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Traub-Gander's family for the simultaneous determination of multiple zeros of polynomials

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Abstract

By combining Traub-Gander's family of third order for finding a multiple zero and suitable corrective approximations od Schröder's and Halley's type, a new family of iterative methods for the simultaneous approximation of multiple zeros of algebraic polynomials is proposed. Taking various forms of a function involved in the iterative formula, a number of different simultaneous methods can be obtained. It is proved that the order of convergence is 4, 5 or 6, depending of the type of employed corrective approximations. Two numerical examples are given to demonstrate the convergence properties of the proposed family of simultaneous methods. Displayed trajectories of the sequences of approximations point to global characteristics of the proposed family of iterative methods.

AMS Mathematical Subject Classification (2010): 65H05

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1 Introduction

The problem of approximating the zeros of algebraic polynomials is one of the most important problems in the theory and practice of iterative processes. For many decades available root-finding algorithms have calculated only one zero of the given polynomial at a time using the process of deflation if more than one zero was needed. This approach often gives "falsified" coefficients of deflated polynomial whose degree is lowered by one. This flaw is consequence of handling with an approximation x_i to the sought zero α_i after removing the corresponding linear factor $x - x_i$ (instead of $x - \alpha_i$). In many cases the above drawback can be overcome by approximating all zeros simultaneously, in a parallel fashion.

Simultaneous methods have been developed about the sixties of the twentieth century in the works of Durand [1], Dochev [2], Kerner [3], Ehrlich [4], Börsch-Supan [5], Aberth [6] and others. For more details see the books [7]–[10] and references cited therein. Simultaneous methods for the determination of multiple zeros of polynomials have been considered in many papers and books, see, e.g., [9], [11]–[25].

In this paper we propose a new family of iterative methods for the simultaneous approximation of multiple zeros of a polynomial f. This family is constructed by combining Traub-Gander's family

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of third order, presented in [26], the so-called Weierstrass' function and suitable corrective approximations $c_{j,r}$ $(r \in \{1,2,3\})$ in order to obtain as fast as possible convergence. The proposed iterative formula is of the form

$$\phi(z_i) = z_i - \frac{m_i h(t_i)}{\dfrac{f'(z_i)}{f(z_i)} - \displaystyle\sum_{j\neq i} m_j \Big(z_i - c_{j,r}\Big)^{-1}},$$

where $z_1, \ldots, z_n$ are approximations to the zeros of f and the function h satisfies the conditions $h(0) = 1,\ h'(0) = \frac{1}{2}$. Taking various forms of h it is possible to produce many different simultaneous methods, which is the main advantage of the proposed family. In Section 3 we prove that the order of convergence is four for ordinary approximations $c_{j,1} = z_j$, five for Schröder's approximations $c_{j,2} = z_j - m_j f(z_j)/f'(z_j)$ and six for Halley-like corrective approximations $c_{j,3} = z_j - f(z_j)/\big((m_j + 1)f'(z_j)/(2m_j) - f(z_j)f''(z_j)/f'(z_j)\big)$. To demonstrate the convergence properties of the proposed family of simultaneous methods, two numerical examples are given. Due to a very high convergence of the proposed methods, two iterations are sufficient to provide a high accuracy of the produced approximations to the zeros in solving most practical problems. It has been also shown that the computational convergence order matches well the theoretical order of convergence.

Finally, note that the implementation of simultaneous methods for multiple zeros of polynomials requires the knowledge of multiplicities in advance, which is not a trivial task. In fact, procedures for finding reasonably good initial approximations and multiplicities, together with an efficient zero-finding method, are parts of a composite algorithm, as discussed in [23], [27] and [28]. Fortunately, an extensive study in Farmer's dissertation [23] and the recent results presented in [25] give satisfactory solution of the aforesaid task concerning multiplicity. In this paper we restrict our research to the development of a new, efficient family of simultaneous methods.

2 Simultaneous methods of Traub-Gander's type

Let α be the zero of f of the known order of multiplicity $m \geq 1$. The following iteration function for finding a single multiple zero, referred to as Traub-Gander's family, has been presented in [26]

$$G_m(f;z) = z - m\frac{f(z)}{f'(z)}h(T_f(z)), \tag{1}$$

where

$$T_f(z) = 1 - m + \frac{mf(z)f''(z)}{f'(z)^2} \tag{2}$$

and h is at least two-times differentiable function of one variable. For simplicity, we will write sometimes only t instead of $T_f(z)$ assuming that t is a function of z given by (2). It has been proved in [26] that necessary and sufficient conditions which guarantee cubic convergence of the family of iterative methods (1) are

$$h(0) = 1, \quad h'(0) = 1/2, \quad |h^{(k)}(0)| \leq M < \infty \ \ (k = 2, 3, \ldots,\ M \text{ is a positive constant}). \tag{3}$$

