



An iterative updating method for damped structural systems using symmetric eigenstructure assignment[☆]

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ABSTRACT

An iterative method for updating finite element models with measured modal results using a symmetric eigenstructure assignment technique is developed. By the method, the updated symmetric damping and stiffness matrices can be obtained within finite iteration steps in the absence of roundoff errors by choosing a special kind of initial matrices and the measured data are embedded in the updated model. The numerical results show that the proposed method is reliable and attractive.

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1. Introduction

Vibrating structures with feedback controls, such as bridges, buildings, airplanes and automobiles, generated by finite-element methods, are most often modeled in a system of second-order differential equations:

$$M_a \ddot{q}(t) + D_a \dot{q}(t) + K_a q(t) = Bu(t), \quad (1)$$

where M_a , D_a and K_a are $n \times n$ symmetric matrices with M_a being nonsingular, which represent the analytical mass, damping and stiffness matrices, respectively. The time-dependent variable $q(t) \in \mathbf{R}^{n \times 1}$ is the position vector, $B \in \mathbf{R}^{n \times m}$ is the full rank control feedback matrix and $u(t) \in \mathbf{R}^{m \times 1}$ is the control vector. In addition, the output or measurement vector $y(t) \in \mathbf{R}^{r \times 1}$ is given by

$$y(t) = Cq(t), \quad (2)$$

where C is a real $r \times n$ output matrix. In discussing the feedback control, we usually assume that the control vector $u(t)$ is defined by the control law

$$u(t) = Fy(t) + G\dot{y}(t), \quad (3)$$

where $F, G \in \mathbf{R}^{m \times r}$ are output feedback gain matrices. Substituting (2) and (3) into (1) yields the following closed-loop system:

$$M_a \ddot{q}(t) + (D_a - BGC)\dot{q}(t) + (K_a - BFC)q(t) = 0. \quad (4)$$

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In order to reduce the numbers of design parameters, it is desirable to set $C = B^T$. Namely, the input and output devices are placed at the same location. To solve the homogeneous second-order system Eq. (4) with $C = B^T$ is known as to solve the quadratic eigenvalue problem (QEP)

$$(\lambda^2 M_a + \lambda(D_a - BGB^T) + K_a - BFB^T)x = 0 \quad (5)$$

by letting $q(t) = xe^{\lambda t}$ in Eq. (4), where $\lambda \in \mathbf{C}$ and $x \in \mathbf{C}^{n \times 1}$ are eigenvalues and eigenvectors of QEP, respectively. It is known that the equation of (5) has $2n$ finite eigenvalues over the complex field, provided that the leading matrix coefficient M_a is nonsingular. Note that the signification of the system (4) usually can be interpreted via the eigenvalues and eigenvectors of Eq. (5). Because of this connection, a lot of efforts have been devoted to the QEP in the literature. Many applications, properties and numerical methods for the QEP are surveyed in the thesis by Tisseur and Meerbergen [1].

Finite element (FE) models are widely used to predict the dynamic characteristics of structures and these models are constructed on the basis of highly idealized engineering blueprints and designs that may not truly represent all the physical aspects of the actual structures. These models often give results that differ from the measured results and therefore need to be updated to match the measured data. FE model updating entails tuning the model so that it can better reflect the measured data from the physical structure being modeled [2]. The problem of how to modify the analytical model from the dynamic measurements is known as the model updating in structural dynamics. Basically, FE model updating is an inverse problem to identify and correct uncertain parameters of FE models and it is usually posed as an optimization problem. The updated model may then be considered a better dynamical representation of the structure and used with greater confidence for the analysis of the structure under different boundary conditions or with physical structural changes.

In the past 30 years, various techniques for updating mass and stiffness matrices for undamped systems using measured response data have been discussed by Baruch [3], Baruch and Bar-Itzhack [4], Berman [5], Berman and Nagy [6], Wei [7–9], Yang et al. [10], Yang and Chen [11], and Yuan [12]. For damped structured systems, the theory and computation have been considered by Friswell et al. [13], Pilkey [14], Kuo et al. [15], Chu et al. [16] and Yuan [17]. The finite element model correction of the closed-loop system (4) using a symmetric eigenstructure assignment was proposed in [18,19]. The method incorporates the measured modal data into the finite element model to produce an adjusted finite element model on damping and stiffness with symmetric low-rank updating that matches the experimental modal data.

In vibration industries, through vibration tests where the excitation and the response of the structure at selected points are measured experimentally, there are identification techniques to extract a portion of eigenpair information from the measurements. However, quantities related to high frequency terms in a finite-dimensional model generally are susceptible to measurement errors due to the finite bandwidth of measuring devices. It is simply unwise to use experimental values of high natural frequencies to reconstruct a model. In fact, in a large and complicated physical system, it is often impossible to acquire knowledge of the entire spectral information. While there is no reasonable analytical tool available to evaluate the entire spectral information, we can attain only partial information through experiments. For this reason, it might be more sensible to consider a model updating using only a few measured eigenvalues and eigenvectors [2,6]. However, in practice, the eigenvectors are measured only at limited degrees of freedom due to hardware limitations. There are ways to deal with incomplete measured data, such as model reduction and model expansion techniques [2]. For the purpose of this paper, we will assume that the eigenvectors have been measured to the full degree of freedom or some measures have been taken so that a comparison with analytical eigenvectors is possible.

The problem of updating damping and stiffness matrices using symmetric eigenstructure assignment can be stated as follows.

