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The Riemann–Hilbert problem and the generalized Neumann kernel on multiply connected regions

Rudolf Wegmann[a,*], Mohamed M.S. Nasser[b]

[a] *Max-Planck-Institut für Astrophysik, D-85748 Garching, Germany*
[b] *Department of Mathematics, Faculty of Science, Ibb University, P.O. Box 70270, Ibb, Yemen*

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Abstract

This paper presents and studies Fredholm integral equations associated with the linear Riemann–Hilbert problems on multiply connected regions with smooth boundary curves. The kernel of these integral equations is the generalized Neumann kernel. The approach is similar to that for simply connected regions (see [R. Wegmann, A.H.M. Murid, M.M.S. Nasser, The Riemann–Hilbert problem and the generalized Neumann kernel, J. Comput. Appl. Math. 182 (2005) 388–415]). There are, however, several characteristic differences, which are mainly due to the fact, that the complement of a multiply connected region has a quite different topological structure. This implies that there is no longer perfect duality between the interior and exterior problems.

We investigate the existence and uniqueness of solutions of the integral equations. In particular, we determine the exact number of linearly independent solutions of the integral equations and their adjoints. The latter determine the conditions for solvability. An analytic example on a circular annulus and several numerically calculated examples illustrate the results.

MSC: 30E25; 45B05

Keywords: Riemann–Hilbert problems; Generalized Neumann kernel; Multiply connected regions; Fredholm integral equations

1. Introduction

Linear Riemann–Hilbert problems (RH problems, for short) play an important role in the investigation of oblique derivative problems of potential theory (see e.g., [12]). They are also very valuable tools in the iterative solution of nonlinear Riemann–Hilbert problems by the Newton method. A prominent example is conformal mapping.

For the theoretical treatment, the method of the regularizing factor reduces the problem successively to simpler problems, so-called normal forms, which can be studied more easily (see e.g., [5,7]). Under favourable conditions this approach can be used also for numerical calculations that yield fast and efficient methods. Here conformal mapping also gives good examples (see [9,10] for the simply and multiply connected case, respectively). The general nonlinear Riemann–Hilbert problems have also been treated this way ([8]).

In previous papers [3,4,11], the RH problems on simply connected regions are reduced to Fredholm integral equations of the second kind with a kernel which can be interpreted as a generalization of the Neumann kernel which is well

* Corresponding author.
E-mail address: ruw@mpa-garching.mpg.de (R. Wegmann).

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known in the study of potential theory. In order to make this method useful for numerical purposes, the questions of solvability and uniqueness have been discussed in detail in [11].

In this paper we extend this integral equation method to multiply connected regions. The situation is quite different from the simply connected case. For a simply connected Jordan region G the exterior region $G^- := \overline{\mathbb{C}}\backslash G$, i.e., the complement of G with respect to the closed complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, is again simply connected. Hence, exterior and interior RH problems are equivalent. For multiply connected regions G, however, the structures of G and G^- are quite different.

In this paper we derive the integral equations for multiply connected regions and study their properties. In particular, we investigate the existence and the uniqueness of solutions of the Fredholm integral equations. We determine the exact dimensions of the null-spaces of the integral operators. These results yield in particular the number of solutions of the homogeneous integral equations and the number of constraints which the right-hand sides of the integral equations must satisfy in order to be solvable.

A simple analytical example on a circular annulus, and several numerically calculated examples illustrate the results.

2. Auxiliary material

2.1. Regions

Let G be a bounded region of connectivity $m + 1$. The boundary $\Gamma = \partial G$ consists of $m + 1$ Jordan curves Γ_j, $j = 0, 1, \ldots, m$. The outer curve Γ_0 has counterclockwise, the inner curves, $\Gamma_1, \ldots, \Gamma_m$, have clockwise orientation (see Fig. 1). The complement $G^- := \overline{\mathbb{C}}\backslash\overline{G}$ of G with respect to the closed complex plane $\overline{\mathbb{C}}$ consists of $m + 1$ simply connected components G_j, $j = 0, 1, \ldots, m$. The component G_0 is unbounded, the components $G_1, \ldots, G_m$ are bounded.

The curve Γ_j is parametrized by a 2π-periodic complex function $\eta_j(s)$. The total parameter domain J is the disjoint union of $m + 1$ intervals $J_j = [0, 2\pi]$. We define a parametrization of the whole boundary as the complex function η defined on J by $\eta(s) := \eta_j(s)$ if $s \in J_j$, i.e.,

$$\eta(s) = \begin{cases} \eta_0(s), & s \in J_0 = [0, 2\pi], \\ \eta_1(s), & s \in J_1 = [0, 2\pi], \\ & \vdots \\ \eta_m(s), & s \in J_m = [0, 2\pi]. \end{cases}$$

One could without loss of generality assume that G is a circular region, i.e., all Γ_j are circles ([7, p. 227]). We do not use this assumption in this paper but we assume for convenience that the functions η_j are twice continuously differentiable and that the first derivative $\dot{\eta}_j \neq 0$. (A dot always denotes the derivative with respect to the parameter s.)

2.2. Cauchy integrals

For a Hölder continuous function h on Γ, the function Ψ defined by

$$\Psi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{h(\eta)}{\eta - z} \, d\eta \tag{1}$$

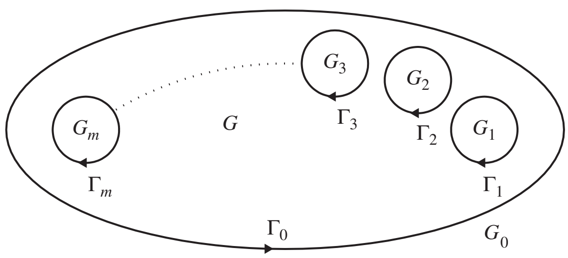

Fig. 1. The bounded multiply connected region of connectivity $m + 1$.

for $z \notin \Gamma$ is an analytic function in G as well as in G^-. The boundary values Ψ^+ from inside and Ψ^- from outside can be calculated by Plemelj's formulas

$$\Psi^{\pm}(\zeta) = \pm\frac{1}{2}h(\zeta) + \frac{1}{2\pi \mathrm{i}}\int_\Gamma \frac{h(\eta)}{\eta - \zeta}\,\mathrm{d}\eta \tag{2}$$

for $\zeta \in \Gamma$. The integral in (2) is a Cauchy principal value integral. Both boundary functions $\Psi^{\pm}$ are Hölder continuous on Γ.

2.3. Kernels

Let A be a complex function on Γ with $A \neq 0$. On the boundary component Γ_j, the function is given in parametric form $A_j(s)$. We assume that all A_j are continuously differentiable.

We define the real kernels M and N as real and imaginary parts

$$M(s,t) := \mathrm{Re}\left(\frac{1}{\pi}\frac{A(s)}{A(t)}\frac{\dot\eta(t)}{\eta(t)-\eta(s)}\right), \tag{3}$$

$$N(s,t) := \mathrm{Im}\left(\frac{1}{\pi}\frac{A(s)}{A(t)}\frac{\dot\eta(t)}{\eta(t)-\eta(s)}\right). \tag{4}$$

The function $N(s,t)$ is called the *generalized Neumann kernel* formed with A and η. For $A \equiv 1$ it reduces to the well-known Neumann kernel.

Lemma 1. (a) *The kernel $N(s,t)$ is continuous with*

$$N(t,t) = \frac{1}{\pi}\left(\frac{1}{2}|\dot\eta(t)|K(t) - \mathrm{Im}\frac{\dot A(t)}{A(t)}\right), \tag{5}$$

where $K(t)$ is the curvature of the boundary Γ.

(b) *When $s, t \in J_j$ are in the same parameter interval J_j then*

$$M(s,t) = -\frac{1}{2\pi}\cot\frac{s-t}{2} + M_1(s,t) \tag{6}$$

with a continuous kernel M_1 which takes on the diagonal the values

$$M_1(t,t) = \frac{1}{\pi}\left(\frac{1}{2}\mathrm{Re}\frac{\ddot\eta(t)}{\dot\eta(t)} - \mathrm{Re}\frac{\dot A(t)}{A(t)}\right). \tag{7}$$

Proof. The proof is the same as in the simply connected case [11, Lemma 1]. □

Part (b) of this lemma says that the singular part of the kernel M is (up to the sign) the kernel of the operator **K** of harmonic conjugation on the unit circle.

We define the Fredholm integral operator with kernel N as

$$\mathbf{N}\mu := \int_J N(s,t)\mu(t)\,\mathrm{d}t \tag{8}$$

and the singular operator with kernel M as

$$\mathbf{M}\mu := \int_J M(s,t)\mu(t)\,\mathrm{d}t. \tag{9}$$

The integral in (9) is a principal value integral.

2.4. The adjoint kernels

It follows from the representation (4) that the adjoint kernel

$$N^*(s,t) := N(t,s) = \frac{1}{\pi}\,\mathrm{Im}\left(\frac{A(t)}{A(s)}\frac{\dot\eta(s)}{\eta(s)-\eta(t)}\right)$$

can be represented as

$$N^*(s,t) = -\frac{1}{\pi}\,\mathrm{Im}\left(\frac{\tilde A(s)}{\tilde A(t)}\frac{\dot\eta(t)}{\eta(t)-\eta(s)}\right) \tag{10}$$

with the 'adjoint' function

$$\tilde A(s) := \frac{\dot\eta(s)}{A(s)}. \tag{11}$$

And similarly

$$M^*(s,t) = -\frac{1}{\pi}\,\mathrm{Re}\left(\frac{\tilde A(s)}{\tilde A(t)}\frac{\dot\eta(t)}{\eta(t)-\eta(s)}\right). \tag{12}$$

Let $\tilde N$ be the generalized Neumann kernel and $\tilde M$ the kernel M formed with the function $\tilde A$ instead of A. Then (10) and (12) mean that

$$N^* = -\tilde N, \quad M^* = -\tilde M \tag{13}$$

holds, i.e., the adjoint kernel N^* of the generalized Neumann kernel formed with the function A is the negative of the generalized Neumann kernel $\tilde N$ formed with the function $\tilde A$. The same holds for the kernel M.

