s - \prod_{k=1}^p(1 - L_k)\right\} \leqslant 0,$$

which contradicts $v(t^*) > 0$. Hence $v(t) \leqslant 0$ on $J$, which completes the proof. □

**Lemma 3.** *Let $u \in \Omega$, $M > 0$, $N \geqslant 0$, $0 \leqslant L_k < 1$ such that*

(B$_1$) *if $u(0) \leqslant u(T)$, then*

$$u'(t) + Mu(t) + Nu(\theta(t)) \leqslant 0, \quad t \in J_0,$$
$$\Delta u(t_k) \leqslant -L_k u(t_k), \quad k = 1, \dots, p.$$

(B$_1$)$'$ *if $u(0) > u(T)$, then*

$$u'(t) + Mu(t) + Nu(\theta(t)) + \frac{Mt + N\theta(t) + 1}{T}[u(0) - u(T)] \leqslant 0, \quad t \in J_0,$$
$$\Delta u(t_k) \leqslant -L_k u(t_k) - L_k \times \frac{t_k}{T}[u(0) - u(T)], \quad k = 1, \dots, p.$$

(B$_2$) $N \int_0^T \prod_{t<t_k<T} e^{M(t-\theta(t))}\, dt \leqslant \prod_{k=1}^p (1 - L_k)$.

*Then $u \leqslant 0$ on $J$.*

**Proof.** In the case of (B$_1$), if $u \geqslant 0$, then $u'(t) \leqslant 0$ on $J_0$ and $u(t_k^+) \leqslant (1 - L_k)u(t_k) \leqslant u(t_k)$, $k = 1, \dots, p$. So $u$ is a nonincreasing function. It follows that we have

$$u(0) \geqslant u(T).$$

Since $u(0) \leqslant u(T)$, hence $u(0) = u(T) = const$, so that $u' \equiv 0$, $L_k \equiv 0$ and also $u \equiv 0$.

If there exists $t^* \in J$ such that $u(t^*) < 0$. The proof demonstrates that $u(0) \leqslant 0$ so that we can apply Lemma 2 and affirm that $u \leqslant 0$. If $u(0) > 0$ then $u(T) > 0$. Let $v(t) = e^{Mt}u(t)$, we obtain that $v(0) > 0$, $v(T) > 0$ and $v(t^*) < 0$. Set $\inf_{t\in[0,T]} v(t) = v(\bar{t}) = -\lambda (\lambda > 0)$, then there exists $t_i < \bar{t} \leqslant t_{i+1}$ for some $i$ such that $v(\bar{t}) = -\lambda$ or $v(t_i^+) = -\lambda$. Assume that $v(\bar{t}) = -\lambda$ (if $v(t_i^+) = -\lambda$, the proof is similar). It follows that

$$v'(t) \leqslant -Ne^{M(t-\theta(t))}v(\theta(t)) \leqslant \lambda N e^{M(t-\theta(t))}, \quad t \in J_0,$$

and

$$v(t_k^+) \leqslant (1 - L_k)v(t_k).$$

By Lemma 1, for $t \in [\bar{t}, T]$, we have

$$\begin{aligned} v(t) &\leqslant v(\bar{t}) \prod_{\bar{t}<t_k<t} (1 - L_k) + \int_{\bar{t}}^t \prod_{s<t_k<t} (1 - L_k)\lambda N e^{M(s-\theta(s))}\, ds \\ &\leqslant -\lambda \prod_{k=1}^p (1 - L_k) + \lambda N \int_0^T \prod_{s<t_k<T} (1 - L_k) e^{M(s-\theta(s))}\, ds. \end{aligned}$$

Let $t = T$, we have

$$v(T) \leqslant \lambda \left[ N \int_0^T \prod_{s<t_k<T} (1 - L_k) e^{M(s-\theta(s))} ds - \prod_{k=1}^p (1 - L_k) \right] \leqslant 0,$$

which contradicts $v(T) > 0$, and so $u(0) \leqslant 0$, and the conclusion follows.