Problem P. Let $B \in \mathbf{R}^{n \times m}$ be a full column rank matrix and $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_p\} \in \mathbf{C}^{p \times p}$, $X = [x_1, \dots, x_p] \in \mathbf{C}^{n \times p}$ be the measured eigenvalue and eigenvector matrices, where $p < n$ and both Λ and X are closed under complex conjugation in the sense that $\lambda_{2j} = \bar{\lambda}_{2j-1} \in \mathbf{C}$, $x_{2j} = \bar{x}_{2j-1} \in \mathbf{C}^{n \times 1}$ for $j = 1, \dots, l$, and $\lambda_k \in \mathbf{R}$, $x_k \in \mathbf{R}^{n \times 1}$ for $k = 2l + 1, \dots, p$. Find $(\hat{G}, \hat{F}) \in \mathbf{S_E}$ such that

$$\|B\hat{G}B^T\|^2 + \|B\hat{F}B^T\|^2 = \min_{(G,F) \in \mathbf{S_E}} (\|BGB^T\|^2 + \|BFB^T\|^2),$$

where

$$\mathbf{S_E} = \{(G, F) \in \mathbf{SR}^{m \times m} \times \mathbf{SR}^{m \times m} | M_a X \Lambda^2 + (D_a - BGB^T)X\Lambda + (K_a - BFB^T)X = 0\}.$$

Note that the studies in the works by Chu and Datta [20], Datta [21], Nichols and Kautsky [22], Datta et al. [23], Datta and Sarkissian [24] and Lin and Wang [25] lead to a feedback design problem for a second-order control system. That consideration eventually results in either a full or a partial eigenstructure assignment problem for the QEP. The proportional and derivative state feedback controller designated in these studies is capable of assigning specific eigenvalues and making the resulting system insensitive to perturbations. Nonetheless, these results cannot meet the basic requirement that the updated matrices should be symmetric. Recently, Kuo, Lin and Xu [26] have presented a direct correcting method for quadratic eigenvalue problems using symmetric eigenstructure assignment and it seems that the algorithm proposed is reliable and attractive. However, we observe that in order to use this method, one must solve the system of linear equation (40) of [26], which involves in intricate matrix computations. In view of the complicated representation of solutions for the direct updating method, our main contribution in this paper is to provide an alternative iterative method to solve Problem P.

By the method, an updating model can be obtained and the measured data are embedded in the new model. The paper is organized as follows. In Section 2, an efficient iterative method is presented and several properties of Algorithm 1 are proved. By using the proposed iterative method, the unique solution of Problem P can be obtained within finite iteration steps in the absence of roundoff errors by choosing a special kind of initial matrices. In Section 3, two numerical examples are used to test the effectiveness of the proposed algorithm.

Throughout this paper, we shall adopt the following notation. $\mathbf{C}^{m \times n}$ and $\mathbf{R}^{m \times n}$ denote the set of all $m \times n$ complex and real matrices, $\mathbf{SR}^{n \times n}$ denotes the set of all $n \times n$ symmetric matrices in $\mathbf{R}^{n \times n}$. A^\top , $\text{tr}(A)$ and $R(A)$ stand for the transpose, the trace and the column space of the matrix A , respectively. I_n represents the identity matrix of order n . For $A, B \in \mathbf{R}^{m \times n}$, an inner product in $\mathbf{R}^{m \times n}$ is defined by $(A, B) = \text{tr}(B^\top A)$, then $\mathbf{R}^{m \times n}$ is a Hilbert space. The matrix norm $\|\cdot\|$ induced by the inner product is the Frobenius norm. Given two matrices $A = [a_{ij}] \in \mathbf{R}^{m \times n}$ and $B \in \mathbf{R}^{p \times q}$, the Kronecker product of A and B is defined by $A \otimes B = [a_{ij}B] \in \mathbf{R}^{mp \times nq}$. Also, for an $m \times n$ matrix $A = [a_1, a_2, \dots, a_n]$, where $a_i, i = 1, \dots, n$, is the i -th column vector of A , the stretching function $\text{vec}(A)$ is defined as $\text{vec}(A) = [a_1^\top, a_2^\top, \dots, a_n^\top]^\top$. Let A, B and X be some matrices with appropriate dimensions, then we have the following well-known identity [27]: $\text{vec}(AXB) = (B^\top \otimes A)\text{vec}(X)$.

2. The solution of Problem P

Let $\alpha_i = \text{Re}(\lambda_i)$ (the real part of the complex number λ_i), $0 < \beta_i = \text{Im}(\lambda_i)$ (the imaginary part of the complex number λ_i), $y_i = \text{Re}(x_i)$, $z_i = \text{Im}(x_i)$ for $i = 1, 3, \dots, 2l-1$, and

$$\tilde{A} = \text{diag} \left\{ \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \dots, \begin{bmatrix} \alpha_{2l-1} & \beta_{2l-1} \\ -\beta_{2l-1} & \alpha_{2l-1} \end{bmatrix}, \lambda_{2l+1}, \dots, \lambda_p \right\} \in \mathbf{R}^{p \times p},$$

$$\tilde{X} = [y_1, z_1, \dots, y_{2l-1}, z_{2l-1}, x_{2l+1}, \dots, x_p] \in \mathbf{R}^{n \times p}.$$

Then, the equation $M_a X \Lambda^2 + (D_a - BGB^\top)X\Lambda + (K_a - BFB^\top)X = 0$ can be equivalently written as

$$BGB^\top \tilde{X} \tilde{A} + BFB^\top \tilde{X} = M_a \tilde{X} \tilde{A}^2 + D_a \tilde{X} \tilde{A} + K_a \tilde{X}, \quad \text{s.t. } G \in \mathbf{SR}^{m \times m}, F \in \mathbf{SR}^{m \times m}. \quad (6)$$

