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Homotopy perturbation–reproducing kernel method for nonlinear systems of second order boundary value problems

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ABSTRACT

In this paper, based on homotopy perturbation method (HPM) and reproducing kernel method (RKM), a new method is presented for solving nonlinear systems of second order boundary value problems (BVPs). HPM is based on the use of traditional perturbation method and homotopy technique. The HPM can reduce a nonlinear problem to a sequence of linear problems and generate a rapid convergent series solution in most cases. RKM is also an analytical technique, which can solve powerfully linear BVPs. Homotopy perturbation–reproducing kernel method (HP–RKM) combines advantages of these two methods and therefore can be used to solve efficiently systems of nonlinear BVPs. Three numerical examples are presented to illustrate the strength of the method.

1. Introduction

In this paper, we consider the following nonlinear system of second order boundary value problems in the reproducing kernel space

$$\begin{cases} u'' + a_1(x)u' + a_2(x)u + N_1(u, v) = f(x), & 0 \le x \le 1, \\ v'' + b_1(x)v' + b_2(x)v + +N_2(u, v) = g(x), & 0 \le x \le 1, \\ u(0) = u(1) = 0, \qquad v(0) = v(1) = 0, \end{cases} \tag{1.1}$$

where N_1, N_2 are nonlinear functions of u and v, $a_j(x), b_j(x)$ are continuous, $j = 1, 2, 3$. Here we only consider $u(0) = 0$, $u(1) = 0$, $v(0) = 0$, $v(1) = 0$ since the boundary conditions $u(0) = \alpha$, $u(1) = \beta$, $v(0) = \gamma$, $v(1) = \delta$ can be reduced to $u(0) = 0$, $u(1) = 0$, $v(0) = 0$, $v(1) = 0$.

Ordinary differential systems are important tools in solving real-world problems. A wide variety of natural phenomena are modelled by second order ordinary differential systems. Ordinary differential systems have been applied to many problems, in physics, engineering, biology and so on. For example, the so-called Emden–Fowler equations arise in the study of gas dynamics, fluid mechanics, relativistic mechanics, nuclear physics and also in the study of chemically reacting systems. However, many classical numerical methods used with second order initial value problems cannot be applied to

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second order boundary value problems. We all know that the finite difference method can be used to solve linear second order boundary value problems, but it can be difficult to solve nonlinear second order boundary value problems using this method. For nonlinear second order boundary value problems, there are few valid methods to obtain numerical solutions. In [1], the authors discussed the existence of solutions to second order systems, including the approximation of solutions via finite difference equations. Recently, Geng and Cui [2] presented an iterative RKM for solving nonlinear systems of second order BVPs. Dehghan [3–5] developed homotopy perturbation method, sinc-collocation and cubic *B*-spline scaling function methods. Lu [6] provided variational iteration method. Caglar [7] proposed *B*-spline method for solving linear systems of second order BVPs.

In this work, we will present a new method for obtaining the analytical approximation to the solution of nonlinear system (1.1) by combining HPM and RKM.

The HPM was proposed originally in [7–12]. This method is based on the use of traditional perturbation method and homotopy technique. Using this method, a rapid convergent series solution can be obtained in most cases. Usually, a few terms of the series solution can be used for numerical purposes with a high degree of accuracy. Furthermore, the HPM does not require the discretization of the problem. Thus it is suitable for finding the approximation of the solution without discretization of the problem. The method was successfully applied to boundary value problems, partial differential equations and other fields [7–19].

Reproducing kernel theory has important application in numerical analysis, differential equation, probability and statistics and so on [2,20–29]. Recently, using the RKM, Cui, Geng, Lin and Chen discussed singular linear two-point boundary value problem, singular nonlinear two-point periodic boundary value problem, nonlinear system of boundary value problems and nonlinear partial differential equations.

The rest of the paper is organized as follows. In the next section, the HPM is introduced. The HP–RKM is proposed for solving (1.1) in Section 3. The numerical examples are presented in Section 4. Section 5 ends this paper with a brief conclusion.

2. Analysis of HPM

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{2.1}$$

with the boundary conditions of

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{2.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω.