2.5. Boundary values

With Hölder continuous real functions γ, μ on Γ we define the function

$$\Phi(z) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{\gamma+\mathrm{i}\mu}{A}\frac{\mathrm{d}\eta}{\eta-z} \tag{14}$$

for $z\notin\Gamma$. With the adjoint function $\tilde A$ defined in (11) we form

$$\tilde\Phi(z) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{\gamma+\mathrm{i}\mu}{\tilde A}\frac{\mathrm{d}\eta}{\eta-z} \tag{15}$$

for $z\notin\Gamma$. Both functions Φ and $\tilde\Phi$ are analytic in G as well as in G^-. They vanish at ∞. On Γ they have boundary values Φ^+ from inside and Φ^- from outside (see Section 2.2). The boundary values satisfy the jump relations

$$A\Phi^+ - A\Phi^- = \gamma + \mathrm{i}\mu, \tag{16}$$

$$\tilde A\tilde\Phi^+ - \tilde A\tilde\Phi^- = \gamma + \mathrm{i}\mu. \tag{17}$$

The boundary values can be represented in terms of the operators **N** and **M**.

Lemma 2. *The boundary values of the functions Φ defined in* (14) *and $\tilde\Phi$ defined in* (15) *can be represented by*

$$2\,\mathrm{Re}[A\Phi^\pm] = \pm\gamma + \mathbf{N}\gamma + \mathbf{M}\mu, \tag{18}$$

$$2\,\mathrm{Im}[A\Phi^\pm] = \pm\mu + \mathbf{N}\mu - \mathbf{M}\gamma, \tag{19}$$

$$2\,\mathrm{Re}[\tilde A\tilde\Phi^\pm] = \pm\gamma + \tilde{\mathbf{N}}\gamma + \tilde{\mathbf{M}}\mu, \tag{20}$$

$$2\,\mathrm{Im}[\tilde A\tilde\Phi^\pm] = \pm\mu + \tilde{\mathbf{N}}\mu - \tilde{\mathbf{M}}\gamma. \tag{21}$$

Proof. Use Plemelj's formulas (2) for the functions Φ and $\tilde{\Phi}$, and the definitions (4), (3), (10) and (12) of the kernels. □

3. Riemann–Hilbert problems

3.1. RH problems

Let A be a twice differentiable complex function on Γ with $A \neq 0$, and let γ be a Hölder continuous real function on Γ.

Interior RH problem: Search a function f analytic in G, continuous on the closure $\overline{G}$, such that the boundary values f^+ satisfy on Γ

$$\mathrm{Re}[Af^+] = \gamma. \tag{22}$$

Vekua [7, Problem A, p. 222] defines the problem with $\mathrm{Re}[\overline{A}f^+] = \gamma$, i.e., with the complex conjugate of the function A. This has the consequence that in some of the later results the index of the function A occurs with the opposite sign as in [7].

Exterior RH problem: Search a function g analytic in G^- with $g(\infty) = 0$, continuous on the closure $\overline{G^-}$, such that the boundary values g^- satisfy on Γ

$$\mathrm{Re}[Ag^-] = \gamma. \tag{23}$$

Vekua [7, p. 235] introduces a problem, 'concomitant' to the homogeneous interior RH problem (22), defined by the condition $\mathrm{Re}[\overline{A}\Phi^-] = 0$ on Γ, where Φ is analytic outside G with $\Phi(\infty) = 0$. This agrees with the homogeneous exterior RH problem (23).

3.2. Adjoint RH problems

The adjoint problems are defined with the function $\tilde{A}$ in place of A.

Interior adjoint RH problem: Search a function f analytic in G, continuous on the closure $\overline{G}$, such that the boundary values f^+ satisfy on Γ

$$\mathrm{Re}[\tilde{A}f^+] = \gamma. \tag{24}$$

Vekua [7, p. 229, Eq. (2.3)], defines the homogeneous adjoint problem by the boundary condition $\mathrm{Re}[A\dot{\eta}f^+] = 0$. Since $1/\overline{A} = A/|A|^2$ this agrees with the homogeneous problem (24).

Exterior adjoint RH problem: Search a function g analytic in G^- with $g(\infty) = 0$, continuous on the closure $\overline{G^-}$, such that the boundary values g^- satisfy on Γ

$$\mathrm{Re}[\tilde{A}g^-] = \gamma. \tag{25}$$

3.3. Range spaces

We define the range spaces of the RH problems. These are the spaces of functions γ for which the RH problems have a solution. Let H be the space of all real Hölder continuous functions on Γ.

In all definitions of this section it is implied that the analytic functions f and g are continuous up to the boundary and have Hölder continuous boundary values.

We define

$$R^+ := \{\gamma \in H : \gamma = \mathrm{Re}[Af^+],\ f \text{ analytic in } G\}, \tag{26}$$

$$R^- := \{\gamma \in H : \gamma = \mathrm{Re}[Ag^-],\ g \text{ analytic in } G^-,\ g(\infty) = 0\}. \tag{27}$$

Similarly for the adjoint problems:

$$\tilde{R}^{+} := \{\gamma \in H : \gamma = \mathrm{Re}[\tilde{A} f^{+}],\ f \text{ analytic in } G\}, \tag{28}$$

$$\tilde{R}^{-} := \{\gamma \in H : \gamma = \mathrm{Re}[\tilde{A} g^{-}],\ g \text{ analytic in } G^{-},\ g(\infty) = 0\}. \tag{29}$$

We define the spaces of the boundary values of solutions of the homogeneous RH problems:

$$S^{+} := \{\gamma \in H : \gamma = A f^{+},\ f \text{ analytic in } G\}, \tag{30}$$

$$S^{-} := \{\gamma \in H : \gamma = A g^{-},\ g \text{ analytic in } G^{-},\ g(\infty) = 0\}. \tag{31}$$

The notation in (30) implies $\mathrm{Im}[Af^{+}] = 0$ and the notation in (31) implies $\mathrm{Im}[Ag^{-}] = 0$. (Note that one can transform the homogeneous problems $\mathrm{Im}[Af^{+}] = 0$ and $\mathrm{Im}[Ag^{-}] = 0$ to our standard form by replacing A by iA.)

These definitions apply also for the adjoint problems:

$$\tilde{S}^{+} := \{\gamma \in H : \gamma = \tilde{A} f^{+},\ f \text{ analytic in } G\}, \tag{32}$$

$$\tilde{S}^{-} := \{\gamma \in H : \gamma = \tilde{A} g^{-},\ g \text{ analytic in } G^{-},\ g(\infty) = 0\}. \tag{33}$$

It is sometimes useful to have a representation of an analytic function by a Cauchy integral of the type (14) with a real density γ (i.e., $\mu = 0$). The conditions can be easily described in terms of the spaces $R^{\pm}$.

Theorem 1. (a) *A function Φ analytic in G can be represented by a Cauchy integral*

$$\Phi(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{\gamma}{A} \frac{\mathrm{d}\eta}{\eta - z} \tag{34}$$

for $z \in G$ with a real density function γ if and only if

$$\mathrm{Im}[A\Phi^{+}] \in R^{+} \cap R^{-}. \tag{35}$$

(b) *A function Φ analytic in G^{-} with $\Phi(\infty) = 0$ can be represented by the Cauchy integral* (34) *for $z \in G^{-}$ with a real density γ if and only if*

$$\mathrm{Im}[A\Phi^{-}] \in R^{+} \cap R^{-}. \tag{36}$$

Proof. (a) Assume that (34) holds. It follows from the jump relation (16) that $A\Phi^{+} = \gamma + A\Phi^{-}$, hence $\mathrm{Im}[A\Phi^{+}] = \mathrm{Im}[A\Phi^{-}] \in R^{+} \cap R^{-}$.

On the other hand if $\mathrm{Im}[A\Phi^{+}] = \mathrm{Im}[Ag^{-}] \in R^{-}$ then the density function $\gamma := A\Phi^{+} - Ag^{-}$ is real and the Cauchy integral (34) with this γ agrees in G with the function Φ.

Statement (b) is proved with the same arguments. □

4. Integral equations for RH problems

4.1. RH problem implies integral equation

There is a close connection between RH problems and integral equations with the generalized Neumann kernel.

Theorem 2. (a) *If f is a solution of the interior RH problem* (22) *with boundary values*

$$Af^{+} = \gamma + \mathrm{i}\mu \tag{37}$$

then the imaginary part μ in (37) *satisfies the integral equation*

$$\mu - \mathbf{N}\mu = -\mathbf{M}\gamma. \tag{38}$$

(b) *If g is a solution of the exterior RH problem* (23) *with boundary values*

$$Ag^{-} = \gamma + \mathrm{i}\mu \tag{39}$$

then the imaginary part μ *in* (39) *satisfies the integral equation*

$$\mu + \mathbf{N}\mu = \mathbf{M}\gamma. \tag{40}$$

(c) *If f is a solution of the interior adjoint RH problem* (24) *with boundary values*

$$\tilde{A} f^{+} = \gamma + \mathrm{i}\mu \tag{41}$$

then the imaginary part μ *in* (41) *satisfies the integral equation*

$$\mu - \tilde{\mathbf{N}}\mu = -\tilde{\mathbf{M}}\gamma. \tag{42}$$

(d) *If g is a solution of the exterior adjoint problem* (25) *with boundary values*

$$\tilde{A} g^{-} = \gamma + \mathrm{i}\mu \tag{43}$$

then the imaginary part μ *in* (43) *satisfies the integral equation*

$$\mu + \tilde{\mathbf{N}}\mu = \tilde{\mathbf{M}}\gamma. \tag{44}$$

Proof. (a) Calculate the function Φ by the Cauchy integral (14) with γ and μ satisfying (37). Then $\Phi = f$ in G and $\Phi \equiv 0$ in G^{-}. In particular $\mathrm{Im}[A\Phi^{-}] = 0$. This is equivalent to (38) in view of (19).