Let QR-decomposition of B be

$$B = Q \begin{bmatrix} R \\ 0 \end{bmatrix}, \quad (7)$$

where $Q = [Q_1, Q_2]$ is an $n \times n$ orthogonal matrix ($Q_1 \in \mathbf{R}^{n \times m}$) and R is an $m \times m$ nonsingular matrix. Substituting (7) into (6), we obtain

$$\tilde{G} Q_1^\top \tilde{X} \tilde{A} + \tilde{F} Q_1^\top \tilde{X} = Q_1^\top H, \quad (8)$$

$$Q_2^\top H = 0, \quad (9)$$

where $H = M_a \tilde{X} \tilde{A}^2 + D_a \tilde{X} \tilde{A} + K_a \tilde{X}$, $\tilde{F} = RFR^\top$, $\tilde{G} = RGR^\top$. Observe that $\|BGB^\top\|^2 + \|BFB^\top\|^2 = \|\tilde{G}\|^2 + \|\tilde{F}\|^2$. Therefore, when the condition (9) is satisfied, we can easily see that solving Problem P is equivalent to finding the minimum Frobenius norm solution of the matrix equation (8). Once the minimum Frobenius norm solution $[\tilde{G}_{\min}, \tilde{F}_{\min}]$ of (8) is obtained, the solution of the matrix optimal approximation Problem P can be computed. In this case, it can be expressed as

$$\hat{G} = R^{-1} \tilde{G}_{\min} R^{-\top}, \quad \hat{F} = R^{-1} \tilde{F}_{\min} R^{-\top}, \quad (10)$$

and the updated damping and stiffness matrices can be expressed as

$$\hat{D} = D_a - B\hat{G}B^\top, \quad \hat{K} = K_a - B\hat{F}B^\top. \quad (11)$$

Now, we can describe an iterative algorithm for solving Eq. (8) as follows.

Algorithm 1. Step 1. Input matrices $\tilde{X} \in \mathbf{R}^{n \times p}$, $\tilde{A} \in \mathbf{R}^{p \times p}$ and $M_a \in \mathbf{SR}^{n \times n}$ and choose arbitrary $m \times m$ symmetric matrices \tilde{G}_1 and \tilde{F}_1 .

Step 2. Calculate

$$R_1 = Q_1^\top H - \tilde{G}_1 Q_1^\top \tilde{X} \tilde{A} - \tilde{F}_1 Q_1^\top \tilde{X};$$

$$P_1 = \frac{1}{2}(R_1 \tilde{A}^\top \tilde{X}^\top Q_1 + Q_1^\top \tilde{X} \tilde{A} R_1^\top);$$

$$W_1 = \frac{1}{2}(R_1 \tilde{X}^\top Q_1 + Q_1^\top \tilde{X} R_1^\top);$$

$$s := 1.$$

Step 3. If $R_s = 0$, then stop and $[\tilde{G}_s, \tilde{F}_s]$ is a solution to the equation of (8); elseif $R_s \neq 0$ but $P_s = 0$ and $W_s = 0$, then stop and the equation of (8) is not consistent over symmetric matrices; else $s := s + 1$.

Step 4. Calculate

$$\begin{aligned}\tilde{G}_s &= \tilde{G}_{s-1} + \frac{\|R_{s-1}\|^2}{\|P_{s-1}\|^2 + \|W_{s-1}\|^2} P_{s-1}; \\ \tilde{F}_s &= \tilde{F}_{s-1} + \frac{\|R_{s-1}\|^2}{\|P_{s-1}\|^2 + \|W_{s-1}\|^2} W_{s-1}; \\ R_s &= Q_1^\top H - \tilde{G}_s Q_1^\top \tilde{X} \tilde{\Lambda} - \tilde{F}_s Q_1^\top \tilde{X} = R_{s-1} - \frac{\|R_{s-1}\|^2}{\|P_{s-1}\|^2 + \|W_{s-1}\|^2} (P_{s-1} Q_1^\top \tilde{X} \tilde{\Lambda} + W_{s-1} Q_1^\top \tilde{X}); \\ P_s &= \frac{1}{2} (R_s \tilde{\Lambda}^\top \tilde{X}^\top Q_1 + Q_1^\top \tilde{X} \tilde{\Lambda} R_s^\top) + \frac{\|R_s\|^2}{\|R_{s-1}\|^2} P_{s-1}; \\ W_s &= \frac{1}{2} (R_s \tilde{X}^\top Q_1 + Q_1^\top \tilde{X} R_s^\top) + \frac{\|R_s\|^2}{\|R_{s-1}\|^2} W_{s-1}.\end{aligned}$$

Step 5. Go to Step 3.

From Algorithm 1, we can easily see that $\tilde{G}_s, P_s \in \mathbf{SR}^{m \times m}$ and $\tilde{F}_s, W_s \in \mathbf{SR}^{m \times m}$ for $s = 1, 2, \dots$.

Definition 1. Assume that $Y, Z \in \mathbf{R}^{m \times n}$. The matrices Y, Z are called orthogonal to each other if $\text{tr}(Y^\top Z) = 0$. About Algorithm 1, we present the following basic properties.

Lemma 1. The sequences $\{R_i\}$, $\{P_i\}$ and $\{W_i\}$ generated by Algorithm 1 satisfy

$$\text{tr}(R_j^\top R_i) = 0, \quad \text{and} \quad \text{tr}(P_j^\top P_i) + \text{tr}(W_j^\top W_i) = 0 \quad \text{for } i, j = 1, 2, \dots, s, \quad i \neq j. \quad (12)$$

The proof of Lemma 1 is presented in the Appendix.

Lemma 2. Let Eq. (8) be consistent, and $[\tilde{G}^*, \tilde{F}^*]$ be an arbitrary solution pair of Eq. (8). Then, for any initial matrix pair $[\tilde{G}_1, \tilde{F}_1]$ with $\tilde{G}_1, \tilde{F}_1 \in \mathbf{SR}^{m \times m}$, we have

$$\text{tr}((\tilde{G}^* - \tilde{G}_i)^\top P_i) + \text{tr}((\tilde{F}^* - \tilde{F}_i)^\top W_i) = \|R_i\|^2 \quad \text{for } i = 1, 2, \dots, \quad (13)$$

where the sequences $\{\tilde{G}_i\}$, $\{\tilde{F}_i\}$, $\{R_i\}$, $\{P_i\}$ and $\{W_i\}$ are generated by Algorithm 1.