Generally speaking, the operator A can be divided into two parts which are L and N, where L is linear, but N is nonlinear. Eq. (2.1) can therefore be rewritten as

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega. \tag{2.3}$$

By the homotopy technique, we construct a homotopy $V(r, p) : \Omega \times [0, 1] \to R$ which satisfies:

$$H(V, p) = (1-p)[L(V) - L(u_0)] + p[A(V) - f(r)] = 0, \quad p \in [0, 1],\ r \in \Omega, \tag{2.4}$$

or

$$H(V, p) = L(V) - L(u_0) + pL(u_0) + p[N(V) - f(r)] = 0, \quad p \in [0, 1],\ r \in \Omega, \tag{2.5}$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (2.1), which satisfies the boundary conditions. Obviously, from (2.4) or (2.5), one obtains

$$H(V, 0) = L(V) - L(u_0) = 0, \tag{2.6}$$

$$H(V, 1) = A(V) - f(r) = 0, \tag{2.7}$$

the changing process of p from zero to unity is just that of $V(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(V) - L(u_0)$ and $A(V) - f(r)$ are called homotopies.

According to HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solution of (2.4) or (2.5) can be written as a power series in p:

$$V = V_0 + pV_1 + p^2V_2 + \cdots. \tag{2.8}$$

Setting $p = 1$ results in the approximate solution of Eq. (2.1):

$$u = \lim_{p\to 1} V = V_0 + V_1 + V_2 + \cdots. \tag{2.9}$$

The combination of perturbation method and homotopy method is called the HPM, which has eliminated the limitations of traditional perturbation methods. On the other hand, this technique has the full advantage of traditional perturbation techniques. The series (2.9) is convergent in most cases. However, the convergent rate depends on the nonlinear operator $A(V)$ (the following opinions are suggested in [12]).

(1) The second derivative of $N(V)$ with respect to V must be small because the parameter may be relatively large, i.e., $p \to 1$.
(2) The norm of $L^{-1}(\partial N/\partial V)$ must be smaller than one so that the series converges.

3. HP–RKM for solving (1.1)

Put

$$L_1 u = u'' + a_1(x)u' + a_2(x)u, \qquad L_2 v = v'' + b_1(x)v' + b_2(x)v.$$

For (1.1), according to the HPM, we construct a homotopy as follows:

$$\begin{cases} H_1(u,p) = L_1 u(x) - f(x) + pN_1(u(x), v(x)) = 0 \\ H_2(u,p) = L_2 v(x) - g(x) + pN_2(u(x), v(x)) = 0 \end{cases} \tag{3.1}$$

where $p \in [0, 1]$ is an embedding parameter. In case $p = 0$, (3.1) becomes a linear system, which is easy to be solved, and when $p = 1$, (3.1) turns out to be the original one, (1.1).

In view of the HPM, we use the homotopy parameter p to expand the solution

$$\begin{cases} u = u_0 + pu_1 + p^2u_2 + p^3u_3 \cdots \\ v = v_0 + pv_1 + p^2v_2 + p^3v_3 \cdots. \end{cases} \tag{3.2}$$

The approximate solution of (1.1) can be obtained by setting $p = 1$

$$\begin{cases} u = u_0 + u_1 + u_2 + u_3 \cdots \\ v = v_0 + v_1 + v_2 + v_3 \cdots. \end{cases} \tag{3.3}$$

Substituting (3.2) into (3.1), and equating coefficients of the identical powers of p yields the following equations:

$$p^0 : \begin{cases} L_1 u_0(x) = f(x), & u_0(0) = 0,\ u_0(1) = 0, \\ L_2 v_0(x) = g(x), & v_0(0) = 0,\ v_0(1) = 0 \end{cases} \tag{3.4}$$

$$p^m : \begin{cases} L_1 u_m(x) = f_m(x), & u_m(0) = 0,\ u_m(1) = 0, \\ L_2 v_m(x) = g_m(x), & v_m(0) = 0,\ v_m(1) = 0 \end{cases} \tag{3.5}$$

where $m \geq 1, f_m(x) = -\frac{d^{m-1}N_1(u,v)}{(m-1)!dp^{m-1}}\Big|_{p=0}, g_m(x) = -\frac{d^{m-1}N_2(u,v)}{(m-1)!dp^{m-1}}\Big|_{p=0}$.

To solve the above equations, we use the RKM presented in [23]. Take the following equation as an example

$$Lw(x) = h(x), \quad w(0) = w(1) = 0, \tag{3.6}$$

where $L : W_2^3[0, 1] \to W_2^1[0, 1]$ is a bounded linear operator.

To solve (3.6), first, we construct a reproducing kernel space $W_2^3[0, 1]$ in which every function satisfies the homogeneous boundary conditions of (3.6).

Reproducing kernel Hilbert space $W_2^3[0, 1]$ is defined as $W_2^3[0, 1] = \{u(x) \mid u''(x)$ is an absolutely continuous real value function, $u'''(x) \in L^2[0, 1], u(0) = 0, u(1) = 0\}$. The inner product and norm in $W_2^3[0, 1]$ are given, respectively, by

$$(u(y), v(y))_{W_2^3} = u(0)v(0) + u'(0)v'(0) + u(1)v(1) + \int_0^1 u'''v'''\mathrm{d}y$$

and

$$\|u\|_{W_2^3} = \sqrt{(u,u)_{W_2^3}}, \quad u, v \in W_2^3[0, 1].$$

By [16,20], it is easy to obtain its reproducing kernel

$$k(x,y) = \begin{cases} k_1(x,y), & y \leq x, \\ k_1(y,x), & y > x, \end{cases}$$

where $k_1(x, y) = -\frac{1}{120}(x-1)y(yx^4 - 4yx^3 + 6yx^2 + (y^4 - 5y^3 - 120y + 120)x + y^4)$.

Put $\varphi_i(x) = \overline{k}(x_i, x)$ and $\psi_i(x) = L^*\varphi_i(x)$ where $\overline{k}(x_i, x)$ is the RK of $W_2^1[0, 1]$, L^* is the adjoint operator of L. The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^3[0, 1]$ can be derived from the Gram–Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad (\beta_{ii} > 0, i = 1, 2, \ldots).$$

By the RKM presented in [23], we have the following theorem.

Theorem 3.1. *For (3.6), if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0, 1]$ and $\psi_i(x) = L_s k_\alpha(x, s)|_{s=x_i}$.*

Table 1
Exact solution and absolute errors for $u(x)$ for Example 4.1.

x	Exact solution	Method of [2]	Method of [4]	Present method ($U_{5,\,21}$)	Present method ($U_{5,\,51}$)
0.08	0.0736	5.0×10^{-4}	1.4×10^{-4}	7.7×10^{-5}	2.0×10^{-5}
0.24	0.1824	1.4×10^{-3}	4.4×10^{-5}	2.2×10^{-4}	5.7×10^{-5}
0.40	0.2400	2.1×10^{-3}	6.7×10^{-5}	3.3×10^{-4}	8.6×10^{-5}
0.56	0.2464	2.2×10^{-3}	9.3×10^{-5}	3.7×10^{-4}	9.8×10^{-5}
0.72	0.2016	1.8×10^{-3}	4.9×10^{-5}	3.1×10^{-4}	9.4×10^{-5}
0.88	0.1056	9.0×10^{-4}	8.6×10^{-5}	1.5×10^{-4}	6.5×10^{-5}
0.96	0.0384	3.0×10^{-4}	7.1×10^{-5}	5.4×10^{-5}	1.4×10^{-5}

Theorem 3.2. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0, 1]$ *and the solution of* (3.6) *is unique, then the solution of* (3.6) *is*

$$w(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} h(x_k)\overline{\psi}_i(x).$$

Using the above method, we can obtain $u_0, v_0, u_1, v_1, \ldots$

$$\begin{cases} u_0(x) = \displaystyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{1ik} f(x_k)\overline{\psi}_{1i}(x), \\ v_0(x) = \displaystyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{2ik} g(x_k)\overline{\psi}_{2i}(x) \end{cases} \tag{3.7}$$

$$\begin{cases} u_m(x) = \displaystyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{1ik} f_m(x_k)\overline{\psi}_{1i}(x), \\ v_m(x) = \displaystyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{2ik} g_m(x_k)\overline{\psi}_{2i}(x). \end{cases} \tag{3.8}$$

Therefore, the approximate solution of (1.1) and m-term approximation to solution of (1.1) to this solution are obtained

$$u = \sum_{k=0}^{\infty} u_k, \qquad v = \sum_{k=0}^{\infty} v_k, \qquad U_m = \sum_{k=0}^{m-1} u_k, \qquad V_m = \sum_{k=0}^{m-1} v_k. \tag{3.9}$$

Now, the approximate solution $U_{m,n}(x)$, $V_{m,n}(x)$ can be obtained by the n-term intercept of the $u_k(x)$, $v_k(x)$, $k = 0, 1, 2, \ldots$, and

$$U_{m,n}(x) = \sum_{k=0}^{m-1}\sum_{i=1}^{n} A_{1ik}\overline{\psi}_{1i}(x), \qquad V_{m,n}(x) = \sum_{k=0}^{m-1}\sum_{i=1}^{n} A_{2ik}\overline{\psi}_{2i}(x) \tag{3.10}$$

where $A_{1ik} = \sum_{j=1}^{i} \beta_{1ij} f_k(x_j)$, $A_{2ik} = \sum_{j=1}^{i} \beta_{2ij} g_k(x_j)$.

4. Numerical examples

In this section, we present and discuss the numerical results by employing the HP–RKM for three examples. Results demonstrate that the present method is remarkably effective.

Example 4.1. Consider the following nonlinear system of BVPs [2,4]:

$$\begin{cases} u''(x) + xu(x) + 2xv(x) + xu^2(x) = f(x), & 0 < x < 1, \\ v'(x) + v(x) + x^2u(x) + \sin x v^2(x) = g(x), & 0 < x < 1, \\ u(0) = 0, \quad u(1) = 0, \quad v(0) = 0, \quad v(1) = 0, \end{cases}$$

where $f(x) = -2 + x(x - x^2) + x(x - x^2)^2 + 2x \sin \pi x$, $g(x) = x^3(1 - x) + \sin(\pi x)(1 + \sin x \sin(\pi x)) + \pi \cos(\pi x)$. It is easy to see that the exact solution is $u(x) = x - x^2$, $v(x) = \sin \pi x$.

Solution: According to (3.7)–(3.10), one can obtain the approximation $U_{m,n}(x)$ and $V_{m,n}(x)$.

When we take $m = 5$, $n = 21, 51$, the numerical results compared with other methods are shown in Tables 1 and 2.

Table 2
Exact solution and absolute errors for $v(x)$ for Example 4.1.

x	Exact solution	Method of [2]	Method of [4]	Present method ($V_{5,\,21}$)	Present method ($V_{5,\,51}$)
0.08	0.248690	2.0×10^{-3}	2.4×10^{-4}	7.1×10^{-4}	1.1×10^{-4}
0.24	0.684547	5.6×10^{-3}	2.3×10^{-3}	1.9×10^{-3}	3.3×10^{-4}
0.40	0.951057	7.9×10^{-3}	8.9×10^{-4}	2.7×10^{-3}	4.6×10^{-4}
0.56	0.982287	8.2×10^{-3}	1.4×10^{-3}	2.8×10^{-3}	4.8×10^{-4}
0.72	0.770513	6.5×10^{-3}	3.1×10^{-3}	2.2×10^{-3}	3.8×10^{-4}
0.88	0.368125	3.1×10^{-3}	1.6×10^{-3}	1.7×10^{-3}	2.9×10^{-4}
0.96	0.125333	1.0×10^{-3}	9.8×10^{-4}	3.6×10^{-4}	6.2×10^{-5}

Table 3
Exact solution and absolute errors for $u(x)$ for Example 4.2.

x	Exact solution	Method of [6] (u_1)	Present method ($U_{5,\,21}$)	Present method ($U_{5,\,51}$)
0.00	0.00	0.00	0.00	0.00
0.10	−0.27812	3.0×10^{-4}	4.4×10^{-5}	7.1×10^{-6}
0.30	−0.56631	7.8×10^{-3}	1.0×10^{-4}	1.6×10^{-5}
0.50	−0.50000	2.7×10^{-2}	1.2×10^{-4}	1.8×10^{-5}
0.70	−0.24271	4.6×10^{-2}	9.6×10^{-5}	1.5×10^{-5}
0.90	−0.03092	3.1×10^{-2}	3.8×10^{-5}	6.0×10^{-6}
1.00	0.00	0.00	0.00	0.00

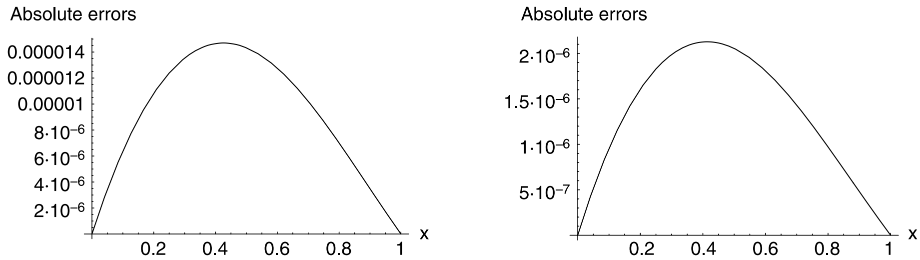

Fig. 1. Figures of absolute errors $|v(x) - V_{5,\,21}(x)|$, $|v(x) - V_{5,\,51}(x)|$ for Example 4.2.

Example 4.2. Consider the following nonlinear system of BVPs [6]:

$$\begin{cases} u''(x) + xu'(x) + \cos \pi x v'(x) = f(x), & 0 < x < 1, \\ v''(x) + xu'(x) + xu^2(x) = g(x), & 0 < x < 1, \\ u(0) = 0, \quad u(1) = 0, \quad v(0) = 0, \quad v(1) = 0, \end{cases}$$

where $f(x) = 2\cos x + (1 - 2x)\cos \pi x - (x - 1)\sin x + x((x - 1)\cos x + \sin x)$, $g(x) = -2 + (1 - 2x)x + (x - 1)^2 x \sin^2 x$. It is easy to see that the exact solution is $u(x) = (x - 1)\sin \pi x$, $v(x) = x - x^2$.

Solution: According to (3.7)–(3.10), one can obtain the approximation $U_{m,n}(x)$ and $V_{m,n}(x)$.

When we take $m = 5$, $n = 21, 51$, the numerical results compared with other methods are shown in Table 3, Fig. 1.

For the variational iteration method presented in [6], when the nonlinear terms are complicated, it is difficult to perform the iteration many times and obtain high accuracy approximations.

Example 4.3. Consider the following nonlinear system of BVPs:

$$\begin{cases} u''(x) + 20u'(x) + 4\cos x u(x) + \sin(u(x)v(x)) = f(x), & 0 < x < 1, \\ v''(x) + 5e^x v'(x) + 6\sinh x v(x) + \cos v(x) = g(x), & 0 < x < 1, \\ u(0) = 1, \quad u(1) = e, \quad v(0) = 0, \quad v(1) = \sinh 1, \end{cases} \tag{4.1}$$

where $f(x) = 21e^x + 4e^x \cos x + \sin(e^x \sinh x)$, $g(x) = \cos(\sinh x) + 5e^x \cosh x + \sinh x + 6\sinh^x$. It is easy to see that the exact solution is $u(x) = e^x$, $v(x) = \sinh x$.

Put $\overline{u} = u + a_0 + a_1 x$, $\overline{v} = v + b_0 + b_1 x$, where a_0, a_1, b_0, b_1 are determined by letting $\overline{u}(0) = \overline{u}(1) = 0$, $\overline{v}(0) = \overline{v}(1) = 0$. Obviously, (4.1) can be reduced to a system for $\overline{u}$, $\overline{v}$ with homogeneous boundary conditions.

Solution: According to (3.7)–(3.10), one can obtain the approximations $U_{m,n}(x)$ and $V_{m,n}(x)$.

When we take $m = 5$, $n = 21, 51$, the numerical results are shown in Figs. 2 and 3.

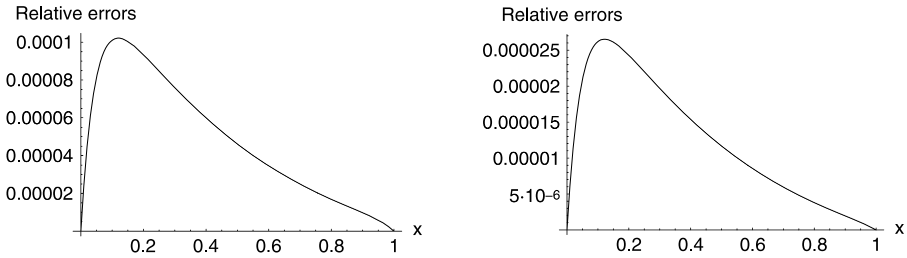

Fig. 2. Figures of relative errors $\left|\frac{u(x)-U_{5,\,21}(x)}{u(x)}\right|$, $\left|\frac{u(x)-U_{5,\,51}(x)}{u(x)}\right|$ for Example 4.3.

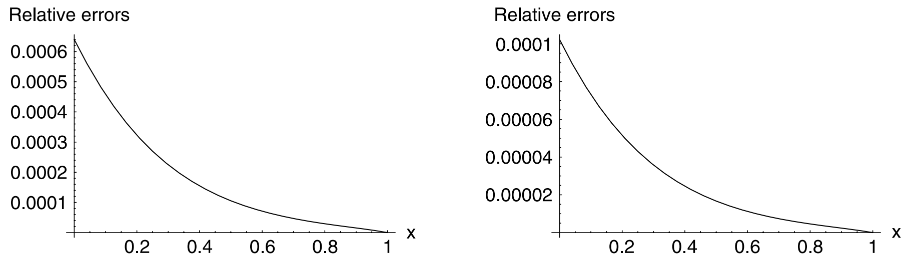

Fig. 3. Figures of relative errors $\left|\frac{v(x)-V_{5,\,21}(x)}{u(x)}\right|$, $\left|\frac{v(x)-V_{5,\,51}(x)}{v(x)}\right|$ for Example 4.3.

5. Conclusion

In this paper, the combination of HPM and RKM was employed successfully for solving nonlinear systems of boundary value problems. The numerical results show that the present method is an accurate and reliable analytical technique for the solutions of systems of boundary value problems.

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References

[1] H.B. Thompson, C. Tisdell, Boundary value problems for systems of diffference equations associated with systems of second-order ordinary differential equations, Applied Mathematics Letters 15 (6) (2002) 761–766.
[2] F.Z. Geng, M.G. Cui, Solving a nonlinear system of second order boundary value problems, Journal of Mathematical Analysis and Applications 327 (2007) 1167–1181.
[3] A. Saadatmandia, M. Dehghan, A. Eftekharia, Application of He's homotopy perturbation method for non-linear system of second-order boundary value problems, Nonlinear Analysis. Real World Applications 10 (2009) 1912–1922.
[4] M. Dehghan, A. Saadatmandi, The numerical solution of a nonlinear system of second-order boundary value problems using the sinc-collocation method, Mathematical and Computer Modelling 46 (2007) 1434–1441.
[5] M. Dehghan, M. Lakestani, Numerical solution of nonlinear system of second-order boundary value problems using cubic *B*-spline scaling functions, International Journal of Computer Mathematics 85 (9) (2008) 1455–1461.
[6] J.F. Lu, Variational iteration method for solving a nonlinear system of second-order boundary value problems, Computers and Mathematics with Applications 54 (2007) 1133–1138.
[7] N. Caglar, H. Caglar, *B*-spline method for solving linear system of second-order boundary value problems, Computers and Mathematics with Applications 57 (2009) 757–762.
[8] J.H. He, Homotopy perturbation technique, Computer Methods in Applied Mechanics and Engineering 178 (1999) 257–262.
[9] J.H. He, Homotopy perturbation method: a new nonlinear analytical technique, Applied Mathematics and Computation 135 (1) (2003) 73–79.
[10] J.H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, Applied Mathematics and Computation 151 (1) (2004) 287–292.
[11] J.H. He, Homotopy perturbation method for bifurcation of nonlinear problems, International Journal of Nonlinear Sciences and Numerical Simulation 6 (2) (2005) 207–208.
[12] J.H. He, Non-perturbative methods for strongly nonlinear problems, Disertation, de-Verlag im GmbH, Berlin, 2006.
[13] J.H. He, Some asymptotic methods for strongly nonlinear equation, International Journal of Modern Physics B 20 (10) (2006) 1141–1199.
[14] M.A. Rana, A.M. Siddiqui, Q.K. Ghori, R. Qamar, Application of He's homotopy perturbation method to Sumudu transform, International Journal of Nonlinear Sciences and Numerical Simulation 8 (2) (2007) 185–190.
[15] E. Yusufoğlu, Homotopy perturbation method for solving a nonlinear system of second order boundary value problems, International Journal of Nonlinear Sciences and Numerical Simulation 8 (3) (2007) 353–358.
[16] A. Ghorbani, J. Saberi-Nadjafi, He's homotopy perturbation method for calculating adomian polynomials, International Journal of Nonlinear Sciences and Numerical Simulation 8 (2) (2007) 229–232.

[17] A. Beléndez, C. Pascual, A. Márquez, D.I. Méndez, Application of He's homotopy perturbation method to the relativistic (an)harmonic oscillator. I: comparison between approximate and exact frequencies, International Journal of Nonlinear Sciences and Numerical Simulation 8 (4) (2007) 483–492.
[18] A. Beléndez, C. Pascual, D.I. Méndez, M.L. Álvarez, C. Neipp, Application of He's homotopy perturbation method to the relativistic (an)harmonic oscillator. II: a more accurate approximate solution, International Journal of Nonlinear Sciences and Numerical Simulation 8 (4) (2007) 493–504.
[19] D.D. Ganji, A. Sadighi, Application of homotopy-perturbation and variational iteration methods to nonlinear heat transfer and porous media equations, Journal of Computational and Applied Mathematics 207 (1) (2007) 24–34.
[20] S. Saitoh, D. Alpay, Joseph A. Ball, Takeo Ohsanwa, Reproducing Kernels and their Applications, Springer, 1999.
[21] A. Daniel, Reproducing Kernel Spaces and Applications, Springer, 2003.
[22] A. Berlinet, C. Thomas-Agnan, Reproducing Kernel Hilbert Space in Probability and Statistics, Kluwer Academic Publishers, 2004.
[23] M.G. Cui, F.Z. Geng, Solving singular two-point boundary value problem in reproducing kernel space, Journal of Computational and Applied Mathematics 205 (2007) 6–15.
[24] F.Z. Geng, M.G. Cui, Solving singular nonlinear second-order periodic boundary value problems in the reproducing kernel space, Applied Mathematics and Computation 192 (2007) 389–398.
[25] F.Z. Geng, M.G. Cui, Solving singular nonlinear two-point boundary value problems in the reproducing kernel space, Journal of the Korean Mathematical Society 45 (3) (2008) 77–87.
[26] M.G. Cui, F.Z. Geng, A computational method for solving one-dimensional variable-coefficient Burgers equation, Applied Mathematics and Computation 188 (2007) 1389–1401.
[27] M.G. Cui, Y.Z. Lin, A new method of solving the coefficient inverse problem of differential equation, Science in China Series A 50 (4) (2007) 561–572.
[28] Y.Z. Lin, M.G. Cui, L.H. Yang, Representation of the exact solution for a kind of nonlinear partial differential equations, Applied Mathematics Letters 19 (2006) 808–813.
[29] M.G. Cui, Z. Chen, The exact solution of nonlinear age-structured population model, Nonlinear Analysis. Real World Applications 8 (2007) 1096–1112.