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Equal order approximations enriched with bubbles for coupled Stokes–Darcy problem

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ABSTRACT

As a remedy to the instability of the Galerkin finite element formulation, symmetric stabilization techniques such as the continuous interior penalty, the subgrid and local projection methods were proposed and analyzed by Burman and Hansbo (2006) [10], Badia and Codina (2009) [11], Becker and Braack (2001) [12], and Nafa and Wathen (2009) [13]. In this work we consider a coupled Stokes–Darcy problem, where in one part of the domain the fluid motion is described by Stokes equations and for the other part the fluid is in a porous medium and described by Darcy law and the conservation of mass. Such systems can be discretized by heterogeneous finite elements in the two parts, such as Taylor-Hood or MINI elements for the Stokes domain, and mixed elements of Raviart–Thomas elements type for the Darcy domain. Here, we discretize by standard equal-order finite elements enriched with bubbles functions and use local projection stabilization technique (LPS) to stabilize the method and control the fluctuation of the velocity divergence vector on the Darcy region. We also suggest a way to control the natural $H(div)$ velocity.

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1. Introduction

The transport of substances between surface water and groundwater has attracted a lot of interest into the coupling of viscous flows and porous media flows ([1–6]). In this work we consider coupled problems in fluid dynamics where the fluid in one part of the domain is described by the Stokes equations and in another (porous media) part by the Darcy equation and mass conservation. Velocity and pressure on these two parts are mutually coupled by interface conditions derived in [7–9]. Such systems can be discretized by heterogeneous finite elements in the two parts, e.g. Taylor-Hood or MINI elements for the Stokes part, and mixed elements of Raviart–Thomas type or Brezzi–Douglas–Marini elements for the Darcy region. Such an approach is analyzed by Layton et al. in [1]. In more recent works, unified approaches become more popular. For instance, discontinuous Galerkin methods were analyzed by Girault and Riviere [3], mixed methods by Karper et al. [4], and stabilized methods by Braack et al. [6]. In this work, we take the same variational formulation of the coupled problem as in [1,4], but we discretize by standard equal-order finite elements enriched with bubbles functions and use local projection stabilization technique (LPS) together with the grad–div term to control the natural $H(div)$ velocity norm on the Darcy region. We note that the various types of stabilization cited in references [10–13] can be applied to the Stokes–Darcy using a similar analysis.

2. Coupled systems of equations

Let $\Omega \subset R^d$ ($d = 2$ or 3) be a bounded domain split into two subdomains Ω_S and Ω_D with $\Omega_S \cap \Omega_D = \varnothing$. The Stokes part Ω_S and the Darcy part Ω_D have a common interface $\Gamma_I = \overline{\Omega}_S \cap \overline{\Omega}_D$. The region Ω_S is filled by a fluid and the Stokes system

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has the velocity and the pressure solutions $\mathbf{v}$ and p defined on Ω_S by

$$-2\nu\, div\, (D(\mathbf{v})) + \nabla p = f, \quad \text{in } \Omega_S \tag{1}$$

$$div\, \mathbf{v} = 0, \quad \text{in } \Omega_S \tag{2}$$

with symmetric deformation tensor $D(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$, external force $\mathbf{f}$ and constant viscosity $\nu > 0$.

In the Darcy region Ω_D the velocity $\mathbf{v}$ and the pressure p are solutions of the Darcy system

$$\mathbf{K}^{-1}\mathbf{v} + \nabla p = \mathbf{f} \quad \text{in } \Omega_D \tag{3}$$

$$div\, \mathbf{v} = g, \quad \text{in } \Omega_D \tag{4}$$

where, the permeability $\mathbf{K} = \mathbf{K}(\mathbf{x})$ is a positive definite symmetric tensor and g denotes an external Darcy force.

2.1. Boundary conditions

On $\Gamma_S = \partial\Omega_S \setminus \Gamma_I$, we prescribe homogeneous Dirichlet conditions for the velocity $\mathbf{v}$.

$$\mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_S. \tag{5}$$

The boundary of Ω_D is split into three parts $\partial\Omega_D = \Gamma_I \cup \Gamma_{D,1} \cup \Gamma_{D,2}$. We prescribe zero flux on $\Gamma_{D,1}$ and a homogeneous Dirichlet condition for the pressure on $\Gamma_{D,2}$.

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{D,1} \tag{6}$$

$$p = 0 \quad \text{on } \Gamma_{D,2}. \tag{7}$$

The boundary condition on $\Gamma_{D,1}$ involves the Euclidean scalar product of the velocity and the outer normal $\mathbf{n}$ on the boundary, pointing from Ω_D into Ω_S. This boundary condition ensures a zero mass flux.