The proof of Lemma 2 is also presented in the Appendix.

From Lemma 2, we can easily see that if there exists a positive number l such that $P_l = 0$ and $W_l = 0$ but $R_l \neq 0$, then the equation of (8) is not consistent over symmetric matrix pair $[\tilde{G}, \tilde{F}]$. Hence, the solvability of the equation of (8) over symmetric matrix pair $[\tilde{G}, \tilde{F}]$ can be determined automatically by Algorithm 1.

Theorem 1. Assume that Eq. (8) is consistent. Then for any arbitrary initial matrix pair $[\tilde{G}_1, \tilde{F}_1]$ with $\tilde{G}_1, \tilde{F}_1 \in \mathbf{SR}^{m \times m}$, a symmetric solution pair of Eq. (8) can be obtained with finite iteration steps in the absence of roundoff errors.

Proof. Assume that $R_l \neq 0$, $l = 1, 2, \dots, mp$. From Lemma 2, we know $P_l \neq 0$ or $W_l \neq 0$. Then we can calculate R_{mp+1} and $[\tilde{G}_{mp+1}, \tilde{F}_{mp+1}]$ by Algorithm 1. From Lemma 1, we have

$$\text{tr}(R_{mp+1}^\top R_t) = 0, \quad t = 1, 2, \dots, mp,$$

and

$$\text{tr}(R_j^\top R_i) = 0, \quad i, j = 1, 2, \dots, mp, \quad i \neq j.$$

Therefore, $\{R_1, R_2, \dots, R_{mp}\}$ forms an orthogonal basis of the real-valued matrix space $\mathbf{R}^{m \times p}$, which implies that $R_{mp+1} = 0$, that is, $[\tilde{G}_{mp+1}, \tilde{F}_{mp+1}]$ is a symmetric solution pair of Problem P.

Lemma 3. The equation of (8) has a symmetric solution pair $[\tilde{G}, \tilde{F}]$ if and only if the matrix equations

$$\begin{aligned}\tilde{G} Q_1^\top \tilde{X} \tilde{\Lambda} + \tilde{F} Q_1^\top \tilde{X} &= Q_1^\top H, \\ \tilde{\Lambda}^\top \tilde{X}^\top Q_1 \tilde{G} + \tilde{X}^\top Q_1 \tilde{F} &= H^\top Q_1,\end{aligned} \quad (14)$$

are consistent.

Proof. If the equation of (8) has a symmetric solution pair $[\tilde{G}^*, \tilde{F}^*]$, then $\tilde{G}^* Q_1^\top \tilde{X} \tilde{\Lambda} + \tilde{F}^* Q_1^\top \tilde{X} = Q_1^\top H$, and $(\tilde{G}^* Q_1^\top \tilde{X} \tilde{\Lambda} + \tilde{F}^* Q_1^\top \tilde{X})^\top = \tilde{\Lambda}^\top \tilde{X}^\top Q_1 \tilde{G}^* + \tilde{X}^\top Q_1 \tilde{F}^* = H^\top Q_1$. That is to say, $[\tilde{G}^*, \tilde{F}^*]$ is a solution of (14). Conversely, if the matrix equations of (14) have a solution, say, $\tilde{G} = U$, $\tilde{F} = V$. Let $\tilde{G}^* = \frac{1}{2}(U + U^\top)$, $\tilde{F}^* = \frac{1}{2}(V + V^\top)$, then \tilde{G}^* and \tilde{F}^* are symmetric matrices, and

$$\begin{aligned}\tilde{G}^* Q_1^\top \tilde{X} \tilde{\Lambda} + \tilde{F}^* Q_1^\top \tilde{X} &= \frac{1}{2}(U Q_1^\top \tilde{X} \tilde{\Lambda} + V Q_1^\top \tilde{X}) + \frac{1}{2}(U^\top Q_1^\top \tilde{X} \tilde{\Lambda} + V^\top Q_1^\top \tilde{X}) \\ &= \frac{1}{2} Q_1^\top H + \frac{1}{2} (H^\top Q_1)^\top = Q_1^\top H.\end{aligned}$$

Hence, $[\tilde{G}^*, \tilde{F}^*]$ is a symmetric solution pair of (8).

and

$$K_a = \begin{bmatrix} 2000 & -1000 & & & & & & & \\ -1000 & 3000 & -1000 & & & & & & \\ & -1000 & 2000 & -1000 & & & & & \\ & & -1000 & 3000 & -1000 & & & & \\ & -1000 & & -1000 & 3000 & -1000 & & & \\ & & & & -1000 & 2000 & -1000 & & \\ & & & & & -1000 & 2000 & -1000 & \\ & & & & & & -1000 & 3000 & -1000 \\ & & & & & & & -1000 & 2000 \\ & & & & & & & & -1000 & 2000 \end{bmatrix}.$$

The measured data for experiment were simulated by reducing stiffness of the spring between masses 2 and 5 to 600 N/m and adding Gaussian noise with $\sigma = 2\%$. The analytical eigenvalue and eigenvector matrices are

$$A_a = \begin{bmatrix} -6.23 & 71.1 & 0 & 0 \\ -71.1 & -6.23 & 0 & 0 \\ 0 & 0 & -3.67 & 65.9 \\ 0 & 0 & -65.9 & -3.67 \end{bmatrix},$$

$$X_a = \begin{bmatrix} 0.142 & 0.001 & -0.161 & -0.001 \\ -0.438 & 0.020 & 0.372 & -0.050 \\ 0.288 & 0.065 & -0.056 & 0.031 \\ -0.502 & -0.206 & -0.191 & 0.087 \\ 0.479 & 0.148 & -0.296 & -0.034 \\ -0.136 & -0.011 & 0.263 & 0.091 \\ -0.066 & 0.011 & -0.346 & -0.145 \\ 0.339 & 0.003 & 0.599 & 0.063 \\ -0.122 & -0.007 & -0.296 & -0.093 \\ 0.040 & 0.010 & 0.115 & 0.075 \end{bmatrix}.$$

The measured eigenvalue and eigenvector matrices are

$$A = \begin{bmatrix} -6.16 & 69.8 & 0 & 0 \\ -69.8 & -6.16 & 0 & 0 \\ 0 & 0 & -4.7 & 64.9 \\ 0 & 0 & -64.9 & -4.7 \end{bmatrix},$$

$$X = \begin{bmatrix} 0.102 & 0.026 & -0.172 & -0.023 \\ -0.283 & -0.061 & 0.401 & 0.023 \\ 0.282 & 0.115 & -0.195 & -0.005 \\ -0.579 & -0.240 & 0.074 & 0.068 \\ 0.341 & 0.242 & -0.354 & -0.202 \\ -0.067 & -0.054 & 0.286 & 0.242 \\ -0.168 & 0.036 & -0.249 & -0.340 \\ 0.508 & -0.042 & 0.362 & 0.382 \\ -0.207 & 0.009 & -0.183 & -0.246 \\ 0.077 & 0.011 & 0.060 & 0.130 \end{bmatrix}.$$

Let control feedback matrix B be

$$B = \begin{bmatrix} -1.4076 & 2.6140 & -0.3581 & -3.6090 \\ -7.1050 & -10.7178 & 8.3604 & 11.2322 \\ -0.0985 & 2.2185 & -0.4643 & -1.3306 \\ 1.0920 & 4.1223 & 1.2657 & -3.1758 \\ 11.8725 & -2.7592 & -8.2772 & -6.6878 \\ 8.3940 & -10.6376 & -12.9489 & 12.5141 \\ 1.4069 & -2.8287 & 1.5955 & -0.3407 \\ -124.6234 & 13.8803 & -105.2349 & -117.0221 \\ 0.6226 & -1.5264 & 1.1456 & -0.1770 \\ -0.2305 & -2.5788 & 3.0744 & -1.8265 \end{bmatrix}.$$

It can easily be seen that the condition (9) holds ($\|Q_2^T H\| = 5.901e - 013$).

Now, choosing initial iterative matrices $\tilde{G}_1 = 0$ and $\tilde{F}_1 = 0$. By Algorithm 1, after 122 iteration steps, we get the minimum Frobenius norm solution $(\tilde{G}_{\min}, \tilde{F}_{\min})$ of Eq. (8) as follows:

$$\tilde{G}_{\min} = \tilde{G}_{123} = 1000 \times \begin{bmatrix} -0.0503 & -0.4265 & -0.1831 & -0.5825 \\ -0.4265 & 0.3324 & -0.9737 & -0.2315 \\ -0.1831 & -0.9737 & -0.6664 & -1.2657 \\ -0.5825 & -0.2315 & -1.2657 & -0.9037 \end{bmatrix},$$

$$\tilde{F}_{\min} = \tilde{F}_{123} = 10000 \times \begin{bmatrix} -0.2565 & 1.7614 & 1.4994 & 0.4466 \\ 1.7614 & -4.0172 & 0.7512 & -1.7907 \\ 1.4994 & 0.7512 & 3.3321 & -1.3357 \\ 0.4466 & -1.7907 & -1.3357 & -4.0758 \end{bmatrix}$$

with corresponding residual

$$\|R_{123}\| = \|Q_1^T H - \tilde{G}_{123} Q_1^T X \Lambda - \tilde{F}_{123} Q_1^T X\| = 6.4597e - 009.$$

Therefore, by (10), we can compute

$$\hat{G} = \begin{bmatrix} 0.5498 & -2.4757 & 0.4476 & -1.2356 \\ -2.4757 & -7.1317 & 8.4822 & -6.2769 \\ 0.4476 & 8.4822 & -5.5259 & 5.6946 \\ -1.2356 & -6.2769 & 5.6946 & -4.8304 \end{bmatrix},$$

$$\hat{F} = \begin{bmatrix} 29.4057 & 85.6980 & -72.0939 & 45.6705 \\ 85.6980 & -515.6965 & 174.6757 & -298.1139 \\ -72.0939 & 174.6757 & -65.7251 & 152.0991 \\ 45.6705 & -298.1139 & 152.0991 & -217.8653 \end{bmatrix},$$

and the resulting updated damping and stiffness matrices are given by (11); the error bound of the residual is estimated by

$$\|M_a X \Lambda^2 + \hat{D} X \Lambda + \hat{K} X\| = 6.5051e - 009.$$

Example 2. Consider a model updating problem. The original model is the statically condensed oil rig model (M_a, D_a, K_a) represented by the triplet BCSSTRUC1 in the Harwell–Boeing collection [30]. In this model, M_a and $K_a \in \mathbf{R}^{66 \times 66}$ are symmetric and positive definite, and $D_a = 1.55I_{66}$. There are 132 eigenpairs. Suppose we want to replace the four eigenvalues $\mu_1 = -34.62 + 574.48i$, $\mu_2 = -34.62 - 574.48i$, $\mu_5 = -12.865 + 465.35i$ and $\mu_6 = -12.865 - 465.35i$ by newly measured eigenvalues $\lambda_1 = -41.544 + 689.37i$, $\lambda_2 = -41.544 - 689.37i$, $\lambda_5 = -10.292 + 372.28i$ and $\lambda_6 = -10.292 - 372.28i$, while keeping the corresponding eigenvectors invariant. Choosing control feedback matrix $B = M_a \tilde{X} \tilde{\Lambda}^2 + D_a \tilde{X} \tilde{\Lambda} + K_a \tilde{X}$ and initial iterative matrices $\tilde{G}_1 = 0$, $\tilde{F}_1 = 0$. By Algorithm 1, after 80 iteration steps, we get the minimum Frobenius norm solution $(\tilde{G}_{\min}, \tilde{F}_{\min})$ of Eq. (8). By (11), the updated damping and stiffness matrices can be computed and the error bound of the residual is estimated by

$$\|M_a X \Lambda^2 + \hat{D} X \Lambda + \hat{K} X\| = 9.7031e - 010,$$

which implies that the measured eigenvalues are embedded in the new model $(\lambda^2 M_a + \lambda \hat{D} + \hat{K})x = 0$.