The statements (b), (c) and (d) are proved in the same way. □

Vekua uses the condition $\mathrm{Re}[A\Phi^{-}] = 0$ and arrives at the equation [7, p. 234, Eq. (2.25)]

$$\mathbf{M}\mu = \gamma - \mathbf{N}\gamma, \tag{45}$$

which is analogous to (38). Vekua interprets this equation as a singular integral equation of the first kind with the operator $\mathbf{M}$ for the function μ. It follows from Lemma 1 that the operator $\mathbf{M}$ differs from the operator $\mathbf{K}$ of the Hilbert transform only by the sign and a compact operator. Therefore, $\mathbf{M}$ has the same index as $\mathbf{K}$, i.e., 0. Hence Noether's theorems are applicable.

4.2. Integral equation implies RH problem

The integral equations of Theorem 2 give necessary conditions which the solutions of the RH problems have to satisfy. We now discuss the relevance of the solutions μ of these integral equations for the RH problems.

Theorem 3. *Let the real function* $\gamma \in H$ *be given.*

(a) *Let* μ *be a solution of* (38) *and* Φ *be defined by* (14). *Then* $f := \Phi$ *in* G *satisfies*

$$Af^{+} = \gamma + \gamma_0 + \mathrm{i}\mu \tag{46}$$

with $\gamma_0 \in S^{-}$.

(b) *Let* μ *be a solution of* (40) *and* Φ *be defined by* (14). *Then* $g := -\Phi$ *in* G^{-} *satisfies*

$$Ag^{-} = \gamma + \gamma_0 + \mathrm{i}\mu, \tag{47}$$

where $\gamma_0 \in S^{+}$.

Proof. (a) Let Φ be the Cauchy integral (14) formed with the functions γ and μ. The integral (38) is equivalent to $\mathrm{Im}[A\Phi^{-}] = 0$ and this is equivalent to $A\Phi^{-} \in S^{-}$. The jump relation (16) gives

$$A\Phi^{+} = \gamma + A\Phi^{-} + \mathrm{i}\mu, \tag{48}$$

which coincides with (46) since $\gamma_0 := A\Phi^{-} \in S^{-}$.

(b) In the exterior case, the integral equation (38) is equivalent to $\mathrm{Im}[A\Phi^+]=0$ and this is equivalent to $A\Phi^+ \in S^+$. The jump relation (16) gives

$$-A\Phi^- = \gamma - A\Phi^+ + \mathrm{i}\mu, \tag{49}$$

which is (47) since $A\Phi^+ \in S^+$. $\square$

Theorem 3 shows how the solutions of the integral equation must be interpreted in cases where the RH problem has no solution.

It follows from Theorem 2 that the integral (38) has a solution when the RH problem (22) is solvable. Therefore, solutions of the RH problem can always be calculated via the integral equation. The next theorem shows that in cases where the solution of the integral equation is not unique, not every solution of the integral equation leads to a solution of the RH problem.

Theorem 4. (a) *Let the real function $\gamma \in R^+$ be given and let μ be a solution of* (38) *and Φ be defined by* (14). *Then $f := \Phi$ in G satisfies*

$$Af^+ = \gamma + \gamma_0 + \mathrm{i}\mu \tag{50}$$

with $\gamma_0 \in R^+ \cap S^-$.

(b) *Let the real function $\gamma \in R^-$ be given and let μ be a solution of* (40) *and Φ be defined by* (14). *Then $g := -\Phi$ in G^- satisfies*

$$Ag^- = \gamma + \gamma_0 + \mathrm{i}\mu, \tag{51}$$

where $\gamma_0 \in R^- \cap S^+$.

Proof. (a) When $\gamma \in R^+$ it follows from (48) that also $\gamma_0 \in R^+$. By Theorem 3 (a), we have $\gamma_0 \in R^+ \cap S^-$.

Statement (b) is proved with the same arguments. $\square$

We will later investigate the space $R^+ \cap S^-$ in more detail and will show that $R^- \cap S^+ = \{0\}$.

Eq. (45) is in a sense more convenient for theoretical purposes since each solution gives a solution of the RH problem:

Lemma 3. *Let the real function γ be given. Let μ be a solution of* (45) *and Φ be defined by* (14). *Then $f := \Phi$ in G satisfies the RH condition* (22).

Proof. Since Eq. (45) is equivalent to $\mathrm{Re}[A\Phi^-]=0$ the jump relation (16) gives immediately $\mathrm{Re}[A\Phi^+]=\gamma$. $\square$

5. Properties of the operators M and N

5.1. Operator identities

Lemma 4. *The operators* **N**, **M** *and the identity operator* **I** *are connected by the following relations*:

$$\mathbf{I} = \mathbf{N}^2 - \mathbf{M}^2, \tag{52}$$

$$\mathbf{NM} + \mathbf{MN} = 0. \tag{53}$$

Proof. Start from functions $\gamma, \mu \in H$ and form the function Φ according to (14). Put $\gamma_1 = 2\,\mathrm{Re}[A\Phi^+]$, $\mu_1 = 2\,\mathrm{Im}[A\Phi^+]$ and form

$$\Phi_1(z) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{\gamma_1 + \mathrm{i}\mu_1}{A}\frac{\mathrm{d}\eta}{\eta - z} = \frac{1}{\pi\mathrm{i}}\int_\Gamma \Phi^+ \frac{\mathrm{d}\eta}{\eta - z}.$$

Then there holds $\Phi_1 \equiv 0$ in G^-. We obtain from (18) that

$$2\,\mathrm{Re}[A\Phi_1^-] = -\gamma_1 + \mathbf{N}\gamma_1 + \mathbf{M}\mu_1 = 0. \tag{54}$$

Using (18) and (19) we represent γ_1, μ_1 in terms of γ, μ and insert this into (54). We obtain the equation

$$-\gamma + \mathbf{N}^2\gamma - \mathbf{M}^2\gamma + \mathbf{NM}\mu + \mathbf{MN}\mu = 0.$$

Since this identity holds for all functions $\gamma, \mu \in H$ the operator identities (52) and (53) follow. □

5.2. *Eigenvalues of* M

The null space of the operator **M** can be represented as a direct sum. We first prove:

Lemma 5. *The spaces S^+ and S^- of solutions of the homogeneous RH problems have only the zero function in common. The same holds for the adjoint RH problems*:

$$S^+ \cap S^- = \{0\}, \quad \tilde{S}^+ \cap \tilde{S}^- = \{0\}. \tag{55}$$

Proof. Let γ be a function in $S^+ \cap S^-$. Form the function Φ according to (14) with $\mu = 0$. As element in S^+ the function γ has the representation $\gamma = Af^+$, hence $\Phi = f$ in G and $\Phi \equiv 0$ in G^-. On the other hand it follows from $\gamma \in S^-$ that $\gamma = Ag^-$ and $\Phi = -g$ in G^-. This entails $g = 0$ and $\gamma = 0$.

The second formula of (55) is proved with the same arguments. □

For simply connected regions Lemma 5 is trivial since one of the spaces S^+ and S^- is always equal to $\{0\}$. For multiply connected regions both spaces can have positive dimensions. For a simple example see Section 10.

Lemma 6. *The null-spaces of the operators* **M** *and* $\tilde{\mathbf{M}}$ *can be represented as direct sums*:

$$\mathrm{Null}(\mathbf{M}) = S^+ \oplus S^-, \quad \mathrm{Null}(\tilde{\mathbf{M}}) = \tilde{S}^+ \oplus \tilde{S}^-. \tag{56}$$

Proof. Form with $\gamma \in \mathrm{Null}(\mathbf{M})$ and with $\mu = 0$ the function Φ according to (14). It follows from (19) that $2\,\mathrm{Im}[A\Phi^\pm] = -\mathbf{M}\gamma = 0$, hence $A\Phi^\pm \in S^\pm$. The jump relation (16) shows that $\gamma = A\Phi^+ - A\Phi^- \in S^+ \oplus S^-$. The sum $S^+ \oplus S^-$ is direct in view of Lemma 5.

For the proof of the opposite inclusion assume $\gamma \in S^+$, i.e., $\gamma = Af^+$ for some function f analytic in G. Form with this γ the function Φ with $\mu = 0$. Then $\Phi \equiv 0$ in G^-, hence $2\,\mathrm{Im}[A\Phi^-] = -\mathbf{M}\gamma = 0$. An analogous argument applies for $\gamma \in S^-$.

The second formula of (56) is proved in the same way. □

Vekua [7, p. 235] obtains also the representation $\mathrm{Null}(\tilde{\mathbf{M}}) = \tilde{S}^+ \oplus \tilde{S}^-$. The functions χ_j defined in Vekua's formula (2.31) are in $\mathrm{i}\tilde{S}^+$, while the functions defined in formula (2.32) are in $\mathrm{i}\tilde{S}^-$.

We will see later that the null-space of $\mathbf{M}^2$ is in general larger than the null-space of **M**, which means that the algebraic multiplicity of the eigenvalue 0 of the operator **M** is larger than the geometric multiplicity. There is, however, in analogy to Lemma 6 a direct sum representation of the null-space of $\mathbf{M}^2$ in terms of the eigenspaces of **N** for the eigenvalues $+1$ and -1.