On the interface Γ_I, the coupling of the two regimes is modeled by the so-called Beaver–Joseph–Saffman conditions. Since velocity and pressure are not necessarily continuous across Γ_I, we use the notations $\mathbf{v}_S := \mathbf{v}|_{\Omega_S}$, $\mathbf{v}_D := \mathbf{v}|_{\Omega_D}$, $p_S := p|_{\Omega_S}$, $p_D := p|_{\Omega_D}$, and $[\phi] := \phi_D - \phi_S$ the jump of a scalar quantity ϕ across Γ_I. With these notation, the boundary conditions on the interface Γ_I read:

$$[\mathbf{v} \cdot \mathbf{n}] = 0, \tag{8}$$

$$-2\nu\, D(\mathbf{v}_S)\mathbf{n} \cdot \mathbf{n} = [p], \tag{9}$$

$$(\mathbf{v}_S - 2\alpha\, \nu D(\mathbf{v}_S)\mathbf{n}) \cdot \mathbf{t} = 0. \tag{10}$$

Eq. (8) ensures mass conservation across the interface, (9) represents a balance of pressure forces and viscous forces acting across the interface. Eq. (10) involving the tangential vector $\mathbf{t}$, is the Beaver–Joseph–Saffman condition ([7–9]) which gives a relation of the tangential slip velocity $\mathbf{v}_S \cdot \mathbf{t}$ and the normal derivative of the tangential velocity component in the Stokes region.

3. Variational formulation

As variational formulation we consider the so-called L2-formulation used by Karper et al. [4]. Using standard notations for function spaces, the variational spaces for velocity and pressure are

$$\begin{aligned} \mathbf{V} &:= \left\{\mathbf{v} \in L^2(\Omega)^d \mid \mathbf{v}_S \in H^1(\Omega_S)^d,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_S,\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_{D,1}\right\} \\ Q &:= \left\{p \in L^2(\Omega)^d \mid p_D \in H^1(\Omega_D),\ p = 0 \text{ on } \Gamma_{D,2}\right\} \\ X &:= \mathbf{V} \times Q. \end{aligned} \tag{11}$$

Due to the positive definiteness of $\mathbf{K}$ with respect to the $L^2(\Omega_D)$ norm $\|.\|_{\Omega_D}$, there exist positive real numbers k_1 and k_2 such that

$$k_1\|\mathbf{v}\|^2_{\Omega_D} \le (\mathbf{K}^{-1}\mathbf{v}, \mathbf{v}) \le k_2\|\mathbf{v}\|^2_{\Omega_D}, \quad \forall\, \mathbf{v} \in V. \tag{12}$$

For convenience, we define the associated bilinear forms on the parts of the domain by

$$\mathcal{A}_S(\mathbf{v}, p; \mathbf{w}, q) := (2\nu\, D(\mathbf{v}), D(\mathbf{w}))_{\Omega_S} - (p, div\, \mathbf{w})_{\Omega_S} + (div\, \mathbf{v}, q)_{\Omega_S} + \int_{\Gamma_I} \frac{1}{\alpha}\, (\mathbf{v}_S \cdot \mathbf{t})\, (\mathbf{w}_S \cdot \mathbf{t})\, ds$$

$$\mathcal{A}_D(\mathbf{v}, p; \mathbf{w}, q) := (\mathbf{K}^{-1}\mathbf{v}, \mathbf{w})_{\Omega_D} + (\nabla p, \mathbf{w})_{\Omega_D} - (\mathbf{v}, \nabla q)_{\Omega_D}.$$

Hence, the bilinear form for the coupled problem is the sum of $\mathcal{A}_S(\mathbf{v}, p; \mathbf{w}, q)$, $\mathcal{A}_D(\mathbf{v}, p; \mathbf{w}, q)$, and a term to enforce the continuity of the normal part of the velocities across the interface.

$$\mathcal{A}(\mathbf{v}, p; \mathbf{w}, q) := \mathcal{A}_S(\mathbf{v}, p; \mathbf{w}, q) + \mathcal{A}_D(\mathbf{v}, p; \mathbf{w}, q) + \int_{\Gamma_I} (p_D\, (\mathbf{w}_S \cdot \mathbf{n}) - (\mathbf{v}_S \cdot \mathbf{n})\, q_D)\, ds. \tag{13}$$

The variational formulation of the coupled Stokes–Darcy system becomes:

find $(\mathbf{v}, p) \in X$ solution of

$$\mathcal{A}(\mathbf{v}, p; \mathbf{w}, q) = (F; \mathbf{w}, q), \quad \forall(\mathbf{w}, q) \in X \tag{14}$$

with right hand side

$$(\mathbf{F}; \mathbf{w}, q) = (\mathbf{f}, \mathbf{w})_{\Omega_S} + (g, q)_{\Omega_D}. \tag{15}$$

It can easily be shown that a sufficiently regular solution $(\mathbf{v}, p) \in X$ of (14), i.e. $\mathbf{v}_S \in H^2(\Omega_s)^d$, $\mathbf{v}_D \in H^1(\Omega_D)^d$, $p \in H^1(\Omega_s \cup \Omega_D)$ is also a classical solution of (1)–(2) and (3)–(4). An alternative formulation is the so-called $H(div)$-formulation which uses the term $-(p; div\,\mathbf{w})_{\Omega_D} + (div\,\mathbf{v}, q)_{\Omega_D}$ instead of $(\mathbf{w}, \nabla p)_{\Omega_D} - (\nabla q, \mathbf{v})_{\Omega_D}$ ([11]).