The iterative formula (1) is quite general since produces a lot of particular methods for finding multiple zeros. Some examples of simple functions h which satisfy the conditions (3) are given below:

$$\begin{cases} h_1(t) = (1+t/4)^2, \\ h_2(t) = 1 + t/2 + bt^2, \quad b \text{ is arbitrary}, \\ h_3(t) = 1 + \dfrac{t}{2(1+bt)}, \quad b \text{ is arbitrary}, \\ h_4(t) = 1/(1-t/2), \quad \text{Halley-like method}, \\ h_5(t) = \dfrac{1+(\frac{1}{2}+b)t+ct^2}{1+bt+dt^2}, \quad b,c,d \text{ are arbitrary}, \\ h_6(t) = 1/\sqrt{1-t}, \quad \text{Ostrowski-like method}, \\ h_7(t) = 2/(1+\sqrt{1-2t}\,), \quad \text{Euler-like method}. \end{cases} \tag{4}$$

Observe that h_3 can be obtained from h_5 setting $c = d = 0$, and h_4 from h_3 taking $b = -1/2$. We note that h_3 leads to the following family of iterative methods for finding multiple zeros

$$V_m(f,\beta;z) = z - m\frac{f(z)}{f'(z)}\left(1 + \frac{T_f(z)}{2(1-\beta T_f(z))}\right) \ (\beta \text{ is an arbitrary parameter}). \tag{5}$$

In the case of simple zeros ($m = 1$) h_3 reduces to the Werner family [29]

$$V(f,\beta;z) = z - \frac{f(z)}{f'(z)}\left(1 + \frac{R_f(z)}{2(1-\beta R_f(z))}\right), \quad R_f(z) = \frac{f(z)f''(z)}{f'(z)^2},$$

so that the family (5) will be called the family of Werner's type.

Let f be a polynomial of order N with simple or multiple zeros $\alpha_1, \ldots, \alpha_n$ ($n \le N$) of the multiplicities $m_1, \ldots, m_n$, respectively, and assume that $c_{1,r}, \ldots, c_{n,r}$ are sufficiently close approximations to the zeros $\alpha_1, \ldots, \alpha_n$. The second subscript index r points to the kind of approximation that will be applied.

Let us introduce

$$\delta_{\lambda,i} = \frac{f^{(\lambda)}(z_i)}{f(z_i)}, \quad S_{\lambda,i} = \sum_{j \ne i} \frac{m_j}{(z_i - c_{j,r})^\lambda} \quad (\lambda = 1,2).$$

In this paper we consider three kinds of *corrective approximations* $c_{j,r}$:

$$\begin{cases} (1) \ \ c_{j,1} = z_j \ \ \text{(ordinary approximation)}, \\ (2) \ \ c_{j,2} = z_j - m_i/\delta_{1,i} \ \ \text{(Schröder's approximation, [30])}, \\ (3) \ \ c_{j,3} = z_j - \dfrac{2\delta_{1,i}}{\frac{m_i+1}{m_i}\delta_{1,i}^2 - \delta_{2,i}} \ \ \text{(Halley-like approximation, [31], [32])}. \end{cases} \tag{6}$$

The name comes from the fact that the approximations $c_{j,2}$ and $c_{j,3}$ improve the order of convergence without additional calculations since they use values $\delta_{1,i}$ and $\delta_{2,i}$ already calculated in the execution of iterative methods presented below by (10). In this way, we attain a high computational efficiency. Among cubically convergent methods we have taken Halley-like methods although we could also choose Ostrowski's, Euler's or Chebyshev's method. However, the first two methods deal with square root increasing computational costs, while Halley's method gives better results than Chebyshev's method for most polynomial equations.

Let us define Weierstrass-like function (see [10, Ch. 1])

$$W_i(z) = \frac{f(z)}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}(z-c_{j,r})^{m_j}} \quad (z \neq c_{j,r};\ i \in \{1,\dots,n\}). \tag{7}$$

Obviously, the function $W_i(z)$ has the same zeros as the polynomial f. Using this fact, Traub-Gander's family (1) can be rewritten in the form

$$\mathcal{G}_m(z_i) = z_i - m_i \frac{W_i(z_i)}{W'(z_i)} h(t_i), \quad t_i = 1 - m_i + m_i \frac{W_i(z_i)W_i''(z_i)}{W_i(z_i)^2} \quad (i = 1,\dots,n), \tag{8}$$

where z_i are some approximations to the zeros α_i $(i = 1,\dots,n)$.