Lemma 7. *The null-spaces of* $\mathbf{M}^2$ *and* $\tilde{\mathbf{M}}^2$ *have the representations*:

$$\mathrm{Null}(\mathbf{M}^2) = \mathrm{Null}(\mathbf{I} - \mathbf{N}) \oplus \mathrm{Null}(\mathbf{I} + \mathbf{N}), \tag{57}$$

$$\mathrm{Null}(\tilde{\mathbf{M}}^2) = \mathrm{Null}(\mathbf{I} - \tilde{\mathbf{N}}) \oplus \mathrm{Null}(\mathbf{I} + \tilde{\mathbf{N}}). \tag{58}$$

Proof. Assume that a function $\mu \neq 0$ satisfies $\mathbf{M}^2\mu = 0$. Then it follows from (52) that $\mathbf{N}^2\mu = \mu$. Put $w := \mathbf{N}\mu$, then $\mathbf{N}w = \mu$. This implies that $\mu + w \in \mathrm{Null}(\mathbf{I} - \mathbf{N})$ and $\mu - w \in \mathrm{Null}(\mathbf{I} + \mathbf{N})$, hence $2\mu = (\mu + w) + (\mu - w) \in \mathrm{Null}(\mathbf{I} - \mathbf{N}) \oplus \mathrm{Null}(\mathbf{I} + \mathbf{N})$.

Conversely, if $\mu = \sigma_1 + \sigma_2$ with $\sigma_1 \in \mathrm{Null}(\mathbf{I} - \mathbf{N})$, $\sigma_2 \in \mathrm{Null}(\mathbf{I} + \mathbf{N})$ then

$$\mathbf{M}^2\mu = (\mathbf{N}^2 - \mathbf{I})\mu = (\mathbf{N} + \mathbf{I})(\mathbf{N} - \mathbf{I})\sigma_1 + (\mathbf{N} - \mathbf{I})(\mathbf{N} + \mathbf{I})\sigma_2 = 0.$$

This proves (57). The representation (58) is proved with the same arguments. □

5.3. Eigenvalues of $\mathbf{N}$

The solutions of the homogeneous RH problems give rise to solutions of the homogeneous integral equations (38) etc. This is the content of the following lemma.

Lemma 8. *The following inclusions hold true*:

$$S^+ \subset \mathrm{Null}(\mathbf{I} - \mathbf{N}), \quad S^- \subset \mathrm{Null}(\mathbf{I} + \mathbf{N}), \tag{59}$$

$$\tilde{S}^+ \subset \mathrm{Null}(\mathbf{I} - \tilde{\mathbf{N}}), \quad \tilde{S}^- \subset \mathrm{Null}(\mathbf{I} + \tilde{\mathbf{N}}). \tag{60}$$

Proof. Form with an element $\gamma \in S^+$ and $\mu = 0$ the function Φ. Since γ has the representation $\gamma = Af^+$ the function Φ vanishes in G^-. Hence $(\mathbf{I} - \mathbf{N})\gamma = -2\,\mathrm{Re}[A\Phi^-] = 0$. This proves the first inclusion of Lemma 8. The other inclusions are verified in the same way. □

We will show later that $S^- = \mathrm{Null}(\mathbf{I} + \mathbf{N})$. The first inclusion $S^+ \subset \mathrm{Null}(\mathbf{I} - \mathbf{N})$, however, may be strict in general.

Lemma 9. *The following equivalences hold*:

(a) $\mathbf{M}\gamma \in S^+$ *if and only if* $(\mathbf{I} + \mathbf{N})\gamma \in S^+$,
(b) $\mathbf{M}\gamma \in S^-$ *if and only if* $(\mathbf{I} - \mathbf{N})\gamma \in S^-$,
(c) $\tilde{\mathbf{M}}\gamma \in \tilde{S}^+$ *if and only if* $(\mathbf{I} + \tilde{\mathbf{N}})\gamma \in \tilde{S}^+$,
(d) $\tilde{\mathbf{M}}\gamma \in \tilde{S}^-$ *if and only if* $(\mathbf{I} - \tilde{\mathbf{N}})\gamma \in \tilde{S}^-$.

Proof. (a) Form the function Φ with $\gamma \in H$ and $\mu = 0$. Then it follows from Lemma 2 that

$$2A\Phi^+ = (\mathbf{I} + \mathbf{N})\gamma - \mathrm{i}\mathbf{M}\gamma. \tag{61}$$

If $(\mathbf{I} + \mathbf{N})\gamma = Af^+ \in S^+$ then $\mathbf{M}\gamma = A\Psi^+ \in S^+$ with $\Psi := \mathrm{i}(2\Phi - f)$.
If $\mathbf{M}\gamma = Af^+ \in S^+$ then $(\mathbf{I} + \mathbf{N})\gamma = A\Psi^+ \in S^+$ with $\Psi := 2\Phi + \mathrm{i}f$. This proves statement (a).
The equivalences (b), (c) and (d) are proved with the same arguments. □

Lemma 10. *If* λ *is an eigenvalue of* $\mathbf{N}$ *with eigenfunction* $v \notin S^+ \oplus S^-$, *then* $-\lambda$ *is also an eigenvalue of* $\mathbf{N}$. *A corresponding eigenfunction is* $w := \mathbf{M}v$.

Proof. When the eigenvalue equation $\mathbf{N}v = \lambda v$ is multiplied by $\mathbf{M}$ from the left and the anticommutation law (53) is used, the equation $\mathbf{N}\mathbf{M}v = -\lambda\mathbf{M}v$ is obtained. Since by hypothesis $v \notin S^+ \oplus S^-$, the function $w := \mathbf{M}v$ is not the zero function (in view of Lemma 6). Hence $-\lambda$ is an eigenvalue of $\mathbf{N}$ with eigenfunction w. □

6. Solvability

With the inner product

$$(\psi, \chi) := \int_J \psi(s)\chi(s)\,\mathrm{d}s \tag{62}$$

the space H is a pre-Hilbert space. We denote the orthogonal complement of a linear subspace B of H with respect to this inner product by $B^\perp$.

6.1. Solvability of the RH problems

Theorem 5. *The solvability of the RH problems is connected with the solution space of the homogeneous adjoint problems by the relations*:

$$R^+ = (\tilde{S}^+)^\perp, \quad R^- = (\tilde{S}^-)^\perp, \tag{63}$$

$$\tilde{R}^+ = (S^+)^\perp, \quad \tilde{R}^- = (S^-)^\perp. \tag{64}$$

Proof. Vekua [7, p. 236, Theorem 4.2] proves $R^+ = (\tilde{S}^+)^\perp$. The equation $R^- = (\tilde{S}^-)^\perp$ follows from the corresponding result for simply connected regions, using the fact that G^- is the union of $m+1$ simply connected regions. □

6.2. Solvability of the integral equations

Theorem 6. *The solvability of the Fredholm integral equations with generalized Neumann kernel is governed by the relations*:

$$\mathrm{Range}(\mathbf{I} - \mathbf{N}) = \mathrm{Null}(\mathbf{I} + \tilde{\mathbf{N}})^\perp, \quad \mathrm{Range}(\mathbf{I} + \mathbf{N}) = \mathrm{Null}(\mathbf{I} - \tilde{\mathbf{N}})^\perp, \tag{65}$$

$$\mathrm{Range}(\mathbf{I} - \tilde{\mathbf{N}}) = \mathrm{Null}(\mathbf{I} + \mathbf{N})^\perp, \quad \mathrm{Range}(\mathbf{I} + \tilde{\mathbf{N}}) = \mathrm{Null}(\mathbf{I} - \mathbf{N})^\perp. \tag{66}$$

Proof. The function γ is in $\mathrm{Range}(\mathbf{I} - \mathbf{N})$ if and only if the integral equation

$$(\mathbf{I} - \mathbf{N})\mu = \gamma \tag{67}$$

has a solution. In view of Fredholm's second theorem (see e.g., the handbook of integral equations [6]), Eq. (67) is solvable if and only if $\int \gamma\psi \, \mathrm{d}s = 0$ for all solutions ψ of the homogeneous transposed equation

$$(\mathbf{I} - \mathbf{N}^*)\psi = 0. \tag{68}$$

Since $N^* = -\tilde{N}$ in view of (13) the first Eq. (65) follows. The other identities are proved with the same arguments. □

Lemma 11. *The following inclusions hold*:

$$\mathbf{M}(R^-) \subset \mathrm{Range}(\mathbf{I} + \mathbf{N}) \subset R^+, \quad \mathbf{M}(R^+) \subset \mathrm{Range}(\mathbf{I} - \mathbf{N}) \subset R^-, \tag{69}$$

$$\tilde{\mathbf{M}}(\tilde{R}^-) \subset \mathrm{Range}(\mathbf{I} + \tilde{\mathbf{N}}) \subset \tilde{R}^+, \quad \tilde{\mathbf{M}}(\tilde{R}^+) \subset \mathrm{Range}(\mathbf{I} - \tilde{\mathbf{N}}) \subset \tilde{R}^-. \tag{70}$$

Proof. It follows from (18) that $\mathrm{Range}(\mathbf{I} + \mathbf{N}) \subset R^+$. On the other hand, let $\gamma \in R^-$. Then there exists $\mu \in H$ such that $\gamma + \mathrm{i}\mu = Ag^-$. Theorem 2 shows that $\mu + \mathbf{N}\mu = \mathbf{M}\gamma$, i.e., $\mathbf{M}\gamma \in \mathrm{Range}(\mathbf{I} + \mathbf{N})$. This proves the first relations of (69). The other inclusions are proved with the same arguments. □

The right-hand sides of the integral equations for the RH problems have a special form. The solvability of these equations is described in the next theorem.