We equip the spaces $\mathbf{V}$, Q, and X with the norms

$$\begin{aligned}
|||\mathbf{v}|||_{\mathbf{V}} &:= \left(2\nu\,\|D(\mathbf{v})\|^2_{\Omega_S} + \left\|\mathbf{K}^{-1}\mathbf{v}\right\|^2_{\Omega_D} + \int_{\Gamma_I} \frac{1}{\alpha}\,(\mathbf{v}_S \cdot \mathbf{t})^2\,ds\right)^{1/2},\\
|||p|||_Q &:= \left(\|p\|^2_{\Omega_S} + \|\nabla p\|^2_{\Omega_D}\right)^{1/2},\\
|||(\mathbf{v}, p)|||_X &:= \left(|||\mathbf{v}|||^2_{\mathbf{V}} + |||p|||^2_Q\right)^{1/2}.
\end{aligned}$$

The existence and uniqueness of the solution of problem (14) follows from Brezzi's conditions for saddle point problems, namely

$$A(\mathbf{v}, p; \mathbf{v}, p) = |||\mathbf{v}|||^2_{\mathbf{V}}, \quad \forall v \in \mathbf{V} \tag{16}$$

and

$$\begin{aligned}
&\inf_{q\in L^2(\Omega_S)} \sup_{\mathbf{v}\in H^1(\Omega_S)^d} \frac{(div\,\mathbf{v}, q)_{\Omega_S}}{\|\nabla\mathbf{v}\|_{\Omega_S}\,\|q\|_{\Omega_S}} \geqslant \beta_S,\\
&\inf_{q\in H^1(\Omega_D)} \sup_{\mathbf{v}\in L^2(\Omega_D)^d} \frac{-(\mathbf{v}, \nabla q)_{\Omega_D}}{\|\nabla\mathbf{v}\|_{\Omega_D}\,\|q\|_{\Omega_D}} \geqslant \beta_D.
\end{aligned} \tag{17}$$

These conditions lead to the coupled inf–sup condition (17)

$$\inf_{q\in Q} \sup_{\mathbf{v}\in\mathbf{V}} \frac{(div\,\mathbf{v}, q)_{\Omega_S} - (\mathbf{v}, \nabla q)_{\Omega_D}}{\left(\|\nabla\mathbf{v}\|^2_{\Omega_S} + \|\mathbf{v}\|^2_{\Omega_D}\right)^{1/2}\|q\|_{\Omega}} \geqslant \beta, \tag{18}$$

with a positive constant $\beta > 0$. Which ensures the existence and uniqueness of the pressure field $p \in Q$.

The next lemma follows from the continuous inf–sup conditions (17) ([6]).

Lemma 1. *For every* $(\mathbf{v}, p) \in X$ *there is* $\mathbf{w} \in \mathbf{V}$ *such that* $\mathbf{w} = \mathbf{0}$ *on* $\partial\Omega_S$ *satisfying*

$$\begin{aligned}
&\mathcal{A}(\mathbf{v}, p; \mathbf{w}, 0) \geq \frac{1}{2}|||p|||^2_Q - c_1|||\mathbf{v}|||^2_{\mathbf{V}}\\
&\textit{with} \quad |||\mathbf{w}|||_{\mathbf{V}} \leq c_2|||p|||_Q,
\end{aligned}$$

with positive constants c_1 *and* c_2.

4. Finite element discretization

Let $\mathcal{T}_h$ be a shape-regular partition of triangular or tetrahedral elements of Ω ([14]). The diameter of element $T \in \mathcal{T}_h$ will be denoted by h_T and the global mesh size is defined by $h := \max\{h_T, T \in \mathcal{T}_h\}$. Let $P_r(T)$ be the space of all polynomials on T with maximal degree $r \geq 1$. Here we will use the continuous finite element space

$$P_r(\mathcal{T}_h) := \left\{v_h \in L^2(\Omega) \cap C(\Omega) : v_h|_T \in P_r(T), T \in \mathcal{T}_h\right\}. \tag{19}$$

For the discrete spaces $\mathbf{V}_h$ and Q_h we use the equal-order finite element spaces of piecewise polynomials of degree r.

4.1. Stabilization

It is known that the standard Galerkin discretizations of Stokes and Darcy systems are not stable for equal-order elements. This instability stems from the violation of the discrete analog of the inf–sup condition. One possibility to circumvent this condition is to work with a modified bilinear form $\mathcal{A}_h(.;.)$ by adding a stabilization term $\mathcal{S}_h(.;.)$, i.e.,

$$\mathcal{A}_h(\mathbf{v}_h, p_h; \mathbf{w}, q) = \mathcal{A}_h(\mathbf{v}_h, p_h; \mathbf{w}, q) + \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{w}, q)\,; \tag{20}$$

such that the stabilized discrete problem reads

$$\mathcal{A}_h(\mathbf{v}_h, p_h; \mathbf{w}, q) = \mathcal{F}(\mathbf{w}, q) \quad \forall(\mathbf{w}, q) \in \mathbf{V}_h \times Q_h. \tag{21}$$

In this work, we will use the one level local projection stabilization of the pressure gradient and the divergence of the velocity on the Darcy domain. This leads to a weaker consistent method, but the consistency error decreases, with mesh size as $h \to 0$, at a faster rate than the optimal order of approximation. In this paper, we will consider the following form of a symmetric stabilization:

$$\mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{w}, q) \;=\; (\alpha_h\, \kappa_h \nabla p, \kappa_h \nabla q)_\Omega + (\tau_h \kappa_h (div\, \mathbf{v}_h), \kappa_h\, (div\, \mathbf{w}))_\Omega\,, \tag{22}$$

where, κ_h is a locally acting fluctuation operator

$$\kappa_h \,:\, L^2(\Omega)^d \to L^2(\Omega)^d. \tag{23}$$

$\alpha_h \,:\, \Omega \to \mathbb{R}^+$ and $\tau_h \,:\, \Omega \to \mathbb{R}^+$ are patch-wise constant functions, on each element, that are chosen such that the consistency error asymptotic rate is greater that the rate of convergence of the method. Hence, they should be carefully selected so that the method gets enough stability.

4.2. Fluctuation and interpolation operators

The general framework developed in [15] and [16] has opened up the way to consider continuous approximations and discontinuous projections which are defined on the same mesh $\mathcal{T}_h$. Let π_h be the L_2^d-projection into the space $\mathbf{D}_h = D_h^d = \left(P_{r-1}^{disc}(\mathcal{T}_h)\right)^d$, $\kappa_h = I - \pi_h$, with I denoting the identity and $P_r^{disc}(\mathcal{T}_h)$ the space of discontinuous polynomial functions of maximal degree $r \geq 1$ on $\mathcal{T}_h$

$$P_r^{disc}(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) : v\,|_T \in P_r(T),\ \forall T \in \mathcal{T}_h \right\}. \tag{24}$$

Let b_T be a bubble function defined on $T \in \mathcal{T}_h$. Then, the velocity and pressure approximation spaces $\mathbf{V}_h$ and Q_h. Here, for the Stokes system we use a generalized MINI element [17,18] and for the Darcy system we use a polynomial approximation of order r supplemented by local bubble functions, i.e.

$$\mathbf{V}_h = P_r^b(\mathcal{T}_h)^d \cap \mathbf{V},$$

$$Q_h = \begin{cases} P_r(\mathcal{T}_h), & \text{if } T \subset \Omega_S \\ P_r^b(\mathcal{T}_h), & \text{if } T \subset \Omega_D, \end{cases}$$

with

$$P_r^b(\mathcal{T}_h) = \{ v \in C(\Omega) :\ v\,|_T \in P_r(T) + b_T P_{r-1}(T), \quad \forall T \in \mathcal{T}_h \}.$$

The linear projection operator π_h: $\left(L^2(\Omega)\right)^d \to \mathbf{D}_h$ defined as patch-wise L^2-projection $\pi_h\,|_T$: $\left(L^2(T)\right)^d \to (P_{r-1}(T))^d$, $T \in \mathcal{T}_h$, such that

$$(\pi_h\,|_T\, \varphi - \varphi, \psi)_T = 0, \quad \forall \psi \in \mathbf{D}_h\,|_T,\ \forall \varphi \in \left(L^2(T)\right)^d. \tag{25}$$

A simple local projection scheme of low order for this class, corresponding to $r = 1$, is to use MINI element approximation for the stokes system and continuous piecewise linear approximation for the Darcy system. Then, we enrich the velocity and pressure spaces of the latter by a cubic bubble function, and use the space of piecewise constant functions as projection space. Hence, one of the merits of the proposed method is that it uses essentially the same type of approximation on both parts of the domain but with slightly different approach.

The following properties hold for the fluctuation operator:

Lemma 2. *The local fluctuation operator $\kappa_T = \kappa_h\,|_T$ is locally L^2-stable, i.e.*

$$\|\kappa_T q\|_{0,T} \leq C \|q\|_{0,T}. \tag{26}$$

In addition, $\kappa_T q$ is small for smooth functions q, in the sense that

$$\|\kappa_T q\|_{0,T} \leq C h^k\, |q|_{k,T}\,; \quad \forall T \in \mathcal{T}_h,\ q \in H^k(T),\ k \leq r. \tag{27}$$

Here and in the error analysis below $C > 0$ is a suitable generic constant independent of the mesh parameter h.

In order to investigate the stability and derive the asymptotic error estimates we use the following interpolation operator [15]

$$j^h : H^1(\Omega) \to P_r^b(\mathcal{T}_h) \cap H^1(\Omega) \tag{28}$$

with the orthogonality property

$$(j^h v - v, w) = 0, \quad \forall v \in H^1(\Omega),\ \forall w \in D_h \tag{29}$$

and the following stability and approximation properties

$$\left\| \nabla j^h v \right\|_T \le c_s \left| \nabla v \right|_T, \quad \forall v \in H^1(\Omega),\ T \in \mathcal{T}_h, \tag{30}$$

$$\left\| j^h v - v \right\|_{m,T} \le c_i h^{r+1-m} \left| v \right|_{m,T}, \quad \forall v \in H^{r+1}(\Omega),\ m = \{1, 2\},\ T \in \mathcal{T}_h. \tag{31}$$

The vector-valued version for velocities is denoted by

$$\mathbf{j}^h :\ V \cap \left(H^1(\Omega)\right)^d \to \mathbf{V}_h. \tag{32}$$

4.3. Stabilization parameters

For the analysis of the method the mesh parameters α_h and τ_h are chosen such that

$$\alpha_h \left.\right|_T := \begin{cases} 0, & \text{if } T \in \mathcal{T}_h \text{ and } T \subset \Omega_S \\ k_T^{-1}, & \text{if } T \in \mathcal{T}_h \text{ and } T \subset \Omega_D \end{cases} \tag{33}$$

with

$$k_T = \inf\{(\mathbf{K}^{-1}\mathbf{v}, \mathbf{v})_T,\ \mathbf{v} \in L^2(\Omega)^d,\ \|\mathbf{v}\|_T = 1\}. \tag{34}$$

Since $\mathbf{K}^{-1}$ is a positive definite, $k_T > 0$. In addition, we choose τ_h such that

$$\tau_h \left.\right|_T := \begin{cases} 0, & \text{if } T \in \mathcal{T}_h \text{ and } T \subset \Omega_S \\ 1, & \text{if } T \in \mathcal{T}_h \text{ and } T \subset \Omega_D. \end{cases} \tag{35}$$

Below we prove the discrete stability of the method with respect to the norm

$$|||(\mathbf{v}, p)|||_h = (|||(\mathbf{v}, p)|||_X + \mathcal{S}_h(\mathbf{v}, p; \mathbf{v}, p))^{\frac{1}{2}}. \tag{36}$$

5. Stability

Theorem 3. *Let $\mathcal{T}_h$ be a quasi-regular partition and assume that the mesh parameters α_h and τ_h be as in (33) and (35). Then, the following discrete inf–sup condition holds for some positive constant $\tilde{\beta}$ independent of the mesh size h.*

$$\inf_{(\mathbf{v}_h, p_h) \in X_h \setminus \{\mathbf{0}\}} \ \sup_{(\mathbf{w}_h, q_h) \in X_h \setminus \{\mathbf{0}\}} \frac{\mathcal{A}(\mathbf{v}_h, p_h; \mathbf{w}_h, q_h)}{|||(\mathbf{v}_h, p_h)|||_{X_h}\ |||(\mathbf{w}_h, q_h)|||_{X_h}} \ge \tilde{\beta}. \tag{37}$$

Proof. First, let $(\mathbf{v}_h, p_h) \in X_h$, then the diagonal testing gives

$$\mathcal{A}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h) = \mathcal{A}(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h) + \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h) = |||\mathbf{v}|||_{\mathbf{V}}^2 + \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h). \tag{38}$$

Second, let $\mathbf{w}$ be as in Lemma 2, associated with $(\mathbf{v}_h, p_h) \in X_h$, and set $\mathbf{z} = j^h\mathbf{w} - \mathbf{w}$. Then,

$$\begin{aligned} \mathcal{A}(\mathbf{v}_h, p_h; \mathbf{j}^h\mathbf{w}, 0) &= \mathcal{A}(\mathbf{v}_h, p_h; \mathbf{w}, 0) + \mathcal{A}(\mathbf{v}_h, p_h; \mathbf{z}, 0) \\ &\ge \frac{1}{2}|||p_h|||_Q^2 - c_2|||\mathbf{v}_h|||_{\mathbf{V}}^2 + \mathcal{A}_S(\mathbf{v}_h, p_h; \mathbf{z}, 0) + \mathcal{A}_D(\mathbf{v}_h, p_h; \mathbf{z}, 0). \end{aligned}$$

Next, we estimate $\mathcal{A}_S(\mathbf{v}_h, p_h; \mathbf{z}, 0)$ and $\mathcal{A}_D(\mathbf{v}_h, p_h; \mathbf{z}, 0)$ as follows:

$$\mathcal{A}_S(\mathbf{v}_h, p_h; \mathbf{z}, 0) = (2\nu D(\mathbf{v}), D(\mathbf{z}))_{\Omega_S} + (\nabla p_h, \mathbf{z})_{\Omega_S} + \int_{\Gamma_I} \frac{1}{\alpha} (\mathbf{v}_{hS} \cdot \mathbf{t})\, (\mathbf{z}_S \cdot \mathbf{t})\, ds. \tag{39}$$

The first two terms bounded using Cauchy inequality together with the approximation, stability, and inverse inequalities

$$\begin{aligned}\left|(\nu D(\mathbf{v}), D(\mathbf{z}))_{\Omega_S}\right| &\le \nu \, \|D(\mathbf{v})\|_{\Omega_S} \, \|D(\mathbf{z})\|_{\Omega_S} \\ &\le \nu^{1/2} |||\mathbf{v}|||_{\mathbf{V}} \, \|\nabla \mathbf{z}\|_{\Omega_S} \\ &\le \nu^{1/2} c_i |||\mathbf{v}|||_{\mathbf{V}} \, \|\nabla \mathbf{w}\|_{\Omega_S} \\ &\le \nu^{1/2} c_2 c_i |||\mathbf{v}|||_{\mathbf{V}} |||p_h|||_Q\end{aligned}$$

and

$$\begin{aligned}(\nabla p_h, \mathbf{z})_{\Omega_S} &\le \left(\sum_{T \in \mathcal{T}_h, T \subset \Omega_S} h_T^{-2} \, \|\mathbf{z}\|_{0,T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h, T \subset \Omega_S} h_T^2 \, \|\nabla p_h\|_{0,T}^2 \right)^{1/2} \\ &\le c_i c_I \, \|\nabla \mathbf{w}\|_{\Omega_S} \, \|p_h\|_{\Omega_S} \\ &\le c_i c_I c_2 |||p_h|||_Q^2.\end{aligned}$$

The integral term is bounded using the trace theorem and the H^1-stability by

$$\begin{aligned}\left| \int_{\Gamma_I} \frac{1}{\alpha} \, (\mathbf{v}_{hS} \cdot \mathbf{t}) \, (\mathbf{z}_S \cdot \mathbf{t}) \, ds \right| &\le c_\gamma \alpha^{-1/2} |||\mathbf{v}_h|||_{\mathbf{V}} \, \|\nabla \mathbf{z}\|_{\Omega_S} \\ &\le c_\gamma c_s \alpha^{-1/2} |||\mathbf{v}_h|||_{\mathbf{V}} |||p_h|||_Q.\end{aligned}$$

Hence, by Young's inequality we obtain

$$\begin{aligned}\mathcal{A}_S(\mathbf{v}_h, p_h; \mathbf{z}, 0) &\le (2\nu^{1/2} c_2 c_i + c_\gamma c_s \alpha^{-1/2}) |||\mathbf{v}|||_{\mathbf{V}} |||p_h|||_Q + c_i c_I c_2 |||p_h|||_Q^2 \\ &\le \frac{1}{8} |||p_h|||_Q^2 + c_3 |||\mathbf{v}|||_{\mathbf{V}}^2\end{aligned} \tag{40}$$

with $c_3 = c_3(\nu, \alpha)$. For the Darcy bilinear form we have

$$\begin{aligned}\mathcal{A}_D(\mathbf{v}_h, p_h; \mathbf{z}, 0) &= (\mathbf{K}^{-1}\mathbf{v}_h + \nabla p_h, \mathbf{z})_{\Omega_D} \\ &= (\mathbf{K}^{-1}\mathbf{v}_h, \mathbf{z})_{\Omega_D} + (\kappa_h \nabla p_h, \mathbf{z})_{\Omega_D} \\ &\le \left\| \mathbf{K}^{-1/2}\mathbf{v}_h \right\|_{\Omega_D} \left\| \mathbf{K}^{-1/2}\mathbf{z} \right\|_{\Omega_D} + \alpha_h^{-1/2} \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h)^{1/2} \, \|\mathbf{z}\|_{\Omega_D} \\ &\le \left(c_K \left\| \mathbf{K}^{-1/2}\mathbf{v}_h \right\|_{\Omega_D} + \alpha_h^{-1/2} \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h)^{1/2} \right) \|\mathbf{z}\|_{\Omega_D} \\ &\le \left(c_K \left\| \mathbf{K}^{-1/2}\mathbf{v}_h \right\|_{\Omega_D} + \alpha_h^{-1/2} \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h)^{1/2} \right) c_2 c_i |||p_h|||_Q \\ &\le \frac{1}{8} |||p_h|||_Q^2 + c_4 |||\mathbf{v}_h|||_{\mathbf{V}}^2 + c_5 \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h).\end{aligned} \tag{41}$$