Starting from (7) and applying the logarithmic derivatives, we find

$$\begin{aligned} \frac{W_i'(z_i)}{W_i(z_i)} &= \delta_{1,i} - S_{1,i}, \\ \frac{W_i''(z_i)}{W_i'(z_i)} &= \delta_{1,i} - S_{1,i} + \frac{\delta_{2,i} - \delta_{1,i}^2 + S_{2,i}}{\delta_{1,i} - S_{1,i}}. \end{aligned} \tag{9}$$

Substituting these expressions in (8), we obtain the following family of iterative methods of Traub-Gander's type

$$\begin{cases} t_i = 1 + \dfrac{m_i(\delta_{2,i} - \delta_{1,i}^2 + S_{2,i})}{(\delta_{1,i} - S_{1,i})^2}, \\ \hat{z}_i = \mathcal{G}_m(z_i) = z_i - \dfrac{m_i\, h(t_i)}{\delta_{1,i} - S_{1,i}}, \quad (i = 1,\dots,n), \end{cases} \tag{10}$$

where $\hat{z}_1,\dots,\hat{z}_n$ denote new approximations to the zeros $\alpha_1,\dots,\alpha_n$.

Let $z_1^{(0)},\dots,z_n^{(0)}$ be initial approximations to the zeros of the polynomial f, then the new family of iterative methods is of the form

$$z_i^{(k+1)} = \mathcal{G}_m(z_i^{(k)}) = z_i^{(k)} - \frac{m_i\, h(t_i^{(k)})}{\delta_{1,i}^{(k)} - S_{1,i}^{(k)}}, \quad (i = 1,\dots,n;\ k = 0,1,\dots), \tag{11}$$

where

$$\delta_{r,i}^{(k)} = \frac{f^{(k)}(z_i)}{f(z_i)}, \quad S_{r,i}^{(k)} = \sum_{j\neq i} \frac{m_j}{\left(z_i^{(k)} - c_{j,r}^{(k)}\right)^{\lambda}} \quad (r \in \{1,2,3\};\ \lambda = 1,2).$$

The family (11) belongs to the class of *total-step* methods ("Jacobi" or parallel mode). For simplicity, in what follows we will omit the iteration index k and use the formula (10).

3 Convergence analysis

For simplicity, let f be a monic polynomial, then

$$f(z) = \prod_{j=1}^{n}(z-\alpha_j)^{m_j}. \tag{12}$$

Applying the logarithmic derivative to (12) we obtain

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{n} \frac{m_j}{z-\alpha_j} \tag{13}$$

and hence

$$\frac{f''(z)f(z) - f'(z)^2}{f(z)^2} = -\sum_{j=1}^{n} \frac{m_j}{(z-\alpha_j)^2}. \tag{14}$$

Setting $z = z_i$ and combining (13) and (14) we get

$$\delta_{1,i} = \sum_{j=1}^{n} \frac{m_j}{z_i-\alpha_j} \tag{15}$$

and

$$\delta_{2,i} - \delta_{1,i}^2 = -\sum_{j=1}^{n} \frac{m_j}{(z_i-\alpha_j)^2}. \tag{16}$$

Let us introduce the errors $\varepsilon_j = z_j - \alpha_j,\ \varepsilon_{j,r} = z_j - c_{j,r}$ and assume that all errors $\varepsilon_{1,r}, \ldots \varepsilon_{n,r}$ have magnitudes of approximately the same size, that is, $|\varepsilon_{i,r}| = O(|\varepsilon_{j,r}|) = O(|\varepsilon_r|)$, where $\varepsilon_r \in \{\varepsilon_{1,r}, \ldots, \varepsilon_{n,r}\}$ is the error of the largest magnitude. If $c_{j,r} = c_{j,1} = z_j$, then it is clearly $\varepsilon_{j,1} = \varepsilon_j$.

Theorem 1. *Let $z_1^{(0)}, \ldots, z_n^{(0)}$ be initial approximations sufficiently close to the zeros $\alpha_1, \ldots, \alpha_n$ of a polynomial f and let the conditions* (3) *hold. Then the family of iterative methods* (11) *with corrective approximations $c_{j,r}$, given by* (6)*, has the order of convergence $r+3,\ r \in \{1,2,3\}$.*

Proof. For simplicity, we omit the iteration index and use the formulas (10). Also, we will write $\sum_{j\neq i} a_j$ instead of $\sum_{\substack{j=1\\ j\neq i}}^{n} a_j$.

