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Some results involving series representations of hypergeometric functions

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ABSTRACT

In this paper, we prove some generalisations of several theorems given in [K.A. Driver, S.J. Johnston, An integral representation of some hypergeometric functions, Electron. Trans. Numer. Anal. 25 (2006) 115–120] and examine some special cases which correspond to a transformation given by Chaundy in [T.W. Chaundy, An extension of hypergeometric functions, I., Quart. J. Maths. Oxford Ser. 14 (1943) 55–78] and other transformations involving the Riemann zeta function and the beta function.

1. Introduction

The general hypergeometric function is defined by

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x)=\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{x^k}{k!},\qquad |x|<1 $$

where

$$ (\alpha)_k=\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} $$

is Pochhammer's symbol and Γ is the gamma function. This function has a branch point at $x=1$. Under suitable conditions on the parameters, convergence may be obtained on the unit circle. The Euler integral representation of the ${}_2F_1$ Gauss hypergeometric function is well known in the literature (cf. [1,2]) and is formulated as follows (cf. [1], p. 65, Theorem 2.2.1).

Theorem 1.1. *If* $\mathrm{Re}(c)>\mathrm{Re}(b)>0$, *then*

$$ {}_2F_1(a,b;c;x)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\mathrm{d}t $$

in the x plane cut along the real axis from 1 to ∞. Here, it is understood that $\arg t=\arg(1-t)=0$ and $(1-xt)^{-a}$ has its principle value.

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This integral may be viewed as the analytic continuation of the ${}_2F_1$ hypergeometric series for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$. The Euler integral representation plays a prominent role in the derivation of transformation identities and the evaluation of ${}_2F_1(a, b; c; 1)$, among other applications and yields the Gauss summation formula given in the following theorem.

Theorem 1.2 *(cf. [1], p. 66, Theorem 2.2.2). For* $\mathrm{Re}(c-a-b) > 0$, *we have*

$$ {}_2F_1(a, b; c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

The case where one of the numerator parameters is a negative integer, thereby making the ${}_2F_1$ a finite sum, is known as the Chu–Vandermonde identity (cf. [1], p. 67, Corollary 2.2.3)

$$ {}_2F_1(-n, a; c; 1) = \frac{(c-a)_n}{(c)_n}. $$

The values of ${}_{q+1}F_q$ functions at 1 with $q \geq 2$ include identities due to Dougall, Dixon, Pfaff–Saalschutz, Ramanujan, Rogers, Whipple and other authors. For further discussion, see [1], Chapters 2 and 3.

The general ${}_{p+k}F_{q+k}$ hypergeometric function has an integral representation (cf. [2], Theorem 38) where the integrand involves ${}_pF_q$. In [3], a simple and direct proof of an Euler integral representation for a special class of ${}_{q+1}F_q$ functions for $q \geq 2$ is given as well as the following results.

Theorem 1.3 *(cf. [3], Theorem 2.1). For* $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$,

$$ {}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^2)^{-a} \mathrm{d}t. $$

Theorem 1.4 *(cf. [3], Theorem 2.3). If* $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ *and* $\mathrm{Re}(c-a-b) > 0$, *then*

$$ {}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)} {}_2F_1(a, b; c-a; -1). $$

Generalising Theorem 1.3 yields the following result.

Theorem 1.5 *(cf. [3], Theorem 2.5). If* $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, *then*

$$ \begin{aligned} &{}_{q+1}F_q\left(a, \frac{b}{q}, \frac{b+1}{q}, \ldots, \frac{b+q-1}{q}; \frac{c}{q}, \frac{c+1}{q}, \ldots, \frac{c+q-1}{q}; x\right) \\ &\quad = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^q)^{-a} \mathrm{d}t. \end{aligned} $$

In this paper, we give generalisations of each of the three theorems given above from [3].