Further, the Cauchy–Schwarz inequality and the L^2-stability of $\mathbf{j}^h$ give

$$\begin{aligned}\mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{j}^h\mathbf{w}, 0) &\le \tilde{C} \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h) \left\| \nabla \mathbf{j}^{\mathbf{h}}\mathbf{w} \right\|_{\Omega_D} \\ &\le \tilde{C} c_2 c_s \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h) |||p_h|||_Q \\ &\le \frac{1}{8} |||p_h|||_Q^2 + c_6 \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h).\end{aligned} \tag{42}$$

Combining (40)–(42) we have

$$\mathcal{A}_h(\mathbf{v}_h, p_h; \mathbf{j}^h\mathbf{w}, 0) \ge \frac{1}{8} |||p_h|||_Q^2 - C_1(|||\mathbf{v}_h|||_{\mathbf{V}}^2 + \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h))$$

scaling $\mathbf{j}^h\mathbf{w}$ we obtain

$$\mathcal{A}_h(\mathbf{v}_h, p_h; \mathbf{j}^h\mathbf{w}, 0) \ge |||p_h|||_Q^2 - C_1(|||v_h|||_{\mathbf{V}}^2 + \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h)). \tag{43}$$

Choosing $(\mathbf{w}_h, q_h) = (\mathbf{v}_h, p_h) + \frac{1}{1+C}(\mathbf{j}^{\mathbf{h}}\mathbf{w}, 0) \in X_h$ we obtain

$$\begin{aligned}\mathcal{A}_h(\mathbf{v}_h, p_h; \mathbf{w}_h, q_h) &= |||\mathbf{v}_h|||_{\mathbf{V}}^2 + \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h) - \frac{C_1}{1 + C_1}(|||\mathbf{v}_h|||_{\mathbf{V}}^2 + \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h)) \\ &= \frac{1}{1 + C_1}(|||\mathbf{v}_h|||_{\mathbf{V}}^2 + |||p_h|||_Q^2 + \mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h)) \\ &= \frac{1}{1 + C_1} |||(\mathbf{v}_h, p_h)|||_h^2\end{aligned}$$

and

$$
\begin{aligned}
|||(\mathbf{w}_h, q_h)|||_h &\le |||(\mathbf{v}_h, p_h)|||_h + \frac{1}{1+C}|||(\mathbf{j}^h\mathbf{w}, 0)|||_h \\
&\le |||(\mathbf{v}_h, p_h)|||_h + \left\|\nabla \mathbf{j}^{\mathbf{h}}\mathbf{w}\right\|_\Omega \\
&\le C_2 |||(\mathbf{v}_h, p_h)|||_h,
\end{aligned}
$$

gives the required result

$$
\inf_{(\mathbf{v}_h,p_h)\in X_h\setminus\{\mathbf{0}\}} \sup_{(\mathbf{w}_h,q_h)\in X_h\setminus\{\mathbf{0}\}} \frac{\mathcal{A}_h(\mathbf{v}_h, p_h; \mathbf{w}_h, q_h)}{|||(\mathbf{v}_h, p_h)|||_h \, |||(\mathbf{w}_h, q_h)|||_h} \ge \tilde{\beta} \tag{44}
$$

with $\tilde{\beta} = C_2^{-1}/(1+C)$. □

6. Error analysis

In this section we derive error estimates.

Theorem 4. *Assume that the solution $(\mathbf{v}, p)$ of the Stokes–Darcy problem (14) is such that $(\mathbf{v}, p) \in \mathbf{V} \cap H^{r+1}(\Omega)^d \times Q \cap H^{r+1}(\Omega)$ and $(\mathbf{v}_h, p_h)$ be the solution of the stabilized problem (21). Then, the following error estimate holds*

$$
|||(\mathbf{v}-\mathbf{v}_h, p-p_h)|||_h \le Ch^r(|\mathbf{v}|_{r+1,\Omega} + |p|_{r+1,\Omega}), \tag{45}
$$

with a constant C independent of h.

Proof. As usual, we split the error into interpolation error and projection error

$$
(\mathbf{v}-\mathbf{v}_h; p-p_h) = (\mathbf{v}-\mathbf{j}^h\mathbf{v}; p-j^hp) + (\mathbf{j}^h\mathbf{v}-\mathbf{v}_h; j^hp-p_h). \tag{46}
$$

Using the stability estimate of Theorem 3 we know there exists $(\mathbf{w}_h, q_h) \in X_h$, with $|||(\mathbf{w}_h, q_h)|||_h \le \tilde{C}$ satisfying

$$
\begin{aligned}
|||(\mathbf{j}^{\mathbf{h}}\mathbf{v}-\mathbf{v}_h, j^hp-p_h)|||_h \le{}& \frac{1}{\tilde{\beta}} \sup_{(\mathbf{w}_h,q_h)\in X_h} \mathcal{A}_h(\mathbf{v}-\mathbf{v}_h, p-p_h; \mathbf{w}_h, q_h) \\
&+ \frac{1}{\tilde{\beta}} \sup_{(\mathbf{w}_h,q_h)\in X_h} \mathcal{A}_h(\mathbf{j}^{\mathbf{h}}\mathbf{v}-\mathbf{v}, j^hp-p; \mathbf{w}_h, q_h).
\end{aligned} \tag{47}
$$