First, we have

$$\delta_{1,i} - S_{1,i} = \sum_{j=1}^{n} \frac{m_j}{z_i-\alpha_j} - \sum_{j\neq i} \frac{m_j}{z_i - c_{j,r}} = \frac{m_i}{\varepsilon_i} - \sum_{j\neq i} a_{ij}\,\varepsilon_{j,r}, \quad a_{ij} = \frac{m_j}{(z_i-\alpha_j)(z_i-c_{j,r})}. \tag{17}$$

In a similar way we find

$$\delta_{2,i} - \delta_{1,i} + S_{2,i} = -\sum_{j=1}^{n} \frac{m_j}{(z_i-\alpha_j)^2} + \sum_{j\neq i} \frac{m_j}{(z_i-c_{j,r})^2} = -\frac{m_i}{\varepsilon_i^2} + \sum_{j\neq i} m_j\Big(\frac{1}{(z_i-c_{j,r})^2} - \frac{1}{(z_i-\alpha_j)^2}\Big),$$

wherefrom

$$\delta_{2,i} - \delta_{1,i} + S_{2,i} = -\frac{m_i}{\varepsilon_i^2} + \sum_{j\neq i} b_{ij}\,\varepsilon_{j,r}, \quad b_{ij} = \frac{m_j(2z_i - \alpha_j - c_{j,r})}{(z_i-\alpha_j)^2(z_i-c_{j,r})^2}. \tag{18}$$

Substituting (17) and (18) in (10), we obtain for t_i

$$t_i = \frac{-2m_i\varepsilon_i A_{i,r}^* + \varepsilon_i^2\big(A_{i,r}^*\big)^2 + m_i\varepsilon_i^2 B_{i,r}^*}{(m_i - \varepsilon_i A_{i,r}^*)^2}, \tag{19}$$

where

$$A_{i,r}^* = \sum_{j\neq i} a_{ij}\,\varepsilon_{j,r}, \quad B_{i,r}^* = \sum_{j\neq i} b_{ij}\,\varepsilon_{j,r}.$$

The errors $\varepsilon_{j,r}$ can be expressed as $\varepsilon_{j,r} = \gamma_{j,r}\varepsilon_r$ where $\gamma_{j,r}$ are some constants and $\varepsilon_r \in \{\varepsilon_{1,r}, \ldots, \varepsilon_{n,r}\}$ is the error of maximal magnitude. Let

$$\mathcal{A}_{i,r} = \sum_{j\neq i} \gamma_{j,r} a_{ij}, \quad \mathcal{B}_{i,r} = \sum_{j\neq i} \gamma_{j,r} b_{ij}.$$

Then

$$A^*_{i,r} = \varepsilon_r \mathcal{A}_{i,r} = O(|\varepsilon_r|), \quad B^*_{i,r} = \varepsilon_r \mathcal{B}_{i,r} = O(|\varepsilon_r|).$$

From (19) we observe that $t_i = O(\varepsilon_i\varepsilon_r)$ so that the function $h(t)$ appearing in (10) can be represented by its Taylor's series at the point $t = 0$. First, we introduce a complex value

$$Q_i = \sum_{k=2}^{\infty} \frac{h^{(k)}(0)}{k!} t_i^{k-2}. \tag{20}$$

Since, using (3),

$$|Q_i| \leq \sum_{k=2}^{\infty} \frac{|h^{(k)}(0)|}{k!} |t_i|^{k-2} \leq \frac{M}{|t_i|^2}\Bigg(\sum_{k=0}^{\infty} \frac{|t_i|^k}{k!} - 1 - |t_i|\Bigg) = M\phi(|t_i|),$$

where

$$\phi(|t_i|) = \frac{e^{|t_i|} - 1 - |t_i|}{|t_i|^2}.$$

The function ϕ is monotonically increasing on the interval $[0, 1]$ with $\phi(0) \to 1/2$ as $|t_i| \to 0$ and $\phi(|t_i|) \leq e - 2 \approx 0.718$ for $|t_i| \in [0, 1]$.

Therefore, Q_i is bounded and we can take $Q_i = O(1)$. Then Taylor's series of h can be represented by

$$h(t_i) = h(0) + h'(0)t_i + Q_i t_i^2, \tag{21}$$

where Q_i is given by (20). Later we will show that the terms of higher order, expressed by $Q_i t_i^2$, do not influence the order of convergence of the family (10).

Now, from (10) and (17) there follows

$$\hat{z}_i = z_i - \frac{m_i\varepsilon_i}{m_i - \varepsilon_i\varepsilon_r \sum_{j\neq i} \gamma_{j,r} a_{ij}} \Big(1 + \tfrac{1}{2}t_i + Q_i t_i^2\Big). \tag{22}$$

Let $\hat{\varepsilon}_i = \hat{z}_i - \alpha_i$. The determination of expression of the errors $\hat{\varepsilon}_i$ requires long and tedious calculation so that we have used symbolic computation in algebra computer system *Mathematica*. Starting from (22) we have found

$$\begin{aligned} \hat{\varepsilon}_i \;\; = \;\; & \frac{\varepsilon_i^3\varepsilon_r}{2(\varepsilon_i\varepsilon_r\mathcal{A}_{i,r} - m_i)^5}\Big\{2\varepsilon_i^3\varepsilon_r^4\mathcal{A}_{i,r}^5 + m_i\varepsilon_i^2\varepsilon_r^3\mathcal{A}_{i,r}^4(2Q_i - 7) \\ & + m_i^2\varepsilon_i\varepsilon_r^2\mathcal{A}_{i,r}^2\Big(\varepsilon_i\mathcal{B}_{i,r}(4Q_i + 1) - 8\mathcal{A}_{i,r}(Q_i - 1)\Big) \\ & + m_i^3\varepsilon_r\Big(2Q_i(\varepsilon_i\mathcal{B}_{i,r} - 2\mathcal{A}_{i,r})^2 - \mathcal{A}_{i,r}(3\mathcal{A}_{i,r} + 2\varepsilon_i B_{i,r})\Big) + \mathcal{B}_{i,r}m_i^4\Big\}, \end{aligned}$$

wherefrom, after some rearrangement,

$$|\hat{\varepsilon}_i| = \frac{|\varepsilon_i|^3|\varepsilon_r|\big|\sum_{j\neq i}\gamma_{j,r}b_{ij}\big|}{2m_i\Big|1 - \dfrac{\varepsilon_i\varepsilon_r\sum_{j\neq i}\gamma_{j,r}a_{ij}}{m_i}\Big|^5} + O(|\varepsilon_i|^3|\varepsilon_r|^2). \tag{23}$$

Note that higher-order terms expressed by $Q_i t_i^2$ (see (21)) are not involved in the main part of (23), which means that they do not influence the order of convergence of the family of methods (11).

Since Schröder's and Halley-like method have the orders two and three, respectively, there follows $|\varepsilon_r| = \gamma_r |\varepsilon|^r \quad (r = 1, 2, 3)$, $\varepsilon_i = \gamma_{i,1}|\varepsilon|$ and (23) becomes

$$|\hat{\varepsilon}_i| = \frac{|\gamma_r||\gamma_{i,1}|^3 \left|\sum_{j \neq i} \gamma_{j,r} b_{ij}\right|}{2m_i \left|1 - \dfrac{\varepsilon_i \varepsilon_r \sum_{j \neq i} \gamma_{j,r} a_{ij}}{m_i}\right|^5} \cdot |\varepsilon|^{r+3} + O(|\varepsilon|^{3+2r}). \tag{24}$$

Let $\varepsilon \to 0$, then the denominator of (24) tends to $2m_i$, while the expression $|\gamma_r||\gamma_{i,1}|^3 \left|\sum_{j \neq i} \gamma_{j,r} b_{ij}\right|$ in the numerator of (24) tends to some constant. As a consequence, from (24) we conclude that the family of iterative methods (11) has the order 4 for $r = 1$ (ordinary approximations), the order 5 for $r = 2$ (Schröder's approximations), and the order 6 for $r = 3$ (Halley-like approximations). $\square$

Remark 1. Let t_i be very small in magnitude, that is, $t_i \approx 0$. Then, according to the conditions (3), it follows that $h(t_i) \approx h(0) = 1$. Setting this value in (10) one obtains approximate formula

$$\hat{z}_i = z_i - \frac{m_i}{\delta_{1,i} - S_{1,i}}. \tag{25}$$

If $c_{j,1} = z_j$, then (25) is the third order method for multiple zeros originated from the method for simple zeros considered by Ehrlich [4] and Aberth [6]. If $c_{j,2} = z_j - 1/\delta_{1,i}$, then (25) is the fourth order method of Ehrlich-Aberth's type with Schröder's corrections. The sum $S_{1,i}$ in (25) could be also calculated with the approximations $c_{j,3}$ to produce a method of fifth order; however, $c_{j,3}$ requires the evaluation of the second derivative f'' so that such method would be inefficient. Altogether, the application of the mentioned errors $c_{j,r}$ in (25) leads to the method of order $r + 2$ ($r \in \{1, 2, 3\}$). Having in mind this result and the assertion of Theorem 1 proved in Section 3, we observe that the factor $h(t_i) = 1 + \frac{1}{2}t_i$ in (10) has a corrective role since it increases the order to $r + 3$.