Theorem 7. *The solvability of the integral equations of Theorem* 2 *is governed by the following conditions*:

(a) *the integral equation* (38) *for* μ *is solvable if and only if* $\gamma \in R^+ + S^-$;
(b) *the integral equation* (40) *for* μ *is solvable if and only if* $\gamma \in R^- + S^+$;
(c) *the integral equation* (42) *for* μ *is solvable if and only if* $\gamma \in \tilde{R}^+ + \tilde{S}^-$;
(d) *the integral equation* (44) *for* μ *is solvable if and only if* $\gamma \in \tilde{R}^- + \tilde{S}^+$.

Proof. (a) Assume $\gamma = \gamma_1 + \gamma_2$ with $\gamma_1 \in R^+$, $\gamma_2 \in S^-$ then it follows from Lemma 6 that $\mathbf{M}\gamma_2 = 0$. Relation (69) shows that the equation $\mu - \mathbf{N}\mu = -\mathbf{M}\gamma = -\mathbf{M}\gamma_1$ is solvable since $\gamma_1 \in R^+$.

Conversely, assume that γ is so that (38) is solvable. The function Φ formed with γ and μ then satisfies (46) which represents $\gamma = \gamma_1 + \gamma_0$ with $\gamma_1 \in R^+$, $\gamma_0 \in S^-$.

The other statements are proved with the same arguments. □

We will see later in Lemma 18 that the sums of vector spaces in (b) and (d) are direct, i.e., $R^- + S^+ = R^- \oplus S^+$. The sums of vector spaces $R^+ + S^-$ in (a) and (c) are in general not direct since $R^+ \cap S^- = \{0\}$ does in general not hold. This is in contrast to the simply connected case (see [11]).

Lemma 12. *The range of* $\mathbf{M}$ *is related to the solution spaces of the RH problems by the equalities*

$$\mathrm{Range}(\mathbf{M}) = R^+ \cap R^-, \quad \mathrm{Range}(\tilde{\mathbf{M}}) = \tilde{R}^+ \cap \tilde{R}^-. \tag{71}$$

Proof. It follows from Fredholm's theorems and the duality (13) that

$$\mathrm{Range}(\mathbf{M}) = [\mathrm{Null}(\mathbf{M}^*)]^\perp = [\mathrm{Null}(\tilde{\mathbf{M}})]^\perp.$$

Lemma 6 and Theorem 5 yield for the latter space the representation

$$[\mathrm{Null}(\tilde{\mathbf{M}})]^\perp = [\tilde{S}^+ \oplus \tilde{S}^-]^\perp = [\tilde{S}^+]^\perp \cap [\tilde{S}^-]^\perp = R^+ \cap R^-.$$

This proves the first relation of (71). The second is proved in the same way. □

7. Index

The *index* κ_j of the function A on the curve Γ_j is defined as the winding number of A with respect to 0

$$\kappa_j := \frac{1}{2\pi} \Delta \arg(A)|_{\Gamma_j}. \tag{72}$$

This is the change of the argument of A along the curve Γ_j divided by 2π. (Care must be taken to use the correct orientation of the curve Γ_j! (See Section 2.1.))

The index κ of the function A on the whole boundary curve Γ is the sum

$$\kappa := \sum_{j=0}^{m} \kappa_j. \tag{73}$$

Definitions (73) and (72) agree with those in [7, p. 238].

The index $\tilde{\kappa}$ of $\tilde{A}$ is defined in the same way. It is related to the index of A by the equations:

$$\tilde{\kappa}_0 = 1 - \kappa_0, \quad \tilde{\kappa}_j = -1 - \kappa_j \quad \text{for } j = 1, \ldots, m, \tag{74}$$

$$\tilde{\kappa} = \sum_{j=0}^{m} \tilde{\kappa}_j = 1 - m - \kappa. \tag{75}$$

This implies that

$$2\tilde{\kappa} + m - 1 = -2\kappa - m + 1. \tag{76}$$

The index of the function A and its adjoint determine the dimensions of several of the function spaces related to the RH problems and the integral equations with generalized Neumann kernel. This will be investigated in the next sections.

8. Solutions of the homogeneous RH problems

8.1. Exterior problem

The exterior region $G^- = G_0 \cup G_1 \cup \cdots \cup G_m$ is the union of $m + 1$ simply connected regions G_j. We can apply the results of our previous paper [11].

Let S_j^- be the subspace of S^- of real functions $\gamma = Ag^-$ where g is analytic in G_j, vanishes outside G_j and satisfies $g(\infty) = 0$. The spaces $\tilde{S}_j^-$ are defined in the same way with the function $\tilde{A}$.

Lemma 13. *The spaces S^- and $\tilde{S}^-$ are direct sums of the corresponding spaces on the regions G_j:*

$$S^- = S_0^- \oplus S_1^- \oplus \cdots \oplus S_m^-, \tag{77}$$

$$\tilde{S}^- = \tilde{S}_0^- \oplus \tilde{S}_1^- \oplus \cdots \oplus \tilde{S}_m^-. \tag{78}$$

Proof. The spaces S_j^- are subspaces of S^- and satisfy $S_j^- \cap S_k^- = \{0\}$ for $j \neq k$. Therefore, the sum on the right side of (77) is direct and is a subspace of S^-.

On the other hand, a function $\gamma \in S^-$ has the representation $\gamma = Ag^-$ with a function g analytic in G^- with $g(\infty) = 0$. This function can in a unique way be represented as a sum $g = g_0 + g_1 + \cdots + g_m$ where g_j is analytic in G_j vanishes outside G_j and g_0 satisfies $g_0(\infty) = 0$. This leads to

$$\gamma = Ag^- = \sum Ag_j^- \in S_0^- \oplus S_1^- \oplus \cdots \oplus S_m^-.$$

Eq. (78) is proved in the same way. □

Theorem 8. *The dimensions of the spaces S^- and $\tilde{S}^-$ are given by the formulas*

$$\dim(S^-) = \max(0, 2\kappa_0 - 1) + \sum_{j=1}^{m} \max(0, 2\kappa_j + 1), \tag{79}$$

$$\dim(\tilde{S}^-) = \max(0, -2\kappa_0 + 1) + \sum_{j=1}^{m} \max(0, -2\kappa_j - 1). \tag{80}$$

Proof. It follows from [11, Theorem 1] (note that the orientation of the inner curves is clockwise, which changes the sign in [11, Eq. (29)]) that

$$\dim(S_0^-) = \max(0, 2\kappa_0 - 1), \tag{81}$$

$$\dim(S_j^-) = \max(0, 2\kappa_j + 1) \quad \text{for } j = 1, \ldots, m. \tag{82}$$

Eq. (79) now follows from (77). Formula (80) follows from (79) using the relation (74). □

Combining (79) and (80) we get

$$\dim(S^-) + \dim(\tilde{S}^-) = |2\kappa_0 - 1| + \sum_{j=1}^{m} |2\kappa_j + 1|, \tag{83}$$

$$\dim(S^-) - \dim(\tilde{S}^-) = 2\kappa_0 - 1 + \sum_{j=1}^{m} (2\kappa_j + 1) = 2\kappa + m - 1. \tag{84}$$

From (79) we derive lower bounds for the dimension of S^-.

Lemma 14. *The number of linearly independent solutions of the homogeneous exterior RH problems can be estimated by*

$$\dim(S^-) \geqslant 2\kappa + m - 1, \quad \dim(\tilde{S}^-) \geqslant -2\kappa - m + 1, \tag{85}$$

$$\dim(S^-) \geqslant m + \kappa, \quad \dim(\tilde{S}^-) \geqslant 1 - \kappa. \tag{86}$$

Proof. The first estimate (85) follows from (79) by the observation that

$$\dim(S^-) \geqslant 2\kappa_0 - 1 + \sum_{j=1}^{m} (2\kappa_j + 1) = 2\kappa + m - 1.$$

For the proof of the first part of (86) we write

$$\begin{aligned}\dim(S^-) - m - \kappa = {} & \max(0, 2\kappa_0 - 1) + \sum_{j=1}^{m} \max(0, 2\kappa_j + 1) - m - \kappa \\ & = [\kappa_0 - 1 + \max(0, 1 - 2\kappa_0)] + \sum_{j=1}^{m} [\kappa_j + \max(0, -2\kappa_j - 1)]\end{aligned}$$

and note that each of the terms in square brackets is nonnegative for integer values of κ_j, $j = 0, 1, \ldots, m$.

The second parts of (85) and (86) are derived from first parts by replacing κ by $\tilde{\kappa}$ and using (75). □

8.2. Interior problem

Theorem 9. *The number of linearly independent solutions of the homogeneous interior RH problems is determined by the index in the following way*:

(a) *If* $\kappa \geqslant 1$ *then* $\tilde{\kappa} \leqslant -m$ *and*

$$\dim(S^+) = 0, \quad \dim(\tilde{S}^+) = 2\kappa + m - 1. \tag{87}$$

(b) *If* $\kappa \leqslant -m$ *then* $\tilde{\kappa} \geqslant 1$ *and*

$$\dim(S^+) = -2\kappa + 1 - m, \quad \dim(\tilde{S}^+) = 0. \tag{88}$$

(c) *If* $-m + 1 \leqslant \kappa \leqslant 0$ *then* $-m + 1 \leqslant \tilde{\kappa} \leqslant 0$ *and*

$$-2\kappa - m + 1 \leqslant \dim(S^+) \leqslant 1 - \kappa, \quad 2\kappa + m - 1 \leqslant \dim(\tilde{S}^+) \leqslant m + \kappa. \tag{89}$$

Proof. (a) and (b) follow from [7, Theorem 4.10, p. 253].

The estimates (c) follow from [7, p. 375, formula (11) and p. 376, formula (12)]. □

Lemma 15. (a) *Let* $\gamma \in S^+$ *be represented as* $\gamma = Af^+$ *with f analytic in G. Assume that the function f has* n_G *zeros in G and* n_Γ *zeros on* Γ (*counting multiplicities*). *Then these numbers are related to the index by*

$$n_\Gamma + 2n_G = \max(0, -2\kappa). \tag{90}$$

On each boundary component Γ_j *there may exist only an even number of zeros of f*.

(b) *Let* $\gamma \in \tilde{S}^+$ *be represented as* $\gamma = \tilde{A}f^+$ *with f analytic in G. Assume that the function f has* n_G *zeros in G and* n_Γ *zeros on* Γ (*counting multiplicities*). *These numbers are related to the index by*

$$n_\Gamma + 2n_G = \max(0, -2\tilde{\kappa}) = \max(0, 2(\kappa + m - 1)). \tag{91}$$

On each boundary component Γ_j *there may exist only an even number of zeros of f*.

Proof. Statement (a) follows from [7, p. 249, Theorem 4.7].

Statement (b) follows from (a) using the relation (76). □

8.3. Relations

Theorem 10. *The number of linearly independent solutions of the homogeneous RH problem and its adjoint are connected by the formulas*:

$$\dim(S^+) - \dim(\tilde{S}^+) = -\kappa + \tilde{\kappa} = 1 - m - 2\kappa, \tag{92}$$

$$\dim(S^-) - \dim(\tilde{S}^-) = \kappa - \tilde{\kappa} = 2\kappa + m - 1, \tag{93}$$

$$\dim(S^+) + \dim(S^-) = \dim(\tilde{S}^+) + \dim(\tilde{S}^-). \tag{94}$$

Proof. The Eq. (92) is from [7, p. 250, Theorem 4.9].

We have proved (93) earlier in (84).

The equality (94) is from [7, p. 250, (4.25)].

Since $\dim(\mathrm{Null}(\mathbf{M})) = \dim(\mathrm{Null}(\mathbf{M}^*)) = \dim(\mathrm{Null}(\tilde{\mathbf{M}}))$ the equality (94) follows also from Lemma 6. □

Lemma 16. *The number of linearly independent solutions of the homogeneous RH problems and the adjoint problems satisfy the estimates*:

$$\dim \tilde{S}^- \geqslant \dim S^+, \quad \dim S^- \geqslant \dim \tilde{S}^+. \tag{95}$$

Proof. The estimates of Lemma 14 show that (95) holds in all three cases listed in Theorem 9. □

Lemma 17. (a) *A function $\gamma \in S^+$ which does not vanish identically has on Γ at most -2κ zeros (counting multiplicities). The number n_j of zeros on boundary component Γ_j is even. At least one of the following relations is satisfied*:
either

$$n_0 \leqslant -2\kappa_0, \tag{96}$$

or for at least one j with $1 \leqslant j \leqslant m$

$$n_j \leqslant -2(\kappa_j + 1). \tag{97}$$

(b) *A function $\gamma \in \tilde{S}^+$ which does not vanish identically has on Γ at most $2(\kappa + m - 1)$ zeros (counting multiplicities). The number n_j of zeros on boundary component Γ_j is even. At least one of the following relations is satisfied*:
either

$$n_0 \leqslant 2(\kappa_0 - 1), \tag{98}$$

or for at least one j with $1 \leqslant j \leqslant m$

$$n_j \leqslant 2\kappa_j. \tag{99}$$

Proof. (b) We can assume that $\kappa > -m$ since otherwise $\tilde{S}^+ = \{0\}$, in view of Theorem 9, and the statement of the lemma is void.

It follows from Lemma 15 that

$$\sum n_j \leqslant 2\kappa + 2m - 2.$$

Using (85) this can be estimated

$$\sum_{j=0}^{m} n_j \leqslant \dim S^- + m - 1 < \dim S_0^- + \sum_{j=1}^{m} (\dim S_j^- + 1).$$

This inequality can hold if and only if one of the summands in the left sum is less than the corresponding summand in the right sum.

Hence either: $n_0 < \dim S_0^- = 2\kappa_0 - 1$ whence $n_0 \leqslant 2(\kappa_0 - 1)$ follows.

Or there exists an index $j \geqslant 1$ such that $n_j < \dim S_j^- + 1 = 2\kappa_j + 2$ whence $n_j \leqslant 2\kappa_j$ follows since n_j is an even number.

Statement (a) is proved with the same arguments. □

The following lemma shows that the exterior RH problem is not solvable if the right-hand side γ is a nontrivial solution of the homogeneous interior RH problem.

Lemma 18. *The spaces R^- and S^+ have only the zero function in common*:

$$R^- \cap S^+ = \{0\}, \quad \tilde{R}^- \cap \tilde{S}^+ = \{0\}. \tag{100}$$

Proof. We prove the second equality. Let $\gamma \in \tilde{R}^- \cap \tilde{S}^+$, and assume that γ is nontrivial. Then γ has n_j sign changes on the boundary component Γ_j. Since the number of sign changes is not larger than the number of zeros, the statement of Lemma 17 (b) holds true for the numbers n_j. Since also $\gamma \in \tilde{R}^-$ it must be orthogonal to all functions $\chi \in S^-$ in view of Theorem 5.

Case 1: Assume (98) holds and γ changes sign at $0 \leqslant t_1 < \cdots < t_{n_0} < 2\pi$. It follows from [11, Corollary 2(a)], that there exists $\chi \in S_0^-$ which changes sign at the n_0 points t_k and is $\neq 0$ otherwise. Hence $(\gamma, \chi) \neq 0$ for this χ.

Case 2: Assume (99) holds and γ changes sign at $0 \leqslant t_1 < \cdots < t_{n_j} < 2\pi$. It follows from [11, Corollary 2(b)] that there exists $\chi \in S_j^-$ which changes sign at the n_j points t_k and is $\neq 0$ otherwise. Hence $(\gamma, \chi) \neq 0$ for this χ.

In both cases it is shown that $\gamma \notin (S^-)^\perp = \tilde{R}^-$. This proves the second part of (100). The first part is proved in the same way using Lemma 17 (a). □

9. Null-spaces

Lemma 19. (a) *The null-space of* $\mathbf{M}^2$ *can be represented as a direct sum*

$$\mathrm{Null}(\mathbf{M}^2) = S^+ \oplus S^- \oplus W,$$

where W is isomorphic via $\mathbf{M}$ *to* $R^+ \cap S^-$.

(b) *The null-space of* $\tilde{\mathbf{M}}^2$ *can be represented as a direct sum*

$$\mathrm{Null}(\tilde{\mathbf{M}}^2) = \tilde{S}^+ \oplus \tilde{S}^- \oplus \tilde{W},$$

where $\tilde{W}$ is isomorphic via $\tilde{\mathbf{M}}$ *to* $\tilde{R}^+ \cap \tilde{S}^-$.

Proof. (1) We first prove the identity

$$R^+ \cap R^- \cap (S^+ \oplus S^-) = R^+ \cap S^-.$$

Let $c := a + b$ with $a \in S^+$, $b \in S^-$ and $c \in R^+ \cap R^-$. It follows from $c \in R^+$ that $b = c - a \in R^+$. It follows from $c \in R^-$ that $a = c - b \in R^-$.

Hence $R^+ \cap R^- \cap (S^+ \oplus S^-) \subset (R^+ \cap S^-) \oplus (R^- \cap S^+) = R^+ \cap S^-$ in view of Lemma 18.

(2) The function μ is in $\mathrm{Null}(\mathbf{M}^2)$ if and only if $\mathbf{M}\mu \in \mathrm{Null}(\mathbf{M}) = S^+ \oplus S^-$ and $\mathbf{M}\mu \in \mathrm{Range}(\mathbf{M}) = R^+ \cap R^-$, i.e.,

$$\mathbf{M}\mu \in R^+ \cap R^- \cap (S^+ \oplus S^-) = R^+ \cap S^-.$$

This proves part (a) of the lemma. Part (b) is proved with the same arguments. □

This lemma implies that

$$\dim(\mathrm{Null}(\mathbf{M}^2)) = \dim(\mathrm{Null}(\mathbf{M})) + \dim(R^+ \cap S^-). \tag{101}$$

Since $\operatorname{codim} R^+ = \dim \tilde{S}^+$ in view of Theorem 5, the inequality $\dim(R^+ \cap S^-) > 0$ holds in all cases where in (95) strict inequality holds. Then the null-space of $\mathbf{M}$ is a proper subspace of the null-space of $\mathbf{M}^2$, hence the algebraic multiplicity of the eigenvalue 0 of $\mathbf{M}$ is larger than the geometric multiplicity. This is in contrast to the simply connected case (see [11, Lemma 7]).

We determine now the solutions of the homogeneous integral equations with operators $\mathbf{I} \pm \mathbf{N}$.

Theorem 11. *The solutions of the homogeneous integral equation with operator* $\mathbf{I}+\mathbf{N}$ *coincide with the solutions of the homogeneous exterior RH problem*:

$$\mathrm{Null}(\mathbf{I}+\mathbf{N}) = S^-, \quad \mathrm{Null}(\mathbf{I}+\tilde{\mathbf{N}}) = \tilde{S}^-. \tag{102}$$

Proof. If $\mu \in \mathrm{Null}(\mathbf{I}+\mathbf{N})$ then $\mu \in \mathrm{Null}(\mathbf{M}^2)$ in view of Lemma 7. It follows from Lemma 9 that $\mathbf{M}\mu \in S^+$ and from Lemma 19 that $\mathbf{M}\mu \in R^+ \cap S^-$, hence $\mathbf{M}\mu = 0$ and $\mu \in S^+ \oplus S^-$ in view of Lemma 6. If $\mu = a + b$ with $a \in S^+$, $b \in S^-$ then Lemma 8 shows that $a \in \mathrm{Null}(\mathbf{I}-\mathbf{N})$, $b \in \mathrm{Null}(\mathbf{I}+\mathbf{N})$ hence $\mathbf{N}\mu = a - b$. On the other hand $\mu \in \mathrm{Null}(\mathbf{I}+\mathbf{N})$ implies $\mathbf{N}\mu = -a - b$, hence $a = 0$.