4. Concluding remarks

One common procedure to mend the discrepancy between a mathematical model and the corresponding real-world system is to modify the model parameters in such a way so as to achieve a good correspondence between the analytic solution and the real data. This paper presents an iterative method for updating finite element models using the symmetric eigenstructure assignment technique. By this method, the optimal approximation solution (\hat{G}, \hat{F}) of Problem P can be obtained within finite iteration steps in the absence of roundoff errors by choosing a special kind of initial matrix pair. The approach is demonstrated by two numerical examples and reasonable results are produced.

Appendix

The proof of Lemma 1. Since $\text{tr}(R_j^T R_i) = \text{tr}(R_i^T R_j)$, $\text{tr}(P_j^T P_i) = \text{tr}(P_i^T P_j)$ and $\text{tr}(W_j^T W_i) = \text{tr}(W_i^T W_j)$, then we only need to show that

$$\text{tr}(R_j^T R_i) = 0, \quad \text{and} \quad \text{tr}(P_j^T P_i) + \text{tr}(W_j^T W_i) = 0 \quad \text{for } 1 \leq i < j \leq s.$$

We use mathematic induction to prove this conclusion, and we do it in two steps. We first show that

$$\text{tr}(R_{i+1}^T R_i) = 0, \quad \text{and} \quad \text{tr}(P_{i+1}^T P_i) + \text{tr}(W_{i+1}^T W_i) = 0 \quad \text{for } i = 1, 2, \dots, s. \quad (\text{A.1})$$

For $i = 1$, by Algorithm 1 and noting that $P_1, W_1 \in \mathbf{SR}^{m \times m}$, we have

$$\begin{aligned} \text{tr}(R_2^\top R_1) &= \text{tr}((R_1 - \delta_1(P_1 Q_1^\top \tilde{X} \tilde{\Lambda} + W_1 Q_1^\top \tilde{X}))^\top R_1) \\ &= \text{tr}(R_1^\top R_1) - \delta_1 \text{tr}(\tilde{\Lambda}^\top \tilde{X}^\top Q_1 P_1^\top R_1 + \tilde{X}^\top Q_1 W_1^\top R_1) \\ &= \|R_1\|^2 - \frac{1}{2} \delta_1 \text{tr}(\tilde{\Lambda}^\top \tilde{X}^\top Q_1 P_1^\top R_1 + R_1^\top P_1 Q_1^\top \tilde{X} \tilde{\Lambda} + \tilde{X}^\top Q_1 W_1^\top R_1 + R_1^\top W_1 Q_1^\top \tilde{X}) \\ &= \|R_1\|^2 - \frac{1}{2} \delta_1 \text{tr}(P_1^\top R_1 \tilde{\Lambda}^\top \tilde{X}^\top Q_1 + P_1^\top Q_1^\top \tilde{X} \tilde{\Lambda} R_1^\top + W_1^\top R_1 \tilde{X}^\top Q_1 + W_1^\top Q_1^\top \tilde{X} R_1^\top) \\ &= \|R_1\|^2 - \delta_1 \text{tr}(P_1^\top P_1 + W_1^\top W_1) \\ &= 0, \end{aligned}$$

where $\delta_1 = \frac{\|R_1\|^2}{\|P_1\|^2 + \|W_1\|^2}$.

$$\begin{aligned} \text{tr}(P_2^\top P_1) + \text{tr}(W_2^\top W_1) &= \frac{1}{2} \text{tr}((R_2 \tilde{\Lambda}^\top \tilde{X}^\top Q_1 + Q_1^\top \tilde{X} \tilde{\Lambda} R_2^\top) P_1) + \frac{\|R_2\|^2}{\|R_1\|^2} \|P_1\|^2 \\ &\quad + \frac{1}{2} \text{tr}((R_2 \tilde{X}^\top Q_1 + Q_1^\top \tilde{X} R_2^\top) W_1) + \frac{\|R_2\|^2}{\|R_1\|^2} \|W_1\|^2 \\ &= \frac{1}{2} \text{tr}(R_2(P_1 Q_1^\top \tilde{X} \tilde{\Lambda} + W_1 Q_1^\top \tilde{X}))^\top + (P_1 Q_1^\top \tilde{X} \tilde{\Lambda} + W_1 Q_1^\top \tilde{X}) R_2^\top \\ &\quad + \frac{\|R_2\|^2}{\|R_1\|^2} \|P_1\|^2 + \frac{\|R_2\|^2}{\|R_1\|^2} \|W_1\|^2 \\ &= \frac{1}{2} \frac{\|P_1\|^2 + \|W_1\|^2}{\|R_1\|^2} \text{tr}(R_2(R_1 - R_2)^\top + (R_1 - R_2) R_2^\top) + \frac{\|R_2\|^2}{\|R_1\|^2} \|P_1\|^2 + \frac{\|R_2\|^2}{\|R_1\|^2} \|W_1\|^2 \\ &= 0. \end{aligned}$$

Suppose that (A.1) holds for $i = t - 1$. For $i = t$, we have

$$\begin{aligned} \text{tr}(R_{t+1}^\top R_t) &= \text{tr}((R_t - \delta_t(P_t Q_1^\top \tilde{X} \tilde{\Lambda} + W_t Q_1^\top \tilde{X}))^\top R_t) \\ &= \text{tr}(R_t^\top R_t) - \delta_t \text{tr}(\tilde{\Lambda}^\top \tilde{X}^\top Q_1 P_t^\top R_t + \tilde{X}^\top Q_1 W_t^\top R_t) \\ &= \|R_t\|^2 - \frac{1}{2} \delta_t \text{tr}(\tilde{\Lambda}^\top \tilde{X}^\top Q_1 P_t^\top R_t + R_t^\top P_t Q_1^\top \tilde{X} \tilde{\Lambda} + \tilde{X}^\top Q_1 W_t^\top R_t + R_t^\top W_t Q_1^\top \tilde{X}) \\ &= \|R_t\|^2 - \frac{1}{2} \delta_t \text{tr}(P_t^\top R_t \tilde{\Lambda}^\top \tilde{X}^\top Q_1 + P_t^\top Q_1^\top \tilde{X} \tilde{\Lambda} R_t^\top + W_t^\top R_t \tilde{X}^\top Q_1 + W_t^\top Q_1^\top \tilde{X} R_t^\top) \\ &= \|R_t\|^2 - \delta_t \text{tr}\left(P_t^\top \left(P_t - \frac{\|R_t\|^2}{\|R_{t-1}\|^2} P_{t-1}\right) + W_t^\top \left(W_t - \frac{\|R_t\|^2}{\|R_{t-1}\|^2} W_{t-1}\right)\right) \\ &= \|R_t\|^2 - \delta_t \text{tr}(P_t^\top P_t + W_t^\top W_t) \\ &= 0, \end{aligned}$$

where $\delta_t = \frac{\|R_t\|^2}{\|P_t\|^2 + \|W_t\|^2}$.

$$\begin{aligned} \text{tr}(P_{t+1}^\top P_t) + \text{tr}(W_{t+1}^\top W_t) &= \frac{1}{2} \text{tr}((R_{t+1} \tilde{\Lambda}^\top \tilde{X}^\top Q_1 + Q_1^\top \tilde{X} \tilde{\Lambda} R_{t+1}^\top) P_t) + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|P_t\|^2 \\ &\quad + \frac{1}{2} \text{tr}((R_{t+1} \tilde{X}^\top Q_1 + Q_1^\top \tilde{X} R_{t+1}^\top) W_t) + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|W_t\|^2 \\ &= \frac{1}{2} \text{tr}(R_{t+1}(P_t Q_1^\top \tilde{X} \tilde{\Lambda} + W_t Q_1^\top \tilde{X}))^\top + (P_t Q_1^\top \tilde{X} \tilde{\Lambda} + W_t Q_1^\top \tilde{X}) R_{t+1}^\top \\ &\quad + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|P_t\|^2 + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|W_t\|^2 \\ &= \frac{1}{2} \frac{\|P_t\|^2 + \|W_t\|^2}{\|R_t\|^2} \text{tr}(R_{t+1}(R_t - R_{t+1})^\top + (R_t - R_{t+1}) R_{t+1}^\top) \\ &\quad + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|P_t\|^2 + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|W_t\|^2 \\ &= 0. \end{aligned}$$

Therefore, (A.1) holds for $i = t$. By the principle of induction, we know (A.1) holds for all i .

Next, assume that

$$\text{tr}(R_{i+d}^\top R_i) = 0, \quad \text{and} \quad \text{tr}(P_{i+d}^\top P_i) + \text{tr}(W_{i+d}^\top W_i) = 0 \quad \text{for } 1 \leq i \leq s \text{ and } 1 < d < s.$$

We will prove

$$\begin{aligned} \text{tr}(R_{i+d+1}^\top R_i) &= 0, \quad \text{and} \quad \text{tr}(P_{i+d+1}^\top P_i) + \text{tr}(W_{i+d+1}^\top W_i) = 0. \\ \text{tr}(R_{i+d+1}^\top R_i) &= \text{tr}((R_{i+d} - \delta_{i+d}(P_{i+d}Q_1^\top \tilde{X}\tilde{\Lambda} + W_{i+d}Q_1^\top \tilde{X}))^\top R_i) \\ &= -\delta_{i+d} \text{tr}(\tilde{\Lambda}^\top \tilde{X}^\top Q_1 P_{i+d}^\top R_i + \tilde{X}^\top Q_1 W_{i+d}^\top R_i) \\ &= -\frac{1}{2} \delta_{i+d} \text{tr}(\tilde{\Lambda}^\top \tilde{X}^\top Q_1 P_{i+d}^\top R_i + R_i^\top P_{i+d} Q_1^\top \tilde{X}\tilde{\Lambda} + \tilde{X}^\top Q_1 W_{i+d}^\top R_i + R_i^\top W_{i+d} Q_1^\top \tilde{X}) \\ &= -\frac{1}{2} \delta_{i+d} \text{tr}(P_{i+d}^\top R_i \tilde{\Lambda}^\top \tilde{X}^\top Q_1 + P_{i+d}^\top Q_1^\top \tilde{X}\tilde{\Lambda} R_i^\top + W_{i+d}^\top R_i \tilde{X}^\top Q_1 + W_{i+d}^\top Q_1^\top \tilde{X} R_i^\top) \\ &= -\delta_{i+d} \text{tr}\left(P_{i+d}^\top \left(P_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2} P_{i-1}\right) + W_{i+d}^\top \left(W_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2} W_{i-1}\right)\right) \\ &= 0, \end{aligned}$$

where $\delta_{i+d} = \frac{\|R_{i+d}\|^2}{\|P_{i+d}\|^2 + \|W_{i+d}\|^2}$.