2. Results

Theorem 2.1. *For* $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ *and* $r \in \mathbb{R} \setminus \{0, -1\}$,

$$ {}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{r+1}\right) = \left(\frac{r+1}{r}\right)^a \sum_{k=0}^{\infty} \binom{-a}{k} r^{-k} {}_3F_2\left(-k, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; 1\right). $$

Proof. Set $x = \frac{1}{r+1}$ in Theorem 1.3 where $r \in \mathbb{R} \setminus \{0, -1\}$, then

$$ \begin{aligned} {}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{r+1}\right) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\left(1 - \frac{1}{r+1}t^2\right)^{-a} \mathrm{d}t \\ &= (r+1)^a \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\left(r+1-t^2\right)^{-a} \mathrm{d}t \\ &= (r+1)^a \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} r^{-a}\left(1 + \frac{1-t^2}{r}\right)^{-a} \mathrm{d}t \\ &= \left(\frac{r+1}{r}\right)^a \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \sum_{k=0}^{\infty} \binom{-a}{k} r^{-k}(1-t^2)^k \mathrm{d}t \end{aligned} $$

from the binomial theorem. Then we have

$$ {}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{r+1}\right) = \left(\frac{r+1}{r}\right)^a \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \binom{-a}{k} r^{-k} \int_0^1 t^{b-1}(1-t)^{c-b-1+k}(1+t)^k \mathrm{d}t. $$

Applying the binomial theorem again,

$$ \begin{aligned} &{}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{r+1}\right) \\ &= \left(\frac{r+1}{r}\right)^a \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \binom{-a}{k} r^{-k} \int_0^1 t^{b-1}(1-t)^{c-b-1+k} \sum_{m=0}^{k} \binom{k}{m} t^m \mathrm{d}t \\ &= \left(\frac{r+1}{r}\right)^a \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{-a}{k} \binom{k}{m} r^{-k} \int_0^1 t^{b+m-1}(1-t)^{c-b-1+k} \mathrm{d}t \\ &= \left(\frac{r+1}{r}\right)^a \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{-a}{k} \binom{k}{m} r^{-k} B(b+m, c-b+k) \\ &= \left(\frac{r+1}{r}\right)^a \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{-a}{k} \binom{k}{m} r^{-k} \frac{\Gamma(b+m)\Gamma(c-b+k)\Gamma(c)}{\Gamma(m+c+k)\Gamma(b)\Gamma(c-b)} \\ &= \left(\frac{r+1}{r}\right)^a \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{-a}{k} \binom{k}{m} r^{-k} \frac{(b)_m (c-b)_k}{(c)_{m+k}} \quad \text{since} \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)} = (\alpha)_p \\ &= \left(\frac{r+1}{r}\right)^a \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{-a}{k} \binom{k}{m} r^{-k} \frac{(b)_m (c-b)_k}{(c)_k (c+k)_m} \quad \text{since} (\alpha)_{a+b} = (\alpha)_a (\alpha+a)_b \\ &= \left(\frac{r+1}{r}\right)^a \sum_{k=0}^{\infty} \binom{-a}{k} r^{-k} \frac{(c-b)_k}{(c)_k} \sum_{m=0}^{k} \binom{k}{m} \frac{(b)_m}{(c+k)_m} \\ &= \left(\frac{r+1}{r}\right)^a \sum_{k=0}^{\infty} \binom{-a}{k} r^{-k} \frac{(c-b)_k}{(c)_k} \sum_{m=0}^{k} \frac{(-1)^m (-k)_m}{m!} \frac{(b)_m}{(c+k)_m} \\ &= \left(\frac{r+1}{r}\right)^a \sum_{k=0}^{\infty} \binom{-a}{k} r^{-k} \frac{(c-b)_k}{(c)_k} {}_2F_1(-k, b; c+k; -1) \\ &= \left(\frac{r+1}{r}\right)^a \sum_{k=0}^{\infty} \binom{-a}{k} r^{-k} {}_3F_2\left(-k, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; 1\right) \quad \text{by Theorem 1.4.} \quad \square \end{aligned} $$

Theorem 2.1 is a special case of a transform of Chaundy (cf. [4], Eq. (25)) obtained by using differential operators, given by

$$ (1-z)^{-a} {}_{q+1}F_q \left(a, b_1, \dots, b_q; c_1, \dots, c_q; \tfrac{xz}{z-1}\right) = \sum_{k=0}^{\infty} \tfrac{(a)_k z^k}{k!} {}_{q+1}F_q \left(-k, b_1, \dots, b_q; c_1, \dots, c_q; x\right) \tag{2.1} $$

with $x = 1, z = -r^{-1}$ and $q = 2$.

Theorem 2.1 also yields several special cases. When $r = 1$, we obtain Corollary 2.8 from [3]. The case when $r = -2$ yields the following corollary.

Corollary 2.2. *For* $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$,

$$ {}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; -1\right) = \left(\frac{1}{2}\right)^a \sum_{k=0}^{\infty} \binom{-a}{k} (-2)^{-k} {}_3F_2\left(-k, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; 1\right). $$

We next present an application of Chaundy's transformation (2.1) to values of the Riemann zeta function $\zeta(s)$. For this we introduce the alternating zeta function given by

$$ \zeta_a(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \quad \mathrm{Re}(s) > 0. \tag{2.2} $$

Corollary 2.3. *For integers* $q \geq 2$,

$$\begin{aligned} {}_{q+1}F_q(1,\ldots,1;2,\ldots,2;-1) &= (1-2^{1-q})\zeta(q) \\ &= (1-z)\sum_{k=0}^{\infty} z^k \, {}_{q+1}F_q\left(-k,1,\ldots,1;2\ldots,2;\frac{1-z}{z}\right). \end{aligned} \tag{2.3}$$

In particular, we have

$$\begin{aligned} {}_{q+1}F_q(1,\ldots,1;2,\ldots,2;-1) &= (1-2^{1-q})\zeta(q) \\ &= \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \, {}_{q+1}F_q(-k,1,\ldots,1;2\ldots,2;1). \end{aligned} \tag{2.4}$$

Proof. From Eq. (2.2) follows the relations for integers $k \geq 2$,

$$\begin{aligned} \zeta_a(k) &= {}_{k+1}F_k(1,\ldots,1;2,\ldots,2;-1) = (1-2^{1-k})\zeta(k) \\ &= (1-2^{1-k})_{k+1}F_k(1,\ldots,1;2\ldots,2;1). \end{aligned}$$

We next apply Eq. (2.1) at $a = 1$ and $x = \frac{1-z}{z}$ and Eqs. (2.3) and (2.4) follow. □

Since for nonnegative integers n, $\zeta(2n) = (2\pi)^{2n}(-1)^{n+1}B_{2n}/2(2n)!$, where B_j are the Bernoulli numbers, Corollary 2.3 provides a hypergeometric summation representation for these special numbers.

This corollary can be extended to include the case of $q = 1$ given ${}_2F_1(1,1;2;-1) = \ln 2$ and the expansions $1-2^{1-s} = \ln 2(s-1) + O[(s-1)^2]$ and $\zeta(s) = 1/(s-1) + O(1)$ as $s \to 1$.

Theorem 2.4. *For* $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$,

$${}_2F_1\left(\frac{b}{2},\frac{b+1}{2};\frac{c+1}{2};1\right) = \frac{\Gamma(c)\Gamma\left(\frac{c}{2}-b\right)}{\Gamma(c-b)\Gamma\left(\frac{c}{2}\right)}2^{-b}.$$

Proof. We have

$${}_2F_1\left(\frac{b}{2},\frac{b+1}{2};\frac{c+1}{2};1\right) = {}_3F_2\left(\frac{c}{2},\frac{b}{2},\frac{b+1}{2};\frac{c}{2};\frac{c+1}{2};1\right). \tag{2.5}$$

Applying Theorem 1.4 to the right side, we obtain

$$\begin{aligned} {}_2F_1\left(\frac{b}{2},\frac{b+1}{2};\frac{c+1}{2};1\right) &= \frac{\Gamma(c)\Gamma\left(\frac{c}{2}-b\right)}{\Gamma(c-b)\Gamma\left(\frac{c}{2}\right)}{}_2F_1\left(\frac{c}{2},b;\frac{c}{2};-1\right) \\ &= \frac{\Gamma(c)\Gamma\left(\frac{c}{2}-b\right)}{\Gamma(c-b)\Gamma\left(\frac{c}{2}\right)}{}_1F_0\left(b;-1\right) \\ &= \frac{\Gamma(c)\Gamma\left(\frac{c}{2}-b\right)}{\Gamma(c-b)\Gamma\left(\frac{c}{2}\right)}2^{-b}. \quad \square \end{aligned}$$

We note that Theorem 1.2 could also be applied to the left side of Eq. (2.5), yielding an equivalent form. These two forms agree according to Legendre's duplication formula satisfied by the gamma function (cf. [2], p. 23). Theorem 2.4 includes the well-known very special case ${}_2F_1\left(\frac{1}{2},1;2;1\right) = 2$. We omit the details of the special case of $b = c$ on the left-hand side of Theorem 2.4.

Theorem 2.5. *If* $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, *then*

$$\begin{aligned} &{}_{q+1}F_q\left(a,\frac{b}{q},\frac{b+1}{q},\ldots,\frac{b+q-1}{q};\frac{c}{q},\frac{c+q}{2},\ldots,\frac{c+q-1}{q};x\right) \\ &\quad = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{k=0}^{\infty}\binom{c-b-1}{k}\frac{(-1)^k}{b+k}{}_2F_1\left(a,\frac{b+k}{q};\frac{b+k}{q}+1;x\right). \end{aligned}$$

Proof. By Theorem 1.5,

$$
\begin{aligned}
&{}_{q+1}F_q\left(a, \frac{b}{q}, \frac{b+1}{q}, \ldots, \frac{b+q-1}{q}; \frac{c}{q}, \frac{c+1}{q}, \ldots, \frac{c+q-1}{q}; x\right)\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^q)^{-a}\mathrm{d}t\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}\left[\sum_{k=0}^{\infty}\binom{c-b-1}{k}(-t)^k\right](1-xt^q)^{-a}\mathrm{d}t\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty}\binom{c-b-1}{k}(-1)^k \int_0^1 t^{b+k-1}(1-xt^q)^{-a}\mathrm{d}t.
\end{aligned}
$$

Making a substitution of $v = t^q$ in the integral gives

$$
\begin{aligned}
&{}_{q+1}F_q\left(a, \frac{b}{q}, \frac{b+1}{q}, \ldots, \frac{b+q-1}{q}; \frac{c}{q}, \frac{c+1}{q}, \ldots, \frac{c+q-1}{q}; x\right)\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty}\binom{c-b-1}{k}(-1)^k \int_0^1 v^{(b+k-1)/q}(1-xv)^{-a}\frac{1}{q}v^{1/q-1}\mathrm{d}v\\
&= \frac{1}{q}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty}\binom{c-b-1}{k}(-1)^k \int_0^1 v^{(b+k)/q-1}(1-xv)^{-a}\mathrm{d}v\\
&= \frac{1}{q}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty}\binom{c-b-1}{k}(-1)^k \frac{\Gamma\left(\frac{b+k}{q}\right)\Gamma(1)}{\Gamma\left(\frac{b+k}{q}+1\right)}\, {}_2F_1\left(a, \frac{b+k}{q}; \frac{b+k}{q}+1; x\right)
\end{aligned}
$$

by Theorem 1.1. Remembering the recursion formula for the gamma function given by $\Gamma(x+1) = x\Gamma(x)$, we have

$$
\begin{aligned}
&{}_{q+1}F_q\left(a, \frac{b}{q}, \frac{b+1}{q}, \ldots, \frac{b+q-1}{q}; \frac{c}{q}, \frac{c+1}{q}, \ldots, \frac{c+q-1}{q}; x\right)\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty}\binom{c-b-1}{k}(-1)^k \frac{1}{q\left(\frac{b+k}{q}\right)}\, {}_2F_1\left(a, \frac{b+k}{q}; \frac{b+k}{q}+1; x\right)\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty}\binom{c-b-1}{k}\frac{(-1)^k}{b+k}\, {}_2F_1\left(a, \frac{b+k}{q}; \frac{b+k}{q}+1; x\right). \qquad \square
\end{aligned}
$$

We give the proof of a special case of Theorem 2.5 where $q = 4$ and $x = 1$. The proof proceeds slightly differently from that of Theorem 2.5.