This proves the first part of (102). The second part is proved with the same arguments. □

It follows from Theorems 5, 6 and 11 that

$$\mathrm{Range}(\mathbf{I}-\mathbf{N}) = \mathrm{Null}(\mathbf{I}+\tilde{\mathbf{N}})^\perp = (\tilde{S}^-)^\perp = R^-, \tag{103}$$

i.e., the range of $\mathbf{I}-\mathbf{N}$ consists just of the functions γ for which the exterior RH problem is solvable.

Lemma 20. (a) *The null-space of* $\mathbf{I}-\mathbf{N}$ *has the representation*

$$\mathrm{Null}(\mathbf{I}-\mathbf{N}) = S^+ \oplus W_1,$$

where W_1 *is isomorphic* (*via* $\mathbf{M}$) *to* $R^+ \cap S^-$.

(b) *The null-space of* $\mathbf{I}-\tilde{\mathbf{N}}$ *has the representation*

$$\mathrm{Null}(\mathbf{I}-\tilde{\mathbf{N}}) = \tilde{S}^+ \oplus \tilde{W}_1,$$

where $\tilde{W}_1 \subset \mathrm{Null}(\tilde{\mathbf{M}}^2)$ *is isomorphic* (*via* $\tilde{\mathbf{M}}$) *to* $\tilde{R}^+ \cap \tilde{S}^-$.

Proof. (a) If $\mu \in \mathrm{Null}(\mathbf{I}-\mathbf{N})$ then $\mu \in \mathrm{Null}(\mathbf{M}^2)$ in view of Lemma 7, and $\mathbf{M}\mu \in S^-$ in view of Lemma 9. Hence, it follows from Lemma 19 that $\mathbf{M}\mu \in R^+ \cap S^-$, hence W_1 is isomorphic to a subspace of $R^+ \cap S^-$.

Since in view of Lemma 7

$$\mathrm{Null}(\mathbf{M}^2) = \mathrm{Null}(\mathbf{I}-\mathbf{N}) \oplus \mathrm{Null}(\mathbf{I}+\mathbf{N})$$

and $\mathrm{Null}(\mathbf{I}+\mathbf{N}) = S^-$ in view of Theorem 11, the space W_1 must have the same dimension as the space W in Lemma 19. Hence W_1 must be isomorphic to $R^+ \cap S^-$.

Part (b) is proved with the same arguments. □

We can obtain a more convenient representation of the null-space of $\mathbf{I}-\mathbf{N}$ as part of the solutions of RH problems with right-hand sides in $R^+ \cap S^-$.

Lemma 21. (a) *The null-space of* $\mathbf{I}-\mathbf{N}$ *consists of all functions* μ *such that there exists a function f analytic in G such that the boundary values satisfy* $Af^+ = \gamma + \mathrm{i}\mu$ *with* $\gamma \in R^+ \cap S^-$.

(b) *The null-space of* $\mathbf{I}-\tilde{\mathbf{N}}$ *consists of all functions* μ *such that there exists a function f analytic in G such that the boundary values satisfy* $\tilde{A}f^+ = \gamma + \mathrm{i}\mu$ *with* $\gamma \in \tilde{R}^+ \cap \tilde{S}^-$.

Proof. (a) Let $Af^+ = \gamma + \mathrm{i}\mu$ with $\gamma \in R^+ \cap S^-$, then Theorem 2 and Lemma 6 show that $(\mathbf{I}-\mathbf{N})\mu = \mathbf{M}\gamma = 0$, hence $\mu \in \mathrm{Null}(\mathbf{I}-\mathbf{N})$.

Conversely, let μ be a solution of $(\mathbf{I}-\mathbf{N})\mu=0$. In view of Theorem 4 applied with $\gamma=0$, there is an analytic function f in G with

$$Af^{+}=\gamma_0+\mathrm{i}\mu \tag{104}$$

with $\gamma_0 \in R^{+}\cap S^{-}$.

Part (b) is proved with the same arguments. □

We determine the exact number of linearly independent solutions of the homogeneous integral equations.

Theorem 12. *The number of linearly independent solutions of the homogeneous integral equations with operator* $\mathbf{I}\pm\mathbf{N}$ *is given by*

$$\dim(\mathrm{Null}(\mathbf{I}+\mathbf{N}))=\dim(\mathrm{Null}(\mathbf{I}-\tilde{\mathbf{N}}))=\max(0,2\kappa_0-1)+\sum_{j=1}^{m}\max(0,2\kappa_j+1), \tag{105}$$

$$\dim(\mathrm{Null}(\mathbf{I}-\mathbf{N}))=\dim(\mathrm{Null}(\mathbf{I}+\tilde{\mathbf{N}}))=\max(0,-2\kappa_0+1)+\sum_{j=1}^{m}\max(0,-2\kappa_j-1). \tag{106}$$

Proof. It follows from Fredholm's theorems using (13) that

$$\dim(\mathrm{Null}(\mathbf{I}-\mathbf{N}))=\dim(\mathrm{Null}(\mathbf{I}+\tilde{\mathbf{N}})), \tag{107}$$

$$\dim(\mathrm{Null}(\mathbf{I}+\mathbf{N}))=\dim(\mathrm{Null}(\mathbf{I}-\tilde{\mathbf{N}})). \tag{108}$$

The dimensions $\dim(\mathrm{Null}(\mathbf{I}+\mathbf{N}))$ and $\dim(\mathrm{Null}(\mathbf{I}+\tilde{\mathbf{N}}))$ can be calculated using Theorem 11 and Eqs. (79) and (80). □

As a corollary we can calculate the dimension of $\mathrm{Null}(\mathbf{M}^2)$ using Lemma 7 and Eq. (83)

$$\begin{aligned}\dim(\mathrm{Null}(\mathbf{M}^2)) &= \dim(\mathrm{Null}(\mathbf{I}-\mathbf{N}))+\dim(\mathrm{Null}(\mathbf{I}+\mathbf{N}))\\ &= \dim(S^{-})+\dim(\tilde{S}^{-})=|2\kappa_0-1|+\sum_{j=1}^{m}|2\kappa_j+1|.\end{aligned} \tag{109}$$

10. Analytic example

Let the doubly connected region G be a circular annulus ($m=1$) with boundary curves parametrized by

$$\Gamma_0 : \eta_0(t)=\mathrm{e}^{\mathrm{i}t},$$

$$\Gamma_1 : \eta_1(t)=R\mathrm{e}^{-\mathrm{i}t},\quad 0\leqslant t\leqslant 2\pi$$

with $0<R<1$.

We distinguish the functions on Γ_0, Γ_1 by subscripts 0, 1. The function A is constant on each boundary component: $A_0=1$, $A_1=\mathrm{e}^{\mathrm{i}\lambda}$.

Since the index $\kappa=0$ satisfies $-m<\kappa<1$, this is a 'special case' of the RH problem in the sense of Vekua [7].

For an easy description of the spaces of real functions we introduce the following notation. We write $(a,b)\cdot\mathbb{R}$ for the one-dimensional space of functions generated by the piecewise constant function $\gamma_0\equiv a$, $\gamma_1\equiv b$.

In particular, we need the two one-dimensional spaces $C_0:=(1,0)\cdot\mathbb{R}$ of functions γ_0= constant on Γ_0 and $\gamma_1\equiv 0$ on Γ_1, and the corresponding space $C_1:=(0,1)\cdot\mathbb{R}$.

We consider two cases:
Case 1: $e^{i\lambda} \neq \pm 1$.
The spaces related to RH problems are

$$R^+ = H, \quad S^+ = \{0\}, \quad R^- = (C_0)^\perp, \quad S^- = C_1,$$
$$\tilde{R}^+ = H, \quad \tilde{S}^+ = \{0\}, \quad \tilde{R}^- = (C_1)^\perp, \quad \tilde{S}^- = C_0.$$

The null-spaces are

$$\mathrm{Null}(\mathbf{M}) = C_1, \quad \mathrm{Null}(\mathbf{M}^2) = C_0 \oplus C_1,$$
$$\mathrm{Null}(\mathbf{I} + \mathbf{N}) = C_1, \quad \mathrm{Null}(\mathbf{I} - \mathbf{N}) = (1, \cos\lambda) \cdot \mathbb{R}.$$

If $\gamma = (\gamma_0, \gamma_1) \in C_0 \oplus C_1$ then $\mathbf{M}\gamma = (0, -\sin\lambda \cdot \gamma_0)$.
This example shows that the following can happen (in contrast to the simply connected case studied in [11]):

(a) The space $R^+ \cap S^-$ has positive dimension:

$$R^+ \cap S^- = S^- = C_1 \neq \{0\}.$$

(b) The algebraic multiplicity of the eigenvalue 0 of $\mathbf{M}$ is larger than the geometric multiplicity:

$$\dim\ \mathrm{Null}(\mathbf{M}^2) = \dim\ \mathrm{Null}(\mathbf{M}) + 1.$$

(c) The operator $\mathbf{N}$ has both eigenvalues $+1$ and -1:

$$\dim(\mathrm{Null}(\mathbf{I} + \mathbf{N})) = 1, \quad \dim(\mathrm{Null}(\mathbf{I} - \mathbf{N})) = 1.$$

Case 2: $e^{i\lambda} = \cos\lambda = \pm 1$.
The spaces $R^-, S^-, \tilde{R}^-, \tilde{S}^-$ are the same as in the first case, however,

$$S^+ = (1, \cos\lambda) \cdot \mathbb{R}, \quad \tilde{S}^+ = (1, -\cos\lambda) \cdot \mathbb{R},$$
$$\mathrm{Null}(\mathbf{M}) = \mathrm{Null}(\mathbf{M}^2) = C_0 \oplus C_1.$$

This shows another peculiarity of the multiply connected case:

(d) Both spaces S^+ and S^- have positive dimension:

$$\dim S^+ = 1, \quad \dim S^- = 1.$$

11. Numerical examples

Since the functions A_i and η_i are 2π-periodic, the integrals in the operator $\mathbf{N}$ can be best discretized on an equidistant grid by the trapezoidal rule, i.e., the integral operator $\mathbf{N}$ is discretized by the Nyström method [1].