By Galerkin orthogonality property we obtain

$$
\mathcal{A}_h(\mathbf{v}-\mathbf{v}_h, p-p_h; \mathbf{w}_h, q_h) = \mathcal{S}_h(\mathbf{v}, p; \mathbf{w}_h, q_h). \tag{48}
$$

So, the first term can be bounded as follows:

$$
\frac{\mathcal{A}_h(\mathbf{v}-\mathbf{v}_h, p-p_h; \mathbf{w}_h, q_h)}{|||(\mathbf{w}_h, q_h)|||_h} = \frac{\mathcal{S}_h(\mathbf{v}, p; \mathbf{w}_h, q_h)}{|||(\mathbf{w}_h, q_h)|||_h}.
$$

Since

$$
\begin{aligned}
\mathcal{S}_h(\mathbf{v}, p; \mathbf{w}_h, q_h) &\le \mathcal{S}_h(\mathbf{v}, p; \mathbf{v}, p)^{\frac{1}{2}} \, \mathcal{S}_h(\mathbf{w}_h, q_h; \mathbf{w}_h, q_h)^{\frac{1}{2}} \\
&\le \mathcal{S}_h(\mathbf{v}, p; \mathbf{v}, p)^{\frac{1}{2}} \, |||(\mathbf{w}_h, q_h)|||_h.
\end{aligned}
$$

Then, the approximation properties of κ_h imply

$$
\begin{aligned}
\frac{\mathcal{A}_h(\mathbf{v}-\mathbf{v}_h, p-p_h; \mathbf{w}_h, q_h)}{|||(\mathbf{w}_h, q_h)|||_h} &\le \tilde{C}(\alpha_h \, \|\kappa\nabla p\|_\Omega^2 + \tau_h \, \|\kappa \, div\, \mathbf{v}\|_\Omega^2)^{1/2} \\
&\le \tilde{C}h^r(\alpha_h^{1/2} \, |p|_{r+1,\Omega} + \tau_h^{1/2} \, |\nabla\mathbf{v}|_{r,\Omega})
\end{aligned}
$$

i.e. the projection error is of order $O(h^r)$.

To estimate the second term above consider separately each individual term

$$
\begin{aligned}
\mathcal{A}_S(\mathbf{j}^{\mathbf{h}}\mathbf{v}-\mathbf{v}, j^hp-p; \mathbf{w}_h, q_h) \le{}& \nu \left\|\nabla(\mathbf{j}^{\mathbf{h}}\mathbf{v}-\mathbf{v})\right\|_{\Omega_S} \|D\mathbf{w}_h\|_{\Omega_S} + \left\|j^hp-p\right\|_{\Omega_S} \|\nabla\mathbf{w}_h\|_{\Omega_S} \\
&+ \left\|\mathbf{j}^{\mathbf{h}}\mathbf{v}-\mathbf{v}\right\|_{\Omega_S} \|q_h\|_{\Omega_S} + C\alpha \left\|\nabla(\mathbf{j}^{\mathbf{h}}\mathbf{v}-\mathbf{v})\right\|_{\Omega_S},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}_D(\mathbf{j}^{\mathbf{h}}\mathbf{v}-\mathbf{v}, j^hp-p; \mathbf{w}_h, q_h) \le{}& \left\|\mathbf{K}^{-1/2}(\mathbf{j}^{\mathbf{h}}\mathbf{v}-\mathbf{v})\right\|_{\Omega_D} \left\|\mathbf{K}^{-1/2}\mathbf{w}_h\right\|_{\Omega_D} + \left\|\nabla(j^hp-p)\right\|_{\Omega_D} \|\mathbf{w}_h\|_{\Omega_D} \\
&+ \|\nabla q_h\|_{\Omega_D} \left\|\mathbf{j}^{\mathbf{h}}\mathbf{v}-\mathbf{v}\right\|_{\Omega_D}.
\end{aligned}
$$

Then, the approximation properties and the Poincare inequality for the integral terms give the required result

$$|||(\mathbf{v}-\mathbf{v}_h, p-p_h)|||_h \le Ch^r(|\mathbf{v}|_{r+1,\Omega}+|p|_{r+1,\Omega}), \quad (49)$$

which proves the estimate of the theorem.

Remark 5. Another type of stabilization that enforces the control of the natural $H(div)$ velocity norm on the Darcy region is to include the term $(\tau_h div\, \mathbf{v}, div\, \mathbf{w})$ in the bilinear form of the variational problem and use

$$\mathcal{S}_h(\mathbf{v}_h, p_h; \mathbf{w}, q) = (\alpha_h \kappa_h \nabla p, \kappa_h \nabla q)_\Omega. \quad (50)$$

Then, using the same tools as above we obtain the same error estimates as in Theorem 4. □

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