For the case (B$_1$)$'$, we consider the function $m(t) = u(t) + t/T[u(0) - u(T)]$. It follows that $m(0) = m(T)$, and for $t \in J_0$

$$m'(t) + Mm(t) + Nm(\theta(t)) = u'(t) + Mu(t) + Nu(\theta(t)) + \frac{Mt + N\theta(t) + 1}{T}[u(0) - u(T)] \leqslant 0$$

and for $t = t_k$

$$\Delta m(t_k) = \Delta u(t_k) \leqslant -L_k u(t_k) - L_k \times t_k/T[u(0) - u(T)] = -L_k m(t_k),$$

by the case (B$_1$), we have $m(t) \leqslant 0$ on $J$, and so $u(t) \leqslant 0$ on $J$. The proof is complete. □

## 3. Existence for linear problem

In this section, we consider the linear problem of (1.1)

$$\begin{aligned} &u'(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in [0, T],\ t \neq t_k, \\ &\Delta u(t_k) = -L_k u(t_k) + \gamma_k, \quad k = 1, \dots, p, \quad u(0) = u(T), \end{aligned} \tag{3.1}$$

where $\sigma(t) \in PC(J, R)$, $\gamma_k \in R$, $k = 1, \dots, p$, $M > 0$, $N \geqslant 0$, $0 \leqslant L_k < 1$. For $\alpha, \beta \in \Omega$, set $[\alpha, \beta] = \{u : \alpha(t) \leqslant u(t) \leqslant \beta(t), t \in J\}$.

**Theorem 1.** *Suppose that there exist $\alpha, \beta \in \Omega$ such that*

(C$_1$) $\alpha \leqslant \beta$ *on $J$.*
(C$_2$)

$$\alpha'(t) + M\alpha(t) + N\alpha(\theta(t)) \leqslant \sigma(t) - a(t), \quad \Delta\alpha(t_k) \leqslant -L_k\alpha(t_k) + \gamma_k - a_k;$$

$$\beta'(t) + M\beta(t) + N\beta(\theta(t)) \geqslant \sigma(t) - b(t), \quad \Delta\beta(t_k) \geqslant -L_k\beta(t_k) + \gamma_k + b_k,$$

*where $a(t), b(t), a_k, b_k$ be defined by* (1.2)–(1.5).
(C$_3$) $N \int_0^T \prod_{t<t_k<T} e^{M(t-\theta(t))}\, dt \leqslant \prod_{k=1}^p (1 - L_k)$.

*Then there exists a unique solution $u$ for problem* (3.1). *Moreover $u \in [\alpha, \beta]$.*

**Proof.** We prove the Theorem in four steps.

*Step* 1. We prove the uniqueness of solution to this problem. If $u_1, u_2$ are solutions of (3.1), set $v_1 = u_1 - u_2$ and $v_2 = u_2 - u_1$, then

$$\begin{aligned} &v_1(0) = v_1(T), \quad v_1'(t) + Mv_1(t) + Nv_1(\theta(t)) = 0, \quad t \in J_0, \\ &\Delta v_1(t_k) = -L_k v_1(t_k), \quad k = 1, \dots, p, \end{aligned}$$

and

$$\begin{aligned} &v_2(0) = v_2(T), \quad v_2'(t) + Mv_2(t) + Nv_2(\theta(t)) = 0, \quad t \in J_0, \\ &\Delta v_2(t_k) = -L_k v_2(t_k), \quad k = 1, \dots, p. \end{aligned}$$

By Lemma 3, we have that $v_1 = u_1 - u_2 \leqslant 0$ and $v_2 = u_2 - u_1 \leqslant 0$ and so $u_1 = u_2$.

*Step* 2. We prove that if $\omega$ is a classical lower solution and $\gamma$ is a classical upper solution for (3.1) with $\omega \leqslant \gamma$, and (C$_3$) is satisfied, then (3.1) has a solution $u \in [\omega, \gamma]$.

Let $u(\cdot; a)$ denotes the unique solution of the following problem:

$$\begin{aligned} &u'(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in J_0, \\ &\Delta u(t_k) = -L_k u(t_k) + \gamma_k, \quad u(0) = a. \end{aligned} \tag{3.2}$$

First we shall show that $\omega(0) \leqslant u(T; \omega(0))$ and $\gamma(0) \geqslant u(T; \gamma(0))$.

Suppose that $\omega(0) > u(T; \omega(0))$, then the function $v$ defined by $v(t) = \omega(t) - u(t; \omega(0))$ satisfies

$$\begin{aligned} &v'(t) + Mv(t) + Nv(\theta(t)) \leqslant 0, \quad t \in J_0, \\ &\Delta v(t_k) \leqslant -L_k v(t_k), \quad k = 1, \dots, p, \\ &v(0) = \omega(0) - \omega(0) < \omega(T) - u(T; \omega(0)) = v(T). \end{aligned}$$

Thus Lemma 3 ensures that $v(t)\leqslant 0$ on $J$. This implies that $v(T)=\omega(T)-u(T;\omega(0))-\omega(T)\leqslant 0$ and therefore $\omega(0)\leqslant\omega(T)\leqslant u(T;\omega(0))$. This is a contradiction and so $\omega(0)\leqslant u(T;\omega(0))$. The same arguments show that $\gamma(0)\geqslant u(T;\gamma(0))$.

Next we shall prove that there exists $c\in[\omega(0),\gamma(0)]$ such that $u(0;c)=u(T;c)$. If $\omega(0)=\gamma(0)$, we have that

$$\omega(0)\leqslant u(T;\omega(0))=u(T;\gamma(0))\leqslant\gamma(0)=\omega(0).$$

Thus $u(T;\omega(0))=\omega(0)$ and we may choose $c=\omega(0)$.

In consequence, we can assume that $\omega(0)<\gamma(0)$, and define the map $F:[\omega(0),\gamma(0)]\to R$ by $F(a)=a-u(T;a)$, thus $F$ is continuous. Then, since $F(\omega(0))\leqslant 0\leqslant F(\gamma(0))$, there must be one point $c\in[\omega(0),\gamma(0)]$ such that $F(c)=0$. Denote $u=u(\cdot;c)$, then $u$ is a solution of (3.1).

*Step* 3. We claim $u\in[\omega,\gamma]$. Let $m_1(t)=\omega(t)-u(t;c)$ and $m_2(t)=u(t;c)-\gamma(t)$. We obtain that $m_1,m_2\in\Omega$, and

$$\begin{aligned}&m_1'(t)+Mm_1(t)+Nm_1(\theta(t))\leqslant 0,\quad t\in J_0,\\&\Delta m(t_k)\leqslant -L_k m_1(t_k),\quad k=1,\ldots,p,\quad m_1(0)\leqslant 0,\end{aligned}$$

and

$$\begin{aligned}&m_2'(t)+Mm_2(t)+Nm_2(\theta(t))\leqslant 0,\quad t\in J_0,\\&\Delta m(t_k)\leqslant -L_k m_2(t_k),\quad k=1,\ldots,p,\quad m_2(0)\leqslant 0,\end{aligned}$$

using Lemma 2, we get that $m_1\leqslant 0$ and $m_2\leqslant 0$ on $J$. Hence $\omega\leqslant u(\cdot;c)\leqslant\gamma$ on $J$.

*Step* 4. Set

$$\bar\alpha(t)=\begin{cases}\alpha(t) & \text{if } \alpha(0)\leqslant\alpha(T),\\ \alpha(t)+\dfrac{t}{T}[\alpha(0)-\alpha(T)] & \text{if } \alpha(0)>\alpha(T),\end{cases}\tag{3.3}$$

and

$$\bar\beta(t)=\begin{cases}\beta(t) & \text{if } \beta(0)\geqslant\beta(T),\\ \beta(t)-\dfrac{t}{T}[\beta(T)-\beta(0)] & \text{if } \beta(0)<\beta(T).\end{cases}\tag{3.4}$$

It is evident that $\alpha\leqslant\bar\alpha$ and $\bar\beta\leqslant\beta$ on $J$. Also $\bar\alpha(0)=\alpha(0)\leqslant\bar\alpha(T)$, and $\bar\beta(0)=\beta(0)\geqslant\bar\beta(T)$. We can check that $\bar\alpha$ and $\bar\beta$ are classical lower and upper solutions, respectively, for problem (3.1) and that $\bar\alpha\leqslant\bar\beta$ so that $[\bar\alpha,\bar\beta]\subseteq[\alpha,\beta]$. In fact, if $\alpha(0)\leqslant\alpha(T)$ then

$$\begin{aligned}&\bar\alpha'(t)+M\bar\alpha(t)+N\bar\alpha(\theta(t))=\alpha'(t)+M\alpha(t)+N\alpha(\theta(t))\leqslant\sigma(t),\quad t\neq t_k,\\&\Delta\bar\alpha(t_k)=\Delta\alpha(t_k)\leqslant -L_k\alpha(t_k)+\gamma_k=-L_k\bar\alpha(t_k)+\gamma_k,\quad k=1,\ldots,p.\end{aligned}$$

If $\alpha(0)>\alpha(T)$, then

$$\begin{aligned}&\bar\alpha'(t)+M\bar\alpha(t)+N\bar\alpha(\theta(t))=\alpha'(t)+M\alpha(t)+N\alpha(\theta(t))+\frac{Mt+N\theta(t)+1}{T}[\alpha(0)-\alpha(T)]\leqslant\sigma(t),\quad t\neq t_k,\\&\Delta\bar\alpha(t_k)=\Delta\alpha(t_k)\leqslant -L_k\alpha(t_k)+\gamma_k-a_k=-L_k\bar\alpha(t_k)+\gamma_k,\quad k=1,\ldots,p.\end{aligned}$$

Hence $\bar\alpha$ is a classical lower solution for (3.1). Similarly, we can show that $\bar\beta$ is a classical upper solution for (3.1). Now, consider the function $m=\bar\alpha-\bar\beta$, it follows that:

$$\begin{aligned}m'(t)+Mm(t)+Nm(\theta(t))&=\bar\alpha'(t)+M\bar\alpha(t)+N\bar\alpha(\theta(t))-\bar\beta'(t)-M\bar\beta(t)-N\bar\beta(\theta(t))\\&\leqslant\sigma(t)-\sigma(t)=0,\quad t\in J_0,\end{aligned}$$

and

$$\begin{aligned}\Delta m(t_k) &= \Delta\bar{\alpha}(t_k) - \Delta\bar{\beta}(t_k)\\ &\leqslant -L_k\bar{\alpha}(t_k) + \gamma_k - (-L_k\bar{\beta}(t_k) + \gamma_k) \quad = -L_k m(t_k).\end{aligned}$$

Also, $m(0) = \bar{\alpha}(0) - \bar{\beta}(0) = \alpha(0) - \beta(0) \leqslant 0$. Using Lemma 2, we obtain that $m \leqslant 0$ on $J$, or equivalently, $\bar{\alpha} \leqslant \bar{\beta}$ on $J$. This proof is complete. □

## 4. Monotone iterative technique

In this section, we establish existence criteria for solutions of the PBVP (1.1) by the method of lower and upper solutions and the monotone iterative technique.

**Theorem 2.** *Suppose that the following conditions hold*:

($H_1$) *$\alpha$ and $\beta$ are lower and upper solutions for* (1.1) *with* $\alpha \leqslant \beta$.
($H_2$) $g(t,x,y) - g(t,u,v) \geqslant -M(x-u) - N(y-v)$ *for every* $t \in J_0, \alpha \leqslant u \leqslant x \leqslant \beta, \alpha(\theta(t)) \leqslant v(\theta(t)) \leqslant y(\theta(t)) \leqslant \beta(\theta(t))$.
($H_3$) $I_k \in C(R,R)$ *satisfies*

$$I_k(x) - I_k(y) \geqslant -L_k(x-y),$$

*for* $\beta(t_k) \leqslant y(t_k) \leqslant x(t_k) \leqslant \alpha(t_k), 0 \leqslant L_k < 1, k = 1, \ldots, p$.
($H_4$) $N\int_0^T \prod_{t<t_k<T}(1-L_k)\mathrm{e}^{M(t-\theta(t))}\,\mathrm{d}t \leqslant \prod_{k=1}^{p}(1-L_k)$.

*Then there exist monotone sequence* $\{\bar{\alpha}_n(t)\}, \{\bar{\beta}_n(t)\}$ *with* $\bar{\alpha}_0 = \bar{\alpha}, \bar{\beta}_0 = \bar{\beta}$, *where* $\bar{\alpha}, \bar{\beta}$ *defined by* (3.3) *and* (3.4) *such that* $\lim_{n\to\infty}\bar{\alpha}_n(t) = r(t)$ *and* $\lim_{n\to\infty}\bar{\beta}_n(t) = \rho(t)$ *uniformly hold on* $J$, *where* $r(t), \rho(t)$ *are minimal and maximal solutions of PBVP* (1.1), *respectively*.

**Proof.** First we prove that $\bar{\alpha}, \bar{\beta}$ are classical lower and upper solution of (1.1), respectively, and $[\bar{\alpha}, \bar{\beta}] \subseteq [\alpha, \beta]$.

Clearly $\bar{\alpha}(0) \leqslant \bar{\alpha}(T)$, $\bar{\beta}(0) \geqslant \bar{\beta}(T)$ and $\alpha \leqslant \bar{\alpha}$, $\beta \geqslant \bar{\beta}$. Now we show $\bar{\alpha} \leqslant \bar{\beta}$. To this end, take $m = \bar{\alpha} - \bar{\beta}$. Then $m(0) = \alpha(0) - \beta(0) \leqslant 0$. In the case $\alpha(0) > \alpha(T)$ and $\beta(0) < \beta(T)$, we have, according to ($H_2$), that

$$\begin{aligned}m'(t) + Mm(t) + Nm(\theta(t)) = \alpha'(t) + M\alpha(t) + N\alpha(\theta(t)) &+ \frac{Mt + N\theta(t) + 1}{T}[\alpha(0) - \alpha(T)]\\ - \beta'(t) - M\beta(t) - N\beta(\theta(t)) &+ \frac{Mt + N\theta(t) + 1}{T}[\beta(T) - \beta(0)]\\ \leqslant g(t,\alpha(t),\alpha(\theta(t))) &+ M\alpha(t) + N\alpha(\theta(t))\\ - g(t,\beta(t),\beta(\theta(t))) &- M\beta(t) - N\beta(\theta(t)) \leqslant 0, \quad t \in J_0.\end{aligned}$$

By ($H_3$) we have

$$\begin{aligned}\Delta m(t_k) &= \Delta\bar{\alpha}(t_k) - \Delta\bar{\beta}(t_k)\\ &= \Delta\alpha(t_k) - \Delta\beta(t_k)\\ &\leqslant I_k(\alpha(t_k)) - I_k(\beta(t_k)) - L_k a_k - L_k b_k\\ &\leqslant -L_k m(t_k).\end{aligned}$$

Thus in view of Lemma 2, $m(t) \leqslant 0$ on $J$ and so $\bar{\alpha} \leqslant \bar{\beta}$ on $J$.

Now we check that $\bar{\alpha}, \bar{\beta}$ are classical lower and upper solution of (1.1), respectively. Indeed, if $\alpha(0) > \alpha(T)$, then for $t \in J_0$,

$$\bar{\alpha}'(t) = \alpha'(t) + \frac{1}{T}[\alpha(0) - \alpha(T)] \leqslant g(t,\alpha(t),\alpha(\theta(t))) - \frac{Mt + N\theta(t)}{T}[\alpha(0) - \alpha(T)].$$

Since $\alpha \leqslant \bar{\alpha} \leqslant \beta$, according to ($H_2$), we get that

$$g(t, \bar{\alpha}(t), \bar{\alpha}(\theta(t))) - g(t, \alpha(t), \alpha(\theta(t))) \geqslant -M(\bar{\alpha}(t) - \alpha(t)) - N(\bar{\alpha}(\theta(t)) - \alpha(\theta(t))),$$

so

$$\begin{aligned}\bar{\alpha}'(t) &\leqslant g(t, \bar{\alpha}(t), \bar{\alpha}(\theta(t))) + M(\bar{\alpha}(t) - \alpha(t)) + N(\bar{\alpha}(\theta(t)) - \alpha(\theta(t))) \\ &\quad - \frac{Mt + N\theta(t)}{T}[\alpha(0) - \alpha(T)] = g(t, \bar{\alpha}(t), \bar{\alpha}(\theta(t))) + \frac{Mt}{T}[\alpha(0) - \alpha(T)] \\ &\quad + \frac{N\theta(t)}{T}[\alpha(0) - \alpha(T)] - \frac{Mt + N\theta(t)}{T}[\alpha(0) - \alpha(T)] = g(t, \bar{\alpha}(t), \bar{\alpha}(\theta(t))).\end{aligned}$$

For $t = t_k$, from ($H_3$), we have

$$\begin{aligned}\Delta\bar{\alpha}(t_k) = \Delta\alpha(t_k) &\leqslant I_k(\alpha(t_k)) - L_k \times \frac{t_k}{T}[\alpha(0) - \alpha(T)] \\ &\leqslant I_k(\bar{\alpha}(t_k)) + L_k[\bar{\alpha}(t_k) - \alpha(t_k)] - L_k \times \frac{t_k}{T}[\alpha(0) - \alpha(T)] = I_k(\bar{\alpha}(t_k)),\end{aligned}$$

and it is trivial when $\alpha(0) \leqslant \alpha(T)$. Thus $\bar{\alpha}$ is a classical lower solution. Analogously, $\bar{\beta} \in [\alpha, \beta]$ is a classical upper solution.

For any $\eta \in [\bar{\alpha}, \bar{\beta}]$ consider

$$\begin{cases} u'(t) + Mu(t) + Nu(\theta(t)) = M\eta(t) + N\eta(\theta(t)) + g(t, \eta(t), \eta(\theta(t))), \\ \Delta u(t_k) + L_k u(t_k) = I_k(\eta(t_k)) + L_k\eta(t_k), \\ u(0) = u(T). \end{cases} \tag{4.1}$$

By Theorem 1, (4.1) has a unique solution $u \in \Omega$. Define operator $A$ by $u = A\eta$. It follows that $A$ possesses the following properties:

(i) $\bar{\alpha} \leqslant A\bar{\alpha}$, $\bar{\beta} \geqslant A\bar{\beta}$;
(ii) $A\eta_1 \leqslant A\eta_2$ for $\eta_1, \eta_2 \in [\bar{\alpha}, \bar{\beta}]$ with $\eta_1 \leqslant \eta_2$.

First we prove (i). Let $m = \bar{\alpha} - \bar{\alpha}_1$, where $\bar{\alpha}_1 = A\bar{\alpha}$. Then we have

$$\begin{aligned}m'(t) = \bar{\alpha}'(t) - \bar{\alpha}_1'(t) &\leqslant g(t, \bar{\alpha}(t), \bar{\alpha}(\theta(t))) + M\bar{\alpha}_1(t) + N\bar{\alpha}_1(\theta(t)) - M\bar{\alpha}(t) - N\bar{\alpha}(\theta(t)) - g(t, \bar{\alpha}(t), \bar{\alpha}(\theta(t)) \\ &= -M(\bar{\alpha}(t) - \bar{\alpha}(t)) + N(\bar{\alpha}(\theta(t) - \bar{\alpha}(\theta(t))) \leqslant -Mm(t) - Nm(\theta(t)), \quad t \in J_0,\end{aligned}$$

$$\Delta m(t_k) = \Delta\bar{\alpha}(t_k) - \Delta\bar{\alpha}_1(t_k) \leqslant I_k(\bar{\alpha}(t_k)) - I_k(\bar{\alpha}(t_k)) - L_k\bar{\alpha}(t_k) + L_k\bar{\alpha}_1(t_k) \leqslant -L_k m(t_k), \quad k = 1, \ldots, p,$$

and

$$m(0) = \bar{\alpha}(0) - \bar{\alpha}_1(0) \leqslant \bar{\alpha}(T) - \bar{\alpha}_1(T) = m(T).$$

According to Lemma 3, we get that $m(t) \leqslant 0$ on $J$, i.e., $\bar{\alpha} \leqslant A\bar{\alpha}$. Analogously, we have $\bar{\beta} \geqslant A\bar{\beta}$.

Now we claim (ii). Setting $v_1 = A\eta_1$, $v_2 = A\eta_2$, where $\eta_1, \eta_2 \in [\bar{\alpha}, \bar{\beta}]$ with $\eta_1 \leqslant \eta_2$. Let $m = v_1 - v_2$, by ($H_2$), ($H_3$) and (4.1) we have

$$\begin{aligned}m'(t) &= v_1'(t) - v_2'(t) \\ &= -Mv_1(t) - Nv_1(\theta(t)) + g(t, \eta_1(t), \eta_1(\theta(t))) + M\eta_1(t) + N\eta_1(\theta(t)) \\ &\quad - [-Mv_2(t) - Nv_2(\theta(t)) + g(t, \eta_2(t), \eta_2(\theta(t))) + M\eta_2(t) + N\eta_2(\theta(t))] \\ &\leqslant -M(v_1(t) - v_2(t)) - N(v_1(\theta(t)) - v_2(\theta(t))) = -Mm(t) - Nm(\theta(t)), \quad t \in J_0,\end{aligned}$$

$$\begin{aligned}\Delta m(t_k) &= \Delta v_1(t_k) - \Delta v_2(t_k) \\ &= [-L_k v_1(t_k) + I_k(\eta_1(t_k)) + L_k\eta_1(t_k)] - [-L_k v_2(t_k) + I_k(\eta_2(t_k)) + L_k\eta_2(t_k)] \\ &\leqslant -L_k m(t_k), \quad k = 1, \ldots, p,\end{aligned}$$

and $m(0)=m(T)$, by Lemma 3 we have $m(t)\leqslant 0$ on $J$, and so $v_1\leqslant v_2$. Thus we may define the sequences $\{\bar{\alpha}_n\}$, $\{\bar{\beta}_n\}$ by $\bar{\alpha}_{n+1}=A\bar{\alpha}_n$, $\bar{\beta}_{n+1}=A\bar{\beta}_n$, $\bar{\alpha}_0=\bar{\alpha}$, $\bar{\beta}_0=\bar{\beta}$. Using (i) and (ii) it is immediate to verify that

$$\bar{\alpha}_0=\bar{\alpha}\leqslant\bar{\alpha}_1\leqslant\cdots\leqslant\bar{\alpha}_n\leqslant\bar{\beta}_n\leqslant\cdots\leqslant\bar{\beta}_0=\bar{\beta},\quad \forall n\in N.$$

Hence we have

$$\lim_{n\to\infty}\bar{\alpha}_n(t)=r(t)\qquad \lim_{n\to\infty}\bar{\beta}_n(t)=\rho(t)\ \text{uniformly on}\ J.$$

Consider the following equations

$$\bar{\alpha}'_{n+1}(t)+M\bar{\alpha}_{n+1}(t)+N\bar{\alpha}_{n+1}(\theta(t))=M\bar{\alpha}_n(t)+N\bar{\alpha}_n(\theta(t))+g(t,\bar{\alpha}_n(t),\bar{\alpha}_n(\theta(t))),\quad t\in J_0,$$

$$\Delta\bar{\alpha}_{n+1}(t_k)+L_k\bar{\alpha}_{n+1}(t_k)=I_k(\bar{\alpha}_n(t_k))+L_k\bar{\alpha}_n(t_k),\quad k=1,\ldots,p,\ \bar{\alpha}_{n+1}(0)=\bar{\alpha}_{n+1}(T),$$

passing to the limit when $n$ tends to $\infty$, we obtain that $r$ is solution of (1.1). Similarly, $\rho$ is also solution of (1.1).

Finally, to prove that $r$ is the minimal solution on $[\bar{\alpha},\bar{\beta}]$, let $u$ be any solution of (1.1) on $[\bar{\alpha},\bar{\beta}]$. It is obvious that $\bar{\alpha}_0\leqslant u$. Now if $\bar{\alpha}_n\leqslant u$, one can see that $\bar{\alpha}_{n+1}\leqslant u$ by considering the function $m=u-\bar{\alpha}_{n+1}$ and applying Lemma 3 again. Thus passing to the limit we may conclude that $r\leqslant u$ on $J$. The same arguments prove that $u\leqslant\rho$. The proof is complete. □

## 5. An example

Consider the equation

$$u'(t)=g(t,u(t),u(\theta(t)))=-u^2(t)-2u(\tfrac{1}{2}t)+\tfrac{1}{2}\mathrm{e}^t,\quad t\in[0,\tfrac{1}{3}],\ \ t\neq t_1,\ \Delta u(t_1)=-\tfrac{2}{7}u(t_1),\quad t_1=\tfrac{1}{4},$$

$$u(0)=u(\tfrac{1}{3}). \tag{5.1}$$

It is easy to verify that $\alpha=-\frac{1}{2}\mathrm{e}$ is a lower solution and $\beta=\frac{11}{20}$ is an upper solution, and

$$g(t,x,y)-g(t,u,v)=-(x^2-u^2)-2(y-v)\geqslant-\tfrac{11}{10}(x-u)-2(y-v),$$

for $\alpha\leqslant u\leqslant x\leqslant\beta$. Taking $M=\frac{11}{10}$, $N=2$, $L_k=\frac{2}{7}$, we get

$$\begin{aligned}2\int_0^{1/3}\prod_{t<t_k<1/3}(1-L_k)\mathrm{e}^{(11/20)t}\,\mathrm{d}t&=2\int_0^{1/4}\mathrm{e}^{(11/20)t}\,\mathrm{d}t+2\int_{1/4}^{1/3}\frac{5}{7}\mathrm{e}^{(11/20)t}\,\mathrm{d}t\\&=\frac{40}{11}\left[\frac{5}{7}\mathrm{e}^{11/60}+\frac{2}{7}\mathrm{e}^{11/80}-1\right]<\frac{5}{7},\end{aligned}$$

which shows the condition ($H_4$) is satisfied. Hence, by Theorem 2, (5.1) has a solution in $[\alpha,\beta]$ (Fig. 1).