4 Numerical results

To illustrate the convergence properties of the methods from the family (11), we have applied five methods to a number of polynomial equations. For demonstration, two algebraic polynomials of relatively high degree have been selected. We have dealt with the following particular functions h (see (4)):

– h_4 (Halley-like method);

– h_3, $b = 0$ (Chebyshev-like method);

– h_6 (Ostrowski-like method);

– h_7 (Euler-like method);

– h_5, $b = c = d = 1$ (Method with rational functions);

As a measure of accuracy of the obtained approximations, we have calculated Euclid's norm

$$e^{(k)} := \left(\sum_{i=1}^{n} \left|z_i^{(k)} - \alpha_i\right|^2\right)^{1/2} \qquad (k = 0, 1, \dots). \tag{26}$$

In practice, the convergence behavior of iterative methods depends on the closeness of initial approximations to the sought zeros, the location of these zeros and the structure of the tested polynomial, as well as some other factors to a smaller extent. To check the theoretical order of convergence ρ we have displayed in Tables 1 and 2 the so-called *computational order of convergence* r_c using the formula

$$r_c = \frac{\log |e^{(k)}/e^{(k-1)}|}{\log |e^{(k-1)}/e^{(k-2)}|}. \tag{27}$$

Jay [33] has derived this formula for a single nonlinear equation $\varphi(x) = 0$ and systems of nonlinear equations, but it gives satisfactory results for simultaneous methods too; large deviations of r_c relative to the theoretical order ρ are seldom in practice.

Example 1. Five selected methods from the family (11) have been applied for the simultaneous approximation to the zeros of the polynomial of degree $N = 39$:

$$f_1(z) = (z-4)^2(z+1)^3(z^4-16)^2(z^2+9)^4(z^2+2z+5)(z^2+2z+2)(z^2-2z+2)^2(z^2-4z+5)^2(z^2-2z+10)^3 .$$

The zeros of this polynomial and their multiplicities can be easily recognized from the above factorization of f_1. The initial approximations have been selected to give $e^{(0)} \approx 0.949$. The choice of the initial approximations to the zeros that guarantee the convergence has been considered in details in [23] and [34] so that we will not discuss this issue in this paper. The error norms $e^{(k)}$ of approximations in the first three iterations are given in Table 1, where $A(-h)$ means $A \times 10^{-h}$.

	Methods	$(10)_{h4}$	$(10)_{h3,b=0}$	$(10)_{h6}$	$(10)_{h7}$	$(10)_{h5,b=c=d=1}$
$c_{j,1}$ order 4	$e^{(1)}$	$1.67(-2)$	$1.19(-2)$	$1.32(-2)$	$2.43(-2)$	$5.16(-2)$
	$e^{(2)}$	$2.25(-9)$	$6.30(-10)$	$1.20(-9)$	$3.39(-8)$	$8.58(-7)$
	$e^{(3)}$	$2.08(-36)$	$3.39(-39)$	$3.20(-37)$	$1.03(-31)$	$1.15(-25)$
	r_c by (27)	3.938	4.022	3.917	4.016	3.949
$c_{j,2}$ order 5	$e^{(1)}$	$5.33(-3)$	$5.22(-3)$	$5.85(-3)$	$6.94(-3)$	$9.52(-3)$
	$e^{(2)}$	$1.76(-13)$	$1.34(-13)$	$2.06(-13)$	$7.96(-13)$	$1.24(-11)$
	$e^{(3)}$	$2.17(-65)$	$3.19(-67)$	$1.70(-65)$	$3.66(-62)$	$3.40(-55)$
	r_c by (27)	4.953	5.064	4.982	4.963	4.903
$c_{j,3}$ order 6	$e^{(1)}$	$2.80(-3)$	$2.40(-3)$	$2.89(-3)$	$3.02(-3)$	$2.65(-3)$
	$e^{(2)}$	$2.63(-17)$	$8.92(-18)$	$3.29(-17)$	$4.41(-17)$	$1.01(-17)$
	$e^{(3)}$	$1.06(-100)$	$1.40(-105)$	$3.69(-100)$	$1.64(-99)$	$2.55(-103)$
	r_c by (27)	5.945	6.085	5.948	5.958	5.936

Table 1: Norms of approximation errors to the zeros of the polynomial f_1.

Example 2. In order to find the zeros of the polynomial of degree $N = 27$

$$f_2(z) = (z-1)^3(z-2)^3(z-3)^4(z-4)^4(z-5)^2(z-6)^2(z-7)^2(z-8)^2(z-9)^2(z-10)(z-11)(z-12),$$

we have applied the same methods as in Example 1. The zeros of this polynomial and their multiplicities are easy to detect from the factorization of f_2. The initial approximations have been taken to give the norm $e^{(0)} \approx 0.775$. The entries of the error norms $e^{(k)}$ in the first three iterations are given in Table 2. Note that the polynomial f_2 is of Wilkinson's type [35], which is ill-conditioned. It is well known that many algorithms work with big efforts with this class of polynomials. However, the presented family of methods exhibits robustness since it deals successfully with this kind of polynomials. Comparing Tables 1 and 2 we observe that better approximations are produced in the case of the polynomial f_1 (Example 1) although it is of higher degree.

From Tables 1 and 2 we observe that all tested methods from the family (11) produce approximations of very high accuracy. The values of r_c calculated by (27) show that the computational order of convergence r_c mainly fits well the theoretical order of convergence in spite of the fact that the formula (27) was not derived directly for simultaneous methods, as noted above.

	Methods	$(10)_{h4}$	$(10)_{h3,b=0}$	$(10)_{h6}$	$(10)_{h7}$	$(10)_{h5,b=c=d=1}$
$c_{j,1}$ order 4	$e^{(1)}$	1.90(−2)	9.31(−3)	2.05(−2)	2.30(−2)	2.79(−2)
	$e^{(2)}$	8.60(−9)	3.46(−10)	1.38(−8)	2.84(−8)	5.97(−8)
	$e^{(3)}$	5.19(−34)	3.62(−40)	8.60(−33)	2.12(−31)	4.15(−30)
	r_c by (27)	3.976	4.034	3.922	3.914	3.909
$c_{j,2}$ order 5	$e^{(1)}$	1.96(−2)	9.02(−3)	3.80(−2)	1.06(−1)	3.11(−2)
	$e^{(2)}$	4.44(−10)	1.09(−12)	4.68(−9)	1.21(−7)	1.09(−9)
	$e^{(3)}$	4.61(−48)	4.43(−62)	1.52(−43)	1.47(−37)	1.72(−46)
	r_c by (27)	4.969	4.980	4.991	5.032	4.935
$c_{j,3}$ order 6	$e^{(1)}$	1.27(−2)	2.30(−2)	1.02(−2)	9.35(−3)	6.02(−2)
	$e^{(2)}$	2.95(−13)	6.68(−14)	1.41(−13)	1.04(−13)	2.87(−11)
	$e^{(3)}$	1.26(−75)	2.92(−80)	1.53(−77)	2.44(−78)	1.03(−63)
	r_c by (27)	5.864	5.759	5.890	5.900	5.626

Table 2: Norms of approximation errors to the zeros of the polynomial f_2.

Two iterative steps of the considered sixth order methods are usually sufficient in solving most practical problems when initial approximations are reasonably good and polynomials are well-conditioned. The third iteration is given only to demonstrate very fast convergence but, most frequently, it is not needed for real-life problems. Besides, we have executed the third iterations to provide the calculation of the computational order of convergence r_c, see the formula (27). Note that the calculation by (27) is not possible in practice since the zeros are unknown. However, we can easily avoid the third iteration by calculating the error norms in (26) as

$$e^{(k)} := \left(\sum_{i=1}^{n} \left|f(z_i^{(k)})\right|^2\right)^{1/2} \quad (k = 0, 1, \dots).$$

The behavior of the family of iterative methods (11) is graphically displayed in Figures 1–4 taking the functions $h_2\big|_{b=0}$ (Chebyshev's method), h_4 (Halley's method), h_6 (Ostrowski's method) and h_7 (Euler's method), see (6). We have considered the polynomial

$$P(z) = \prod_{r=1}^{15}(x - r)$$

of Wilkinson's type with simple zeros $\alpha_j = r,\ r = 1, 2, \dots, 15$. The stopping criterion has been given by the inequality

$$M(h_i^{(k)}) = \max_{1\le j\le 15} |z_j^{(k)} - \alpha_j| < 10^{-2} \quad (i = 2, 4, 6, 7).$$

Aberth's initial approximations $z_\nu^{(0)}$ (see [6]) have been taken equidistantly from the circle with the radius $r_0 = 20$ and the center

$$c = -\frac{a_1}{n} = \frac{\alpha_1 + \cdots + \alpha_{15}}{15} = 8 \ \text{ (the barycenter of zeros)},$$

where a_1 is the coefficient by z^{n-1} of the polynomial $P(z) = z^n + a_1 z^{n-1} + \cdots + a_0$. Therefore,

$$z_\nu^{(0)} = c + r_0 \cdot \exp\Big(i\frac{\pi}{n}(2\nu - 3/2)\Big) = 8 + 20 \cdot \exp\Big(i\frac{\pi}{15}(2\nu - 3/2)\Big) \quad (\nu = 1, \dots, 15;\ i = \sqrt{-1}).$$

All four methods have shown approximately the same convergence behavior of global type satisfying the stopping criterion after 10 iterations with the following maximal errors

$$M(h_2\big|_{b=0}) = 1.22 \times 10^{-6}, \quad M(h_4) = 1.70 \times 10^{-12}, \quad M(h_6) = 1.55 \times 10^{-26}, \quad M(h_7) = 2.71 \times 10^{-3}.$$

At the beginning, all tested methods converge linearly but almost straightforwardly toward the zeros of P, showing very fast convergence in final iterations. From Figures 1–4 we can observe that the sequences of approximations are radially distributed toward the zeros.

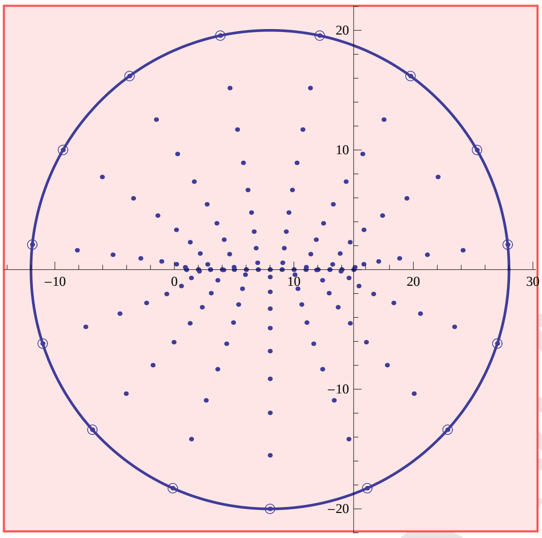

Fig. 1 Trajectory of (11) for $h_2\big|_{b=0}$

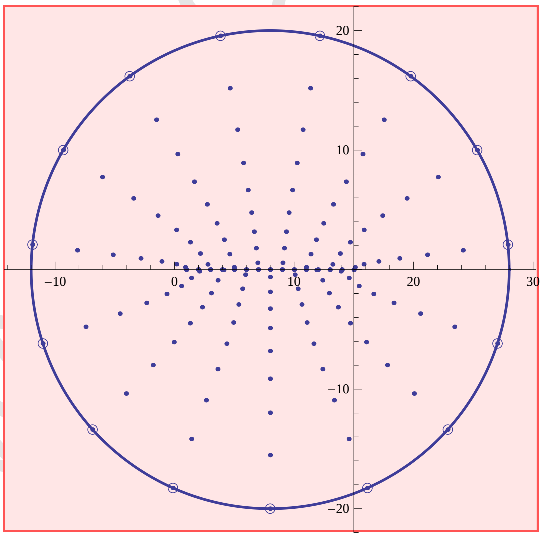

Fig. 2 Trajectory of (11) for h_4

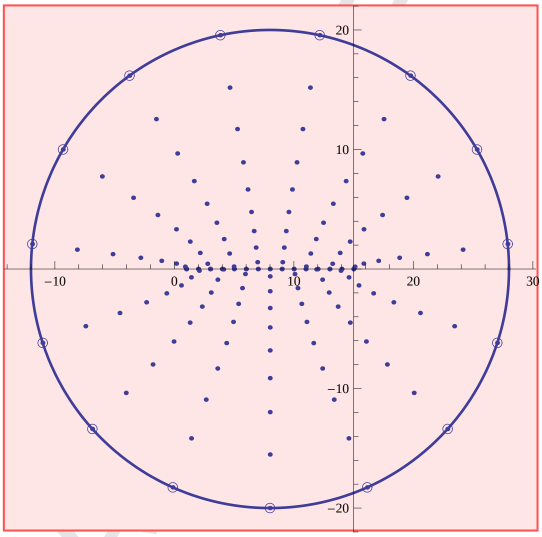

Fig. 3 Trajectory of (11) for h_6

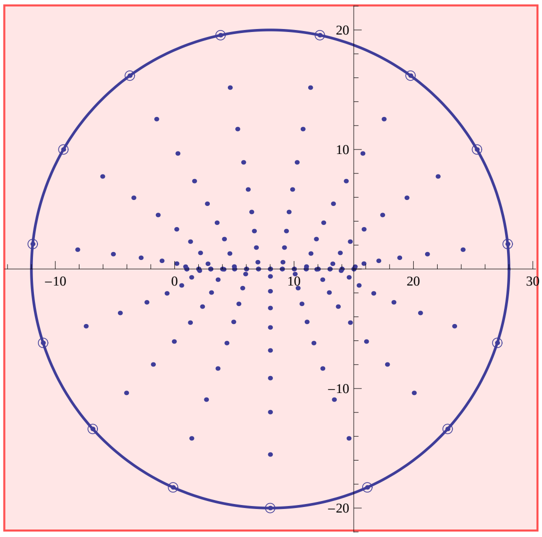

Fig. 4 Trajectory of (11) for h_7

We emphasize that many authors proclaim their own methods as the best ones in the class of methods of the same order of convergence. Such assertions are maybe true for a set of suitably selected polynomials, but a number of numerical examples has shown that the methods of the same order and constructed in the same/similar manner produce approximations of approximately the same quality and that none of these methods can be the best for all examples. For this reason we have not compared the proposed methods with existing fourth, fifth and sixth order methods.

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