Theorem 2.6. *For* $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$,

$$
\begin{aligned}
&{}_5F_4\left(a, \frac{b}{4}, \frac{b+1}{4}, \frac{b+2}{4}, \frac{b+3}{4}; \frac{c}{4}, \frac{c+1}{4}, \frac{c+2}{4}, \frac{c+3}{4}; 1\right)\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{r=0}^{\infty}\binom{-a}{r}\frac{\Gamma(b+2r)\Gamma(c-a-b)}{\Gamma(c-a+2r)}\, {}_2F_1(a, b+2r; c-a+2r; -1).
\end{aligned}
$$

Proof. Let $q = 4$ and $x = 1$ in Theorem 1.5. Then

$$
\begin{aligned}
&{}_5F_4\left(a, \frac{b}{4}, \frac{b+1}{4}, \frac{b+2}{4}, \frac{b+3}{4}; \frac{c}{4}, \frac{c+1}{4}, \frac{c+2}{4}, \frac{c+3}{4}; 1\right)\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t^4)^{-a}\mathrm{d}t\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1-a}(1+t)^{-a}(1+t^2)^{-a}\mathrm{d}t\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1-a}(1+t)^{-a}\sum_{r=0}^{\infty}\binom{-a}{r}t^{2r}\mathrm{d}t
\end{aligned}
$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{r=0}^{\infty} \binom{-a}{r} \int_0^1 t^{2r+b-1}(1-t)^{c-b-1-a}(1+t)^{-a}\mathrm{d}t$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{r=0}^{\infty} \binom{-a}{r} \frac{\Gamma(b+2r)\Gamma(c-a-b)}{\Gamma(c-a+2r)} {}_2F_1(a, b+2r; c-a+2r; -1)$$

using Theorem 1.1. □

Corollary 2.7. *If* $\mathrm{Re}(1-a) > 0$, *then*

$$ {}_{q+1}F_q\left(a, \frac{b}{q}, \frac{b+1}{q}, \ldots, \frac{b+q-1}{q}; \frac{c}{q}, \frac{c+1}{q}, \ldots, \frac{c+q-1}{q}; 1\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \binom{c-b-1}{k} \frac{(-1)^k}{q} B\left(\frac{b+k}{q}, 1-a\right)$$

where B is the beta function.

Proof. Applying the Gauss summation formula (Theorem 1.2) to the result in Theorem 2.5 with $x = 1$, we obtain

$$ {}_{q+1}F_q\left(a, \frac{b}{q}, \frac{b+1}{q}, \ldots, \frac{b+q-1}{q}; \frac{c}{q}, \frac{c+1}{q}, \ldots, \frac{c+q-1}{q}; 1\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \binom{c-b-1}{k} \frac{(-1)^k}{b+k} \frac{\Gamma\left(\frac{b+k}{q}+1\right)\Gamma(1-a)}{\Gamma\left(\frac{b+k}{q}+1-a\right)}.$$

Using the recursion formula for the gamma function on the first gamma function in the numerator of the summand and remembering the representation of a beta function in terms of gamma functions given by $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, our result emerges. □

We recover the following known result for the beta function (see, for instance, [5]) as a direct consequence of Theorem 2.5.

Corollary 2.8. $B(a, b) = \sum_{k=0}^{\infty} \frac{(1-a)_k}{(b+k)k!}$.

Proof. Letting $x = 0$ in Theorem 2.5, both hypergeometric functions are equal to 1. Thus,

$$1 = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \binom{c-b-1}{k} \frac{(-1)^k}{b+k}$$

$$\begin{aligned} \Rightarrow \quad B(b, c-b) &= \sum_{k=0}^{\infty} \binom{c-b-1}{k} \frac{(-1)^k}{b+k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k(b-c+1)_k}{k!} \frac{(-1)^k}{b+k} \\ &= \sum_{k=0}^{\infty} \frac{(b-c+1)_k}{(b+k)k!}. \end{aligned}$$

Replacing $(c-b)$ with a and using $B(b, a) = B(a, b)$ give the corollary. □

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