Let $n > 0$ be a given integer and define the n equidistant collocation points

$$s_j := (j-1)\frac{2\pi}{n}, \quad j = 1, 2, \ldots, n, \tag{110}$$

in the interval $[0, 2\pi]$. For $0 \leqslant i, j \leqslant m$ let $\mathbf{N}_{ij}$ be the $n \times n$ matrix with elements

$$(\mathbf{N}_{ij})_{kl} := \frac{2}{n}\,\mathrm{Im}\left(\frac{A_i(s_k)}{A_j(s_l)}\frac{\dot{\eta}_j(s_l)}{\eta_j(s_l) - \eta_i(s_k)}\right) \tag{111}$$

for $1 \leqslant k, l \leqslant n$ when either $i \neq j$ or $k \neq l$. For $i = j$ and $k = l$, we put according to (5)

$$(\mathbf{N}_{ii})_{kk} := \frac{2}{n}\left(\frac{1}{2}|\dot{\eta}_i(s_k)|K_i(s_k) - \mathrm{Im}\,\frac{A_i(s_k)}{A_i(s_k)}\right), \tag{112}$$

where $K_i(t)$ is the curvature of the curve Γ_i.

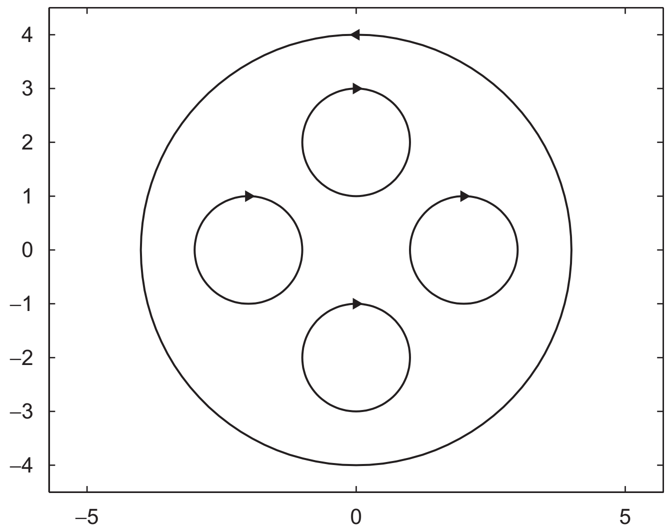

Fig. 2. The curve Γ for the Examples 1–4.

For $s_k \in J_i = [0, 2\pi]$ the integral over J_j is approximated by a sum with the elements of the matrix $\mathbf{N}_{ij}$:

$$\int_{J_j} N(s_k, t)\mu_j(t)\,\mathrm{d}t \approx \sum_{l=1}^{n} (\mathbf{N}_{ij})_{kl}\mu_j(s_l). \tag{113}$$

Therefore, the $(m+1)n \times (m+1)n$ matrix $\mathbf{N}_n$,

$$\mathbf{N}_n = \begin{pmatrix} \mathbf{N}_{00} & \mathbf{N}_{01} & \cdots & \mathbf{N}_{0m} \\ \mathbf{N}_{10} & \mathbf{N}_{11} & \cdots & \mathbf{N}_{1m} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{N}_{m0} & \mathbf{N}_{m1} & \cdots & \mathbf{N}_{mm} \end{pmatrix} \tag{114}$$

built from the blocks $\mathbf{N}_{ij}$, is a discretization of the operator $\mathbf{N}$. We consider the eigenvalues of the matrix $\mathbf{N}_n$ as approximations for the eigenvalues of the operator $\mathbf{N}$ (see e.g., [2, Chapter 3]). In the numerical examples below, the eigenvalues of the matrix $\mathbf{N}_n$ are calculated using the MATLAB function *eig*.

We consider four examples which differ by the choice of the function A. We assume that G is the region of connectivity $m+1=5$ with boundary $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ given by (see Fig. 2)

$$\Gamma_0 : \eta_0(t) = 4\mathrm{e}^{\mathrm{i}t},$$

$$\Gamma_1 : \eta_1(t) = 2 + \mathrm{e}^{-\mathrm{i}t},$$

$$\Gamma_2 : \eta_2(t) = 2\mathrm{i} + \mathrm{e}^{-\mathrm{i}t},$$

$$\Gamma_3 : \eta_3(t) = -2 + \mathrm{e}^{-\mathrm{i}t},$$

$$\Gamma_4 : \eta_4(t) = -2\mathrm{i} + \mathrm{e}^{-\mathrm{i}t}, \quad 0 \leqslant t \leqslant 2\pi.$$

We calculate the dimensions of S^+ and the null-spaces $\mathrm{Null}(\mathbf{I} \pm \mathbf{N})$ with the formulas of Theorems 9 and 12. For comparison we list in Table 1 the numerically calculated eigenvalues of $\mathbf{N}$. These results also illustrate nicely Lemma 10: all eigenvalues $\lambda \neq \pm 1$ occur in pairs $\pm\lambda$.

Example 1. Let $A_i(s) = \eta_i(s)$. Then $\kappa_0 = 1$ and $\kappa_i = 0$ for $i = 1, 2, 3, 4$ and $\kappa = 1$. Hence,

$$\dim S^+ = 0,$$

Table 1
The first 15 eigenvalues with largest moduli for the generalized Neumann kernel for the Examples 1–4

Example 1	Example 2	Example 3	Example 4
−1.0000	1.0000	1.0000	−1.0000
−1.0000	1.0000	1.0000	−1.0000
−1.0000	1.0000	1.0000	−1.0000
−1.0000	1.0000	1.0000	−1.0000
−1.0000	−1.0000	1.0000	−1.0000
−0.5181	0.5181	0.5181	−1.0000
−0.5181	0.5181	0.5181	−1.0000
0.5181	−0.5181	−0.5181	−1.0000
0.5181	−0.5181	−0.5181	−1.0000
−0.4644	0.4644	0.4644	−1.0000
0.4644	−0.4644	−0.4644	−1.0000
−0.3776	0.3776	0.3776	1.0000
0.3776	−0.3776	−0.3776	−1.0000
−0.3282	0.3282	0.3282	−0.1482
0.3282	−0.3282	−0.3282	0.1482

$$\dim\ \mathrm{Null}(\mathbf{I}+\mathbf{N}) = 5,$$

$$\dim\ \mathrm{Null}(\mathbf{I}-\mathbf{N}) = 0.$$

Example 2. Let $A_i(s) = \dot{\eta}_i(s)$. Then $\kappa_0 = 1$ and $\kappa_i = -1$ for $i = 1, 2, 3, 4$ and $\kappa = -3$. Hence,

$$\dim\ \mathrm{Null}(\mathbf{I}+\mathbf{N}) = 1,$$

$$\dim\ \mathrm{Null}(\mathbf{I}-\mathbf{N}) = 4.$$

Since $-m+1 \leqslant \kappa \leqslant 0$ this is a so-called 'special case' of the RH problem. Theorem 9 gives only the estimates

$$3 \leqslant \dim S^+ \leqslant 4, \quad 0 \leqslant \dim \tilde{S}^+ \leqslant 1.$$

However, since $\tilde{A}_i(s) = \dot{\eta}_i(s)/A_i(s) = 1$, it is easily seen that the space $\tilde{S}^+$ consists of all constant real functions on Γ. Hence, we have $\dim \tilde{S}^+ = 1$ and from Theorem 10 it follows that

$$\dim S^+ = 4, \quad \dim \tilde{S}^+ = 1.$$

Example 3. Let $A_i(s) = \dot{\eta}_i(s)/\eta_i(s)$. (This is the adjoint of the function A used in Example 1.) Then $\kappa_0 = 0$ and $\kappa_i = -1$ for $i = 1, 2, 3, 4$ and $\kappa = -4$. Hence,

$$\dim S^+ = 5,$$

$$\dim\ \mathrm{Null}(\mathbf{I}+\mathbf{N}) = 0,$$

$$\dim\ \mathrm{Null}(\mathbf{I}-\mathbf{N}) = 5.$$

Example 4. Let $A_i(s) = \eta_i(s)/\dot{\eta}_i(s)$. Then $\kappa_0 = 0$ and $\kappa_i = 1$ for $i = 1, 2, 3, 4$ and $\kappa = 4$. Hence,

$$\dim S^+ = 0,$$

$$\dim\ \mathrm{Null}(\mathbf{I}+\mathbf{N}) = 12,$$

$$\dim\ \mathrm{Null}(\mathbf{I}-\mathbf{N}) = 1.$$

12. Conclusions

We have defined and studied the boundary integral equation method for the solution of Riemann–Hilbert problems on multiply connected regions. This method has been developed and studied for simply connected regions in previous papers [3,4,11]. The results of this paper show that there are several characteristic differences between the simply and the multiply connected case.

We have determined the eigenspaces to the eigenvalues ± 1 of the integral operator with generalized Neumann kernel and its adjoint. The dimensions of these spaces yield the number of solutions of the homogeneous Fredholm equations, as well as the number of constraints, the right-hand side must satisfy for the equation to be solvable. This information about the existence and uniqueness of solutions makes these Fredholm integral equations a useful tool for the numerical solution of Riemann–Hilbert problems on multiply connected regions.

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