Consider the function $\beta_1(t)=\frac{9}{26}(\mathrm{e}^t-\pi/100)$, $t\in[0,\frac{1}{3}]$. $\beta_1(t)$ is an upper solution for (5.1). Indeed, $\beta_1(0)=\frac{9}{26}(1-\pi/100)<\frac{9}{26}(\mathrm{e}^{1/3}-\pi/100)=\beta_1(\frac{1}{3})$, and take $M=1$, $N=2$,

$$\Delta\beta_1\left(\frac{1}{4}\right)=0>-\frac{2}{7}\cdot\frac{9}{26}\left(\mathrm{e}^{1/4}-\frac{\pi}{100}\right)+\frac{3}{14}\cdot\frac{9}{26}(\mathrm{e}^{1/3}-1),$$

$$\beta_1'(t)=\frac{9}{26}\mathrm{e}^t\geqslant-\left[\frac{9}{26}\left(\mathrm{e}^t-\frac{\pi}{100}\right)\right]^2-\frac{9}{13}\left[\mathrm{e}^{t/2}-\frac{\pi}{100}\right]+\frac{1}{2}\mathrm{e}^t+\frac{27}{26}(2t+1)(\mathrm{e}^{1/3}-1),$$

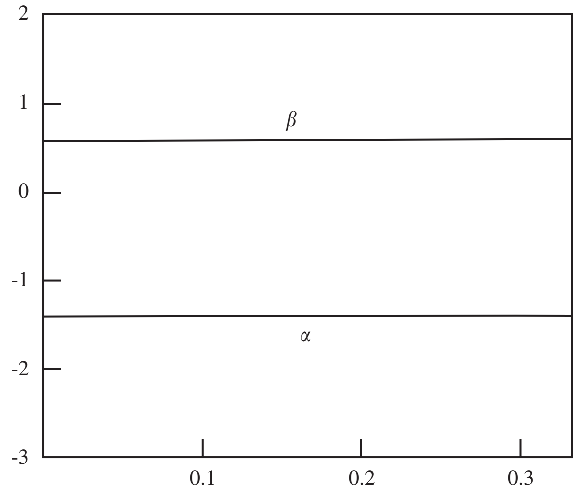


Fig. 1. Functional interval $[\alpha, \beta]$.

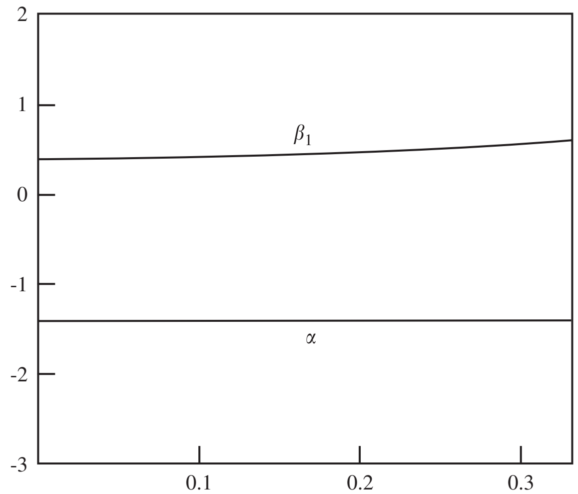


Fig. 2. Functional interval $[\alpha, \beta_1]$.

furthermore, condition $(H_4)$ is satisfied, since

$$2\int_0^{1/3} \prod_{t<t_k<1/3} (1-L_k)\mathrm{e}^{(1/2)t}\,\mathrm{d}t = 2\int_0^{1/4} \mathrm{e}^{(1/2)t}\,\mathrm{d}t + 2\int_{1/4}^{1/3} \frac{5}{7}\mathrm{e}^{(1/2)t}\,\mathrm{d}t = 4\left[\frac{5}{7}\mathrm{e}^{1/6} + \frac{2}{7}\mathrm{e}^{1/8} - 1\right] < \frac{5}{7}.$$

By Theorem 2, we obtain the existence of monotone sequences that approximate the extremal solutions of (5.1) in a functional interval contained in $[\alpha, \beta_1]$ (Fig. 2).


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