From the above results, we have $\text{tr}(R_{i+d+1}^\top R_i) = 0$ and $\text{tr}(R_{i+d+1}^\top R_{i+1}) = 0$. Hence we can get

$$\begin{aligned} \text{tr}(P_{i+d+1}^\top P_i) + \text{tr}(W_{i+d+1}^\top W_i) &= \frac{1}{2} \text{tr}((R_{i+d+1} \tilde{\Lambda}^\top \tilde{X}^\top Q_1 + Q_1^\top \tilde{X}\tilde{\Lambda} R_{i+d+1}^\top) P_i) + \frac{1}{2} \text{tr}((R_{i+d+1} \tilde{X}^\top Q_1 + Q_1^\top \tilde{X} R_{i+d+1}^\top) W_i) \\ &= \frac{1}{2} \text{tr}(R_{i+d+1} (P_i Q_1^\top \tilde{X}\tilde{\Lambda} + W_i Q_1^\top \tilde{X})^\top + (P_i Q_1^\top \tilde{X}\tilde{\Lambda} + W_i Q_1^\top \tilde{X}) R_{i+d+1}^\top) \\ &= \frac{1}{2} \xi \text{tr}(R_{i+d+1} (R_i - R_{i+1})^\top + (R_i - R_{i+1}) R_{i+d+1}^\top) \\ &= 0 \end{aligned}$$

where $\xi = \frac{\|P_i\|^2 + \|W_i\|^2}{\|R_i\|^2}$.

Thus the conclusion (12) holds by the principle of induction. The proof is completed.

The proof of Lemma 2. We prove the conclusion by induction. For $i = 1$, we have

$$\begin{aligned} \text{tr}((\tilde{G}^* - \tilde{G}_1)^\top P_1) + \text{tr}((\tilde{F}^* - \tilde{F}_1)^\top W_1) &= \frac{1}{2} \text{tr}((\tilde{G}^* - \tilde{G}_1)^\top (R_1 \tilde{\Lambda}^\top \tilde{X}^\top Q_1 + Q_1^\top \tilde{X}\tilde{\Lambda} R_1^\top)) \\ &\quad + \frac{1}{2} \text{tr}((\tilde{F}^* - \tilde{F}_1)^\top (R_1 \tilde{X}^\top Q_1 + Q_1^\top \tilde{X} R_1^\top)) \\ &= \frac{1}{2} \text{tr}(Q_1^\top H R_1^\top - \tilde{G}_1 Q_1^\top \tilde{X}\tilde{\Lambda} R_1^\top - \tilde{F}_1 Q_1^\top \tilde{X} R_1^\top) \\ &\quad + \frac{1}{2} \text{tr}(H^\top Q_1 R_1 - \tilde{\Lambda}^\top \tilde{X}^\top Q_1 \tilde{G}_1^\top R_1 - \tilde{X}^\top Q_1 \tilde{F}_1^\top R_1) \\ &= \frac{1}{2} \text{tr}(R_1 R_1^\top) + \frac{1}{2} \text{tr}(R_1^\top R_1) \\ &= \|R_1\|^2. \end{aligned}$$

Now assume the conclusion (13) holds for $1 \leq i \leq t-1$. Then we can get

$$\begin{aligned} \text{tr}((\tilde{G}^* - \tilde{G}_t)^\top P_t) + \text{tr}((\tilde{F}^* - \tilde{F}_t)^\top W_t) &= \text{tr}\left(\left(\tilde{G}^* - \tilde{G}_{t-1} - \frac{\|R_{t-1}\|^2}{\|P_{t-1}\|^2 + \|W_{t-1}\|^2} P_{t-1}\right)^\top P_t\right) \\ &\quad + \text{tr}\left(\left(\tilde{F}^* - \tilde{F}_{t-1} - \frac{\|R_{t-1}\|^2}{\|P_{t-1}\|^2 + \|W_{t-1}\|^2} W_{t-1}\right)^\top W_t\right) \\ &= \text{tr}((\tilde{G}^* - \tilde{G}_{t-1})^\top P_t) + \text{tr}((\tilde{F}^* - \tilde{F}_{t-1})^\top W_t) \\ &= \frac{1}{2} \text{tr}((\tilde{G}^* - \tilde{G}_{t-1})^\top (R_t \tilde{\Lambda}^\top \tilde{X}^\top Q_1 + Q_1^\top \tilde{X}\tilde{\Lambda} R_t^\top)) \\ &\quad + \frac{\|R_t\|^2}{\|R_{t-1}\|^2} \text{tr}((\tilde{G}^* - \tilde{G}_{t-1})^\top P_{t-1}) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \text{tr}((\tilde{F}^* - \tilde{F}_{t-1})^\top (R_t \tilde{X}^\top Q_1 + Q_1^\top \tilde{X} R_t^\top)) \\
& + \frac{\|R_t\|^2}{\|R_{t-1}\|^2} \text{tr}((\tilde{F}^* - \tilde{F}_{t-1})^\top W_{t-1}) \\
& = \frac{1}{2} \text{tr}((\tilde{G}^* - \tilde{G}_{t-1})^\top (R_t \tilde{\Lambda}^\top \tilde{X}^\top Q_1 + Q_1^\top \tilde{X} \tilde{\Lambda} R_t^\top)) \\
& + \frac{1}{2} \text{tr}((\tilde{F}^* - \tilde{F}_{t-1})^\top (R_t \tilde{X}^\top Q_1 + Q_1^\top \tilde{X} R_t^\top)) + \|R_t\|^2 \\
& = \frac{1}{2} \text{tr}(H^\top Q_1 R_t - \tilde{\Lambda}^\top \tilde{X}^\top Q_1 \tilde{G}_{t-1}^\top R_t - \tilde{X}^\top Q_1 \tilde{F}_{t-1}^\top R_t) \\
& + \frac{1}{2} \text{tr}(Q_1^\top H R_t^\top - \tilde{G}_{t-1} Q_1^\top \tilde{X} \tilde{\Lambda} R_t^\top - \tilde{F}_{t-1} Q_1^\top \tilde{X} R_t^\top) + \|R_t\|^2 \\
& = \frac{1}{2} \text{tr}(R_{t-1}^\top R_t) + \frac{1}{2} \text{tr}(R_{t-1} R_t^\top) + \|R_t\|^2 \\
& = \|R_t\|^2.
\end{aligned}$$

Thus we complete the proof of Lemma 2 by the principle of induction.

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