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Reconstruction of a time dependent source term from a single boundary measurment in Maxwell's equations with nonlinear generalized Ohm's law

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Abstract

Hyperbolic Maxwell's equation with an unknown time dependent source is investigated. We consider a nonlinear generalized Ohm's law in our model. The source is reconstructed from a single boundary measurement over a part of the boundary. We use well-known Rothe's method to show the existence of a solution. In the case of regular solution, we provide a uniqueness proof as well. To support theoretical results a numerical experiment is provided.

Keywords: Maxwell's equations, inverse source problem, Rothe's method, boundary measurement
2010 MSC: 65M32, 65M20

1. Introduction

The domain $\Omega \subset \mathbb{R}^3$ is assumed to be either smooth $\Omega \in C^{1,1}$ or convex. The boundary of Ω is denoted by $\partial\Omega = \Gamma$. The symbol $\boldsymbol{n}$ stands for the unit outward normal vector on the boundary Γ. We derive our model from the traditional Maxwell equations (for further details, see [1, 2])

$$\begin{array}{ll} \nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, & \nabla \cdot \boldsymbol{B} = 0, \\ \nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D} + \boldsymbol{J} + \boldsymbol{J}_{app}, & \nabla \cdot \boldsymbol{D} = \rho, \end{array} \tag{1}$$

where $\boldsymbol{E}$ stands for the electric field, $\boldsymbol{B}$ denotes the magnetic induction field, $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{D}$ represents the electric displacement, $\boldsymbol{J}$ is the total current density, ρ is the density of electrical charge and $\boldsymbol{J}_{app}$ expresses the source term.

The constitutive relations between the four vector fields $\boldsymbol{B}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{E}$ are generally expressed in the following manner

$$\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{E}, \boldsymbol{H}), \qquad \boldsymbol{D} = \boldsymbol{D}(\boldsymbol{E}, \boldsymbol{H}).$$

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Their exact form depends on the particular physical phenomenon we are modelling. In many cases the present values of solutions depend on their previous values. These dependencies are expressed with a memory term. There are plenty of applications, for instance in chiral media [3], meta-materials [4, 5], nonlinear optics [6, 7, 8] or geophysics [9, 10, 11, 12]. The physical phenomenon which is often observed in geophysics is charge accumulation in rocks that serve as capacitors. The charge then decays, and this introduces an effective change to the traditional Ohm's law that does not assume capacitance. Hence it becomes a convolution in time. The authors of [11] considered a generalized Ohm's law

$$\boldsymbol{J}(t) = (\sigma * \boldsymbol{E})(t),$$

where the symbol $*$ stands for the usual convolution in time, e.g. $(f * g(\boldsymbol{x}))(t) = \int_0^t f(t-s) g(\boldsymbol{x}, s)\, \mathrm{d}s$.

In our paper we adopt a generalized Ohm's law in the following nonlinear form

$$\boldsymbol{J}(t) = (\sigma * \boldsymbol{E})(t) - (1 * \boldsymbol{N}(\boldsymbol{E}))(t).$$

The electric conductivity term σ is assumed to be separable, i.e.

$$\sigma(\boldsymbol{x}, t) = \hat{\sigma}(\boldsymbol{x}) \sigma(t).$$

Both $\hat{\sigma}(\boldsymbol{x})$ and $\sigma(t)$ are known. Therefore to improve the readability of our paper we assume $\hat{\sigma}(\boldsymbol{x})$ to be a positive constant. Time dependent part $\sigma(t)$ is assumed to be Lipschitz continuous and bounded, i.e. $0 < \sigma_* \leqslant \sigma \leqslant \sigma^* < \infty$. The nonlinear function $\boldsymbol{N}$ is supposed to be globally Lipschitz continuous and it also fulfills the following boundary condition

$$\boldsymbol{N}(\boldsymbol{E}) \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma. \tag{2}$$

Next, we consider a homogeneous dielectric material, i.e.

$$\boldsymbol{B} = \mu \boldsymbol{H}, \qquad \boldsymbol{D} = \varepsilon \boldsymbol{E},$$

where μ and ε are positive constants. Elimination of $\boldsymbol{H}$ in (1) then yields

$$\varepsilon \partial_t^2 \boldsymbol{E} + \hat{\sigma} \partial_t (\sigma * \boldsymbol{E}) + \nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E} = \boldsymbol{N}(\boldsymbol{E}) - \partial_t \boldsymbol{J}_{app}.$$

The tangential component of $\boldsymbol{E}$ is supposed to be continuous across the boundary, i.e.

$$\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma. \tag{3}$$

The initial data are prescribed as follows

$$\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0(\boldsymbol{x}), \qquad \partial_t \boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{W}_0(\boldsymbol{x}). \tag{4}$$

The source term $\partial_t \boldsymbol{J}_{app}$ is assumed to be separable, i.e.

$$-\partial_t \boldsymbol{J}_{app} = h(t) \boldsymbol{f}(\boldsymbol{x}).$$

Here, the function $\boldsymbol{f}(\boldsymbol{x})$ is given but $h(t)$ is unknown. We suppose that

$$\boldsymbol{f} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma.$$

For the sake of simplicity, we assume $\mu \equiv \varepsilon \equiv \hat{\sigma} \equiv 1$. Then our PDE becomes

$$\partial_t^2 \boldsymbol{E} + \partial_t (\sigma * \boldsymbol{E}) + \nabla \times \nabla \times \boldsymbol{E} = h(t) \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{N}(\boldsymbol{E}). \tag{5}$$

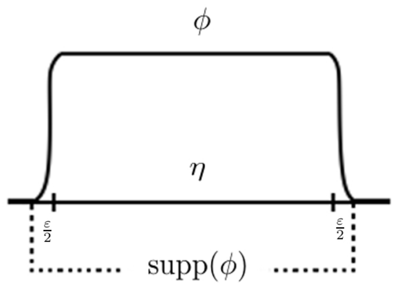

Figure 1. Vertical cut of the measured part of the boundary.

The measurement introduced below will be used to recover the time dependent part of the source term $h(t)$

$$\int_\Gamma \phi \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma = m(t). \tag{6}$$

With ϕ being a function from $C^\infty(\overline{\Omega})$ with $meas\{\mathrm{supp}(\phi) \cap \Gamma\} > 0$. After applying the measurement operator to equation (5)[1] and assuming that $\int_\Gamma \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{n}\phi \, \mathrm{d}\Gamma \neq 0$, we eliminate $h(t)$ to obtain

$$h(t) = \frac{m''(t) + (\sigma * m)'(t) + \int_\Omega \nabla \times \nabla \times \boldsymbol{E} \cdot \nabla\phi \, \mathrm{d}\boldsymbol{x} - \int_\Gamma \boldsymbol{N}(\boldsymbol{E}) \cdot \boldsymbol{n}\phi \, \mathrm{d}\Gamma}{\int_\Gamma \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{n}\phi \, \mathrm{d}\Gamma}. \tag{7}$$

We used the fact that $\partial_t (f * g)(t) = (f * \partial_t g)(t) + f(t)g(0)$ for any given functions f, g and applied the Green theorem in the following way

$$\int_\Gamma (\nabla \times \nabla \times \boldsymbol{E} \cdot \boldsymbol{n})\phi \, \mathrm{d}\Gamma = \int_\Omega \nabla \times \nabla \times \boldsymbol{E} \cdot \nabla\phi \, \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla \cdot (\nabla \times \nabla \times \boldsymbol{E})\phi \, \mathrm{d}\boldsymbol{x} = \int_\Omega \nabla \times \nabla \times \boldsymbol{E} \cdot \nabla\phi \, \mathrm{d}\boldsymbol{x}. \tag{8}$$

The inverse source problem reads as finding a pair $\{\boldsymbol{E}(\boldsymbol{x}, t), h(t)\}$ such that equations (5) and (7) are satisfied.

Remark 1. *In the theoretical part of our paper, we do not assume any noise to be present in the measurement. Otherwise* (6) *and* (7) *would not hold.*

There is a broad range of literature considering Inverse Problems. Many papers are dealing with Inverse Source Problems in hyperbolic setting. There are different techniques used to reconstruct the source term, for instance Carleman estimates have been used in [13, 14]. If the source is also space dependent then an extra measurement in space (e.g. solution in the final time) is needed, cf. [15, 16]. Linear problems have been addressed in [17, 18, 19]. Boundary measurements were used in [20, 21, 22].

We measure only the normal component of $\boldsymbol{E}$ on a part of the boundary Γ which is modeled by the function ϕ. Let us explain the purpose of this function in more detail. Assume that the measurement is done on a part of the boundary denoted as η. Naturally, η is a subset of Γ, i.e. $\eta \subset \Gamma$. Then the function ϕ is defined as $\phi(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in \eta$ and $meas\{\mathrm{supp}(\phi) \cap \eta\} = |\overline{\eta}|$, moreover, $meas\{\mathrm{supp}(\phi) \cap \Gamma\} = |\overline{\eta}| + \varepsilon$, for some small and positive ε. For better interpretation see Fig. 1. The implementation of ϕ in our measurement is solely due to mathematical reasons. With this addition, we can use the Green theorem as in equation (8).

Structure of our paper is organized as follows. In Section 2, we propose a weak formulation of our problem and then in Section 3, we discretize the time variable and approximate (3),(4),(5) and (7). Moreover, we prove the existence of a unique solution at each time step. A numerical scheme for obtaining this solution is also provided. In Section 4, we provide some stability results for our solution. Then in Section 5, we use Rothe's functions cf. [23] to

[1] We multiply (5) with $\phi \in C^\infty(\overline{\Omega})$, take a dot product with a unit outward normal vector $\boldsymbol{n}$ and integrate over Γ.

approximate our discretized system in the continuous form and state an existence theorem. Uniqueness of the solution is then proved in Section 6. To support our numerical scheme, we provide a numerical experiment in the last section.

2. Weak Formulation

Let us introduce some functional spaces and basic notations which will be used in the sequel of the paper. We denote the traditional inner product of $L^2(\Omega)$ by $(\cdot,\cdot)$. Norm induced by it is denoted as $\|\cdot\|$. Functional space $C([0,\mathcal{T}];Y)$ is a set of abstract functions $q : [0,\mathcal{T}] \to Y$ with the usual norm $\max_{t\in[0,\mathcal{T}]} \|\cdot\|_Y$. For $p > 1$ we define a norm in the space $L^p((0,\mathcal{T});Y)$ as $\left(\int_0^{\mathcal{T}} \|\cdot\|_Y^p\right)^{1/p}$. The dual space of a given functional space Y is denoted as Y^*.

Functional spaces $\mathbf{H}(\mathbf{curl}\,;\Omega) = \{\boldsymbol{\varphi};\boldsymbol{\varphi} \in \mathbf{L}^2(\Omega), \nabla\times\boldsymbol{\varphi} \in \mathbf{L}^2(\Omega)\}$ and $\mathbf{H}(\mathrm{div};\Omega) = \{\boldsymbol{\varphi};\boldsymbol{\varphi} \in \mathbf{L}^2(\Omega), \nabla\cdot\boldsymbol{\varphi} \in L^2(\Omega)\}$ are commonly associated with the solutions of problems derived from classical Maxwell's equations. We will be working in the following spaces (see [24, 25])

$$
\begin{aligned}
\boldsymbol{X} &= \mathbf{H}(\mathbf{curl}\,;\Omega) \cap \mathbf{H}(\mathrm{div};\Omega),\\
\boldsymbol{X}_N &= X \cap \{\boldsymbol{\varphi};\boldsymbol{\varphi}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\} = \mathbf{H}_0(\mathbf{curl}\,;\Omega) \cap \mathbf{H}(\mathrm{div};\Omega).
\end{aligned}
$$

Norm in the space $\boldsymbol{X}_N$ is defined as $\|\boldsymbol{w}\|_{\boldsymbol{X}_N} = \|\boldsymbol{w}\| + \|\nabla\times\boldsymbol{w}\| + \|\nabla\cdot\boldsymbol{w}\|$. According to [24, Theorem 2.12] $\boldsymbol{X}_N \subset H^1(\Omega)$ if $\Omega \in C^{1,1}$. The same embedding is also obtained from [24, Theorem 2.17] if Ω is convex. This embedding is very important since we will be using it extensively throughout the paper.

Space $\boldsymbol{X}_N$ is associated with the solution of (5). To obtain the weak formulation of (5), we multiply it by a test function $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$. Then integrate over Ω, take into account boundary condition (3) and use Green's theorem to obtain

$$
\left(\partial_t^2\boldsymbol{E},\boldsymbol{\varphi}\right) + (\partial_t(\sigma*\boldsymbol{E}),\boldsymbol{\varphi}) + (\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{\varphi}) = h(t)\,(\boldsymbol{f}(\boldsymbol{x}),\boldsymbol{\varphi}) + (\boldsymbol{N}(\boldsymbol{E}),\boldsymbol{\varphi})\,, \tag{9}
$$

for any $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$.

Then, the weak formulation of problem (3), (4), (5) and (6) reads as:
Find a solution pair $\{h(t),\boldsymbol{E}(\boldsymbol{x},t)\}$ satisfying (3),(4),(7) and (9) such that $h(t) \in L^2((0,\mathcal{T})), \boldsymbol{E} \in C([0,\mathcal{T}];\mathbf{L}^2(\Omega)) \cap L^\infty((0,\mathcal{T});\boldsymbol{X}_N)$ with its first order time derivative $\partial_t\boldsymbol{E} \in L^2((0,\mathcal{T});\mathbf{L}^2(\Omega)) \cap C([0,\mathcal{T}];\boldsymbol{X}_N^*)$ and second order time derivative $\partial_t^2\boldsymbol{E} \in L^2((0,\mathcal{T});(\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*)$.

3. Time Discretization

To discretize our continuous formulation (5), (7), we start by splitting the time interval $[0,\mathcal{T}]$ into $n \in \mathbb{N}$ equidistant parts with the time step $\tau = \mathcal{T}/n$. We use the following notation (w is an arbitrary function)

$$
t_i = i\tau, \qquad w_i = w(t_i), \qquad \delta w_i = \frac{w_i - w_{i-1}}{\tau}, \qquad \delta^2 w_i = \frac{\delta w_i - \delta w_{i-1}}{\tau^2}
$$

The discretized convolution for given functions f, g is then defined as

$$
(f*g)_i = \sum_{k=0}^{i} f_{i-k}g_k\tau.
$$

This also implies

$$
\delta(f*g)_i = \frac{(f*g)_i - (f*g)_{i-1}}{\tau} = f_0 g_i + \sum_{k=0}^{i-1}\delta f_{i-k}g_k\tau, \text{ for } i \geqslant 1.
$$

Now, we consider a system with unknown variables $\{\boldsymbol{e}_i, h_i\}$ and approximate our ISP at each time step t_i for $i = 1, \ldots, n$

$$\begin{aligned} \delta^2 \boldsymbol{e}_i + (\sigma * \delta \boldsymbol{e})_i + \nabla \times \nabla \times \boldsymbol{e}_i &= \boldsymbol{N}(\boldsymbol{e}_{i-1}) + h_i \boldsymbol{f} - \sigma_i \boldsymbol{E}_0 && \text{in } \Omega \\ \boldsymbol{e}_i \times \boldsymbol{n} &= 0 && \text{on } \Gamma \\ \boldsymbol{e}_0 &= \boldsymbol{E}_0 \\ \delta \boldsymbol{e}_0 &= \boldsymbol{W}_0 \end{aligned} \tag{DPi}$$

and

$$h_i = \frac{m_i'' + (\sigma * m_i')_i + (\nabla \times \nabla \times \boldsymbol{e}_{i-1}, \nabla \phi) - \int_\Gamma \boldsymbol{N}(\boldsymbol{e}_{i-1}) \cdot \boldsymbol{n} \phi \, \mathrm{d}\Gamma}{\int_\Gamma \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{n} \phi \, \mathrm{d}\Gamma} \tag{DMPi}$$

The scheme above is nonlinear and decoupled. The abbreviations DP*i* and DMP*i* stands for the Discretized Problem at the time step $t = t_i$ and the Discretized Measured Problem at the time step $t = t_i$, respectively. The pseudo-algorithm for obtaining the solution pair $\{\boldsymbol{e}_i, h_i\}$ at each time step t_i reads as

Algorithm 1 Implicit Euler

Require: $m, \sigma, \boldsymbol{f}, \delta \boldsymbol{e}_0 = \boldsymbol{E}_0, \delta \boldsymbol{e} = \boldsymbol{W}_0, n \in \mathbb{N}$
1: **for** $i = 1, i \leqslant n$ **do**
2: $\quad h_i \leftarrow$ Solve: $(DMPi)$
3: $\quad \boldsymbol{e}_i \leftarrow$ Solve: (DPi)
4: $\quad i \leftarrow i + 1$
5: **return** $\{h_1, \boldsymbol{e}_1\}, \ldots, \{h_n, \boldsymbol{e}_n\}$

We now proceed with a Lemma which guarantees the existence of a unique solution pair $\{\boldsymbol{e}_i, h_i\}$ at each time step t_i for $i = 1, \ldots, n$.

Remark 2. *From now on, the inequalities of a type $a \leqslant Cb$, where $C > 0$ is a generic constant will be denoted as $a \lesssim b$. The symbol C will always denote a positive constant.*

Lemma 1. *Let $\Omega \in C^{1,1}$ or Ω be convex. Moreover assume that $\boldsymbol{N}$ is globally Lipschitz continuous, $\phi \in C^\infty(\overline{\Omega})$ with meas$\{supp(\phi) \cap \Gamma\} > 0$, $\boldsymbol{f} \in \boldsymbol{X}_N, \boldsymbol{E}_0 \in \boldsymbol{X}_N, \boldsymbol{W}_0 \in \boldsymbol{X}_N, m \in C^2([0, \mathcal{T}])$ and $\int_\Gamma \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{n} \phi \, d\Gamma \neq 0$ and also $0 < \sigma_* \leqslant \sigma(t) \leqslant \sigma^* < \infty$ for any $t \in [0, \mathcal{T}]$. Then for any $i = 1, \ldots, n$ there exists a unique pair $\{\boldsymbol{e}_i, h_i\}$ solving* (DPi) *and* (DMPi). *Furthermore, $h_i \in \mathbb{R}, \boldsymbol{e}_i \in \boldsymbol{X}_N, \nabla \times \nabla \times \boldsymbol{e}_i \in \mathbf{L}^2(\Omega)$ and $\nabla \times \nabla \times \boldsymbol{e}_i \times \boldsymbol{n} = \boldsymbol{0}$ on Γ.*

Proof. For a given $\nabla \times \nabla \times \boldsymbol{e}_{i-1} \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{e}_{i-1} \in \boldsymbol{X}_N$ we can compute h_i from (DMPi). We also see that

$$\begin{aligned} |h_i|^2 &\lesssim \left(1 + \|\nabla \phi\|^2_{\mathbf{C}(\overline{\Omega})} \|\nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 + \|\phi\|^2_{\mathbf{C}(\overline{\Omega})} \|\boldsymbol{N}(\boldsymbol{e}_{i-1})\|^2_{\mathbf{L}^2(\Gamma)}\right) \\ &\lesssim \left(1 + \|\nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 + \|\boldsymbol{e}_{i-1}\|^2_{\mathbf{L}^2(\Gamma)}\right) \overset{\mathbf{H}^1(\Omega) \subset \mathbf{L}^2(\Gamma)}{\lesssim} \left(1 + \|\nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 + \|\boldsymbol{e}_{i-1}\|^2_{\mathbf{H}^1(\Omega)}\right) \\ &\overset{\boldsymbol{X}_N \subset \mathbf{H}^1(\Omega)}{\lesssim} \left(1 + \|\nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 + \|\boldsymbol{e}_{i-1}\|^2_{\boldsymbol{X}_N}\right) \leqslant C_i. \end{aligned}$$

Now, assume that $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{i-1} \in \boldsymbol{X}_N$ and let us take a look at (DPi)

$$\boldsymbol{e}_i \left(\sigma_0 + \frac{1}{\tau^2}\right) + \nabla \times \nabla \times \boldsymbol{e}_i = \frac{\delta \boldsymbol{e}_{i-1}}{\tau} + \boldsymbol{e}_{i-1} \left(\sigma_0 + \frac{1}{\tau^2}\right) + \boldsymbol{N}(\boldsymbol{e}_{i-1}) + h_i \boldsymbol{f} - \sum_{k=0}^{i-1} \sigma_{i-k} \delta \boldsymbol{e}_k \tau - \sigma_i \boldsymbol{E}_0. \tag{10}$$

Now, let us state the weak formulation of (10):

Find $\boldsymbol{e}_i \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$ with $\boldsymbol{e}_0 = \boldsymbol{E}_0$ and $\delta\boldsymbol{e}_0 = \boldsymbol{W}_0$ such that

$$\left(\sigma_0 + \frac{1}{\tau^2}\right)(\boldsymbol{e}_i, \boldsymbol{\varphi}) + (\nabla \times \boldsymbol{e}_i, \nabla \times \boldsymbol{\varphi}) = \frac{1}{\tau}(\delta\boldsymbol{e}_{i-1}, \boldsymbol{\varphi}) + \left(\sigma_0 + \frac{1}{\tau^2}\right)(\boldsymbol{e}_{i-1}, \boldsymbol{\varphi}) + (\boldsymbol{N}(\boldsymbol{e}_{i-1}), \boldsymbol{\varphi}) + h_i(\boldsymbol{f}, \boldsymbol{\varphi}) - \sum_{k=0}^{i-1} \sigma_{i-k}\tau(\delta\boldsymbol{e}_k, \boldsymbol{\varphi}) - \sigma_i(\boldsymbol{E}_0, \boldsymbol{\varphi})$$

is true for any $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$. Since the right hand side of the weak formulation above is from the space $\mathbf{L}^2(\Omega)$, we can apply the Lax-Milgram lemma to obtain a unique $\boldsymbol{e}_i \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$.

To conclude the proof, we need to obtain some additional estimates for our solution $\boldsymbol{e}_i$. After employing a divergence operator to the equation (10), we also see that

$$\nabla \cdot \boldsymbol{e}_i = \nabla \cdot \boldsymbol{e}_{i-1} + \frac{\tau}{1+\sigma_0\tau^2}\nabla \cdot \delta\boldsymbol{e}_{i-1} + \frac{\tau^2}{1+\sigma_0\tau^2}\left(\nabla \cdot \boldsymbol{N}(\boldsymbol{e}_{i-1}) + h_i \nabla \cdot \boldsymbol{f} - \sum_{k=0}^{i-1} \sigma_{i-1}\nabla \cdot \delta\boldsymbol{e}_k \tau - \sigma_i \nabla \cdot \boldsymbol{E}_0\right).$$

Since $\left|\frac{\tau}{1+\sigma_0\tau^2}\right| \leqslant C$ and $\left|\frac{\tau^2}{1+\sigma_0\tau^2}\right| \leqslant C$, we obtain the following estimate for $\nabla \cdot \boldsymbol{e}_i$

$$\begin{aligned}\|\nabla \cdot \boldsymbol{e}_i\| &\lesssim \|\nabla \cdot \boldsymbol{e}_{i-1}\| + \left(1 + \|\nabla \cdot \delta\boldsymbol{e}_{i-1}\| + \|\boldsymbol{e}_{i-1}\|_{\mathbf{H}^1(\Omega)} + C_i\|\nabla \cdot \boldsymbol{f}\| + \sum_{k=0}^{i-1}\|\nabla \cdot \delta\boldsymbol{e}_k\|\tau + \|\nabla \cdot \boldsymbol{E}_0\|\right) \\ &\overset{\boldsymbol{X}_N \subset \mathbf{H}^1(\Omega)}{\leqslant} C_i(1 + \|\boldsymbol{e}_{i-1}\|_{\boldsymbol{X}_N}) \leqslant C_i.\end{aligned}$$

Therefore, $\boldsymbol{e}_i \in \boldsymbol{X}_N$. Furthermore, we can also see that

$$\begin{aligned}&\nabla \times \nabla \times \boldsymbol{e}_i = \boldsymbol{N}(\boldsymbol{e}_{i-1}) + h_i\boldsymbol{f} - \sigma_i\boldsymbol{E}_0 - \delta^2\boldsymbol{e}_i - (\sigma * \delta\boldsymbol{e})_i \in \mathbf{L}^2(\Omega) \\ &\nabla \times \nabla \times \boldsymbol{e}_i \times \boldsymbol{n} = \boldsymbol{N}(\boldsymbol{e}_{i-1}) \times \boldsymbol{n} + h_i\boldsymbol{f} \times \boldsymbol{n} - \sigma_i\boldsymbol{E}_0 \times \boldsymbol{n} - \delta^2\boldsymbol{e}_i \times \boldsymbol{n} - (\sigma * \delta\boldsymbol{e} \times \boldsymbol{n})_i = \mathbf{0} \text{ on } \Gamma,\end{aligned}$$

which concludes our proof. □

4. A priori energy estimates

In this section, we provide some energy estimates for $\boldsymbol{e}_i$ and h_i. Both discrete and continuous version of Grönwall's lemma will be key in the proofs of the following sections. For that reason, we state them here.

Lemma 2 (Discrete version of Grönwall's lemma, from [26])**.** *Let $\{y_n\}$ and $\{g_n\}$ be non-negative sequences and C a non-negative constant. If*

$$y_n \leqslant C + \sum_{i=0}^{n-1} g_i y_i \quad \textit{for } n \geqslant 0,$$

then

$$y_n \leqslant C \exp\left(\sum_{i=0}^{n-1} g_i\right) \quad \textit{for } n \geqslant 0.$$

Lemma 3 (Continuous version of Grönwall's lemma, from [26])**.** *Let y and g be non-negative integrable functions and C a non-negative constant. If*

$$y(t) \leqslant C + \int_0^t g(s)y(s)\,ds \quad \textit{for } t \geqslant 0,$$

then

$$y(t) \leqslant C \exp\left(\int_0^t g(s)\, ds\right) \quad \textit{for } t \geqslant 0.$$

Lemma 4. *Let $\boldsymbol{N}$ be a global Lipschitz continuous function. Assume that $\boldsymbol{f} \in \mathbf{L}^2(\Omega), \boldsymbol{W}_0 \in \mathbf{L}^2(\Omega), \boldsymbol{E}_0 \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$ and $0 < \sigma_* \leqslant \sigma \leqslant \sigma^* < \infty$. Then we have the following estimate*

$$\max_{1\leqslant j\leqslant n} \|\delta \boldsymbol{e}_j\|^2 + \max_{1\leqslant j\leqslant n} \|\nabla \times \boldsymbol{e}_j\|^2 + \sum_{i=1}^n \|\delta\boldsymbol{e}_i - \delta\boldsymbol{e}_{i-1}\|^2 + \sum_{i=1}^n \|\nabla \times \boldsymbol{e}_i - \nabla \times \boldsymbol{e}_{i-1}\|^2 \lesssim \left(1 + \sum_{i=1}^n h_i^2 \tau\right).$$

Proof. We multiply (scalar multiplication) the first equation in (DPi) by $\delta\boldsymbol{e}_i\tau$, integrate over Ω, use Green's theorem and sum up for $i = 1, \ldots, j$ to get

$$\sum_{i=1}^j \left(\delta^2\boldsymbol{e}_i, \delta\boldsymbol{e}_i\right)\tau + \sum_{i=1}^j \left((\sigma * \delta\boldsymbol{e})_i, \delta\boldsymbol{e}_i\right)\tau + \sum_{i=1}^j \left(\nabla \times \boldsymbol{e}_i, \delta\nabla \times \boldsymbol{e}_i\right)\tau = \sum_{i=1}^j \left(\boldsymbol{N}(\boldsymbol{e}_{i-1}), \delta\boldsymbol{e}_i\right)\tau + \sum_{i=1}^j \left(h_i\boldsymbol{f}, \delta\boldsymbol{e}_i\right)\tau - \sum_{i=1}^j \sigma_i \left(\boldsymbol{E}_0, \delta\boldsymbol{e}_i\right)\tau.$$

The convolution term on the left hand side can be bounded in the following way

$$\left|\sum_{i=1}^j \left((\sigma * \delta\boldsymbol{e})_i, \delta\boldsymbol{e}_i\right)\tau\right| \lesssim 1 + \sum_{i=1}^j \|\delta\boldsymbol{e}_i\|^2\tau.$$

In the next step, we take into account Abel's summation formula

$$\sum_{i=1}^j b_i(b_i - b_{i-1}) = \frac{1}{2}\left\{b_j^2 - b_0^2 + \sum_{i=1}^j (b_i - b_{i-1})^2\right\}.$$

Now, using the above-mentioned formula, we can rewrite the terms on the left hand side as follows

$$\begin{aligned}
\sum_{i=1}^j \left(\delta^2\boldsymbol{e}_i, \delta\boldsymbol{e}_i\right)\tau &= \sum_{i=1}^j \left(\delta\boldsymbol{e}_i - \delta\boldsymbol{e}_{i-1}, \delta\boldsymbol{e}_i\right) = \frac{\|\delta\boldsymbol{e}_j\|^2}{2} - \frac{\|\boldsymbol{W}_0\|^2}{2} + \frac{1}{2}\sum_{i=1}^j \|\delta\boldsymbol{e}_i - \delta\boldsymbol{e}_{i-1}\|^2,\\
\sum_{i=1}^j \left(\nabla \times \boldsymbol{e}_i, \delta\nabla \times \boldsymbol{e}_i\right)\tau &= \frac{\|\nabla \times \boldsymbol{e}_j\|^2}{2} - \frac{\|\nabla \times \boldsymbol{E}_0\|^2}{2} + \frac{1}{2}\sum_{i=1}^j \|\nabla \times \boldsymbol{e}_i - \nabla \times \boldsymbol{e}_{i-1}\|^2.
\end{aligned}$$

The first term on the right hand side is handled via the Lipschitz continuity of $\boldsymbol{N}(\cdot)$, Young's and Cauchy's inequalities and the identity $\boldsymbol{e}_i = \boldsymbol{E}_0 + \sum_{k=1}^i \delta\boldsymbol{e}_k\tau$

$$\begin{aligned}
\sum_{i=1}^j \left(\boldsymbol{N}(\boldsymbol{e}_{i-1}), \delta\boldsymbol{e}_i\right)\tau &\lesssim \sum_{i=1}^j \left(1 + \|\boldsymbol{e}_{i-1}\|\right)\|\delta\boldsymbol{e}_i\|\tau = \sum_{i=1}^j \left[1 + \left\|\boldsymbol{E}_0 + \sum_{k=1}^{i-1}\delta\boldsymbol{e}_k\tau\right\|\right]\|\delta\boldsymbol{e}_i\|\tau\\
&\lesssim \sum_{i=1}^j \left[1 + \|\boldsymbol{E}_0\|\right]\|\delta\boldsymbol{e}_i\|\tau + \sum_{i=1}^j\sum_{k=1}^i \|\delta\boldsymbol{e}_k\|\,\|\delta\boldsymbol{e}_i\|\tau^2\\
&\lesssim 1 + \sum_{i=1}^j \|\delta\boldsymbol{e}_i\|^2\tau.
\end{aligned}$$

The rest of the right hand side is estimated using Young's and Cauchy's inequalities once again

$$\left|\sum_{i=1}^{j}(h_i\boldsymbol{f},\delta\boldsymbol{e}_i)\,\tau-\sum_{i=1}^{j}\sigma_i\,(\boldsymbol{E}_0,\delta\boldsymbol{e}_i)\,\tau\right|\lesssim\sum_{i=1}^{j}h_i^2\,\|\boldsymbol{f}\|^2\,\tau+\|\boldsymbol{E}_0\|^2+\sum_{i=1}^{j}\|\delta\boldsymbol{e}_i\|^2\,\tau$$
$$\lesssim\left(1+\sum_{i=1}^{j}h_i^2\tau\right)+\sum_{i=1}^{j}\|\delta\boldsymbol{e}_i\|^2\,\tau.$$

Collecting all partial results above, we obtain

$$\|\delta\boldsymbol{e}_j\|^2+\|\nabla\times\boldsymbol{e}_j\|^2+\sum_{i=1}^{j}\|\delta\boldsymbol{e}_i-\delta\boldsymbol{e}_{i-1}\|^2+\sum_{i=1}^{j}\|\nabla\times\boldsymbol{e}_i-\nabla\times\boldsymbol{e}_{i-1}\|^2\lesssim\left(1+\sum_{i=1}^{j}h_i^2\tau\right)+\sum_{i=1}^{j}\|\delta\boldsymbol{e}_i\|^2\,\tau.$$

The rest of the proof follows from the application of the discrete version of Grönwall's Lemma 2. □

Remark 3. *The identity $\boldsymbol{e}_j=\boldsymbol{E}_0+\sum_{i=1}^{j}\delta\boldsymbol{e}_i\tau$ and Lemma 4 above also imply*

$$\max_{1\leqslant j\leqslant n}\|\boldsymbol{e}_j\|^2\lesssim\left(1+\sum_{i=1}^{n}h_i^2\tau\right).$$

Lemma 5. *Assume that $\boldsymbol{N}$ is a global Lipschitz continuous function. Moreover, we suppose that $\boldsymbol{f}\in\mathbf{H}(\mathrm{div};\Omega)$, $\boldsymbol{W}_0\in\mathbf{H}(\mathrm{div};\Omega)$, $\boldsymbol{E}_0\in\boldsymbol{X}_N$ and $0<\sigma_*\leqslant\sigma\leqslant\sigma^*<\infty$. Then we have the following estimate*

$$\max_{1\leqslant j\leqslant n}\|\nabla\cdot\delta\boldsymbol{e}_j\|^2+\sum_{i=1}^{n}\|\nabla\cdot\delta\boldsymbol{e}_i-\nabla\cdot\delta\boldsymbol{e}_{i-1}\|^2\lesssim\left(1+\sum_{i=1}^{n}h_i^2\tau\right).$$

Proof. First, we apply the divergence operator to the first equation in (DPi), then multiply by $\nabla\cdot\delta\boldsymbol{e}_i\tau$, integrate over Ω and sum up for $i=1,\ldots,j$. We obtain the following

$$\sum_{i=1}^{j}\left(\nabla\cdot\delta^2\boldsymbol{e}_i,\nabla\cdot\delta\boldsymbol{e}_i\right)\tau=\sum_{i=1}^{j}(\nabla\cdot\boldsymbol{N}(\boldsymbol{e}_{i-1}),\nabla\cdot\delta\boldsymbol{e}_i)\,\tau+\sum_{i=1}^{j}h_i\,(\nabla\cdot\boldsymbol{f},\nabla\cdot\delta\boldsymbol{e}_i)\,\tau-\sum_{i=1}^{j}(\sigma_i\nabla\cdot\boldsymbol{E}_0,\nabla\cdot\delta\boldsymbol{e}_i)\,\tau$$
$$-\sum_{i=1}^{j}((\sigma*\nabla\cdot\delta\boldsymbol{e})_i,\nabla\cdot\delta\boldsymbol{e}_i)\,\tau.$$

The left hand side can be rewritten using Abel's summation formula

$$\sum_{i=1}^{j}\left(\nabla\cdot\delta^2\boldsymbol{e}_i,\nabla\cdot\delta\boldsymbol{e}_i\right)\tau=\frac{\|\nabla\cdot\delta\boldsymbol{e}_j\|^2}{2}-\frac{\|\nabla\cdot\boldsymbol{W}_0\|^2}{2}+\frac{1}{2}\sum_{i=1}^{j}\|\nabla\cdot\delta\boldsymbol{e}_i-\nabla\cdot\delta\boldsymbol{e}_{i-1}\|^2.$$

First term on the right hand side is estimated as

$$\begin{aligned}
\sum_{i=1}^{j} (\nabla\cdot \boldsymbol{N}(\boldsymbol{e}_{i-1}), \nabla\cdot\delta\boldsymbol{e}_i)\,\tau &\leqslant \sum_{i=1}^{j} \|\nabla\cdot \boldsymbol{N}(\boldsymbol{e}_{i-1})\|\,\|\nabla\cdot\delta\boldsymbol{e}_i\|\,\tau \lesssim \sum_{i=1}^{j} \|\boldsymbol{e}_{i-1}\|_{\mathbf{H}^1(\Omega)}\,\|\nabla\cdot\delta\boldsymbol{e}_i\|\,\tau\\
&\lesssim \sum_{i=1}^{j} \|\boldsymbol{e}_{i-1}\|_{\boldsymbol{X}_N}\,\|\nabla\cdot\delta\boldsymbol{e}_i\|\,\tau = \sum_{i=1}^{j} (\|\boldsymbol{e}_{i-1}\| + \|\nabla\times\boldsymbol{e}_{i-1}\| + \|\nabla\cdot\boldsymbol{e}_{i-1}\|)\,\|\nabla\cdot\delta\boldsymbol{e}_i\|\,\tau\\
&\overset{\text{Lemma 4,Remark 3}}{\lesssim} \left(1+\sum_{i=1}^{j} h_i^2\tau\right) + \sum_{i=1}^{j}\|\nabla\cdot\delta\boldsymbol{e}_i\|^2\,\tau + \sum_{i=1}^{j}\|\nabla\cdot\boldsymbol{e}_i\|\,\|\nabla\cdot\delta\boldsymbol{e}_i\|\,\tau\\
&\lesssim \left(1+\sum_{i=1}^{j} h_i^2\tau + \sum_{i=1}^{j}\|\nabla\cdot\delta\boldsymbol{e}_i\|^2\,\tau\right) + \sum_{i=1}^{j}\left\|\nabla\cdot\boldsymbol{E}_0 + \sum_{k=1}^{i}\nabla\cdot\delta\boldsymbol{e}_k\tau\right\|\,\|\nabla\cdot\delta\boldsymbol{e}_i\|\,\tau\\
&\lesssim \left(1+\sum_{i=1}^{j} h_i^2\tau + \sum_{i=1}^{j}\|\nabla\cdot\delta\boldsymbol{e}_i\|^2\,\tau\right).
\end{aligned}$$

The other terms on the right hand side are estimated using Cauchy's and Young's inequalities. Therefore, gathering all partial results, we arrive at

$$\|\nabla\cdot\delta\boldsymbol{e}_j\|^2 + \sum_{i=1}^{j}\|\nabla\cdot\delta\boldsymbol{e}_i - \nabla\cdot\delta\boldsymbol{e}_{i-1}\|^2 \lesssim \left(1+\sum_{i=1}^{j} h_i^2\tau + \sum_{i=1}^{j}\|\nabla\cdot\delta\boldsymbol{e}_i\|^2\,\tau\right).$$

An application of discrete version of the Grönwall Lemma 2 and taking maximum over $1 \leqslant j \leqslant n$ yields to the desired result. □

Remark 4. *Using the identity $\nabla\cdot\boldsymbol{e}_i = \nabla\cdot\boldsymbol{E}_0 + \sum_{j=1}^{i}\nabla\cdot\delta\boldsymbol{e}_j\tau$ and Lemma 5 above, we obtain an estimate for $\nabla\cdot\boldsymbol{e}_i$*

$$\max_{1\leqslant j\leqslant n}\|\nabla\cdot\boldsymbol{e}_j\|^2 \lesssim \left(1+\sum_{i=1}^{n} h_i^2\tau\right).$$

Lemma 6. *Let $\boldsymbol{N}$ be a global Lipschitz continuous function and also assume $\boldsymbol{f}\in\mathbf{H}_0(\mathbf{curl};\Omega)$, $\boldsymbol{W}_0\in\mathbf{H}_0(\mathbf{curl};\Omega)$, $\boldsymbol{E}_0\in\boldsymbol{X}_N$, $\nabla\times\nabla\times\boldsymbol{E}_0\in\mathbf{L}^2(\Omega)$ and $0<\sigma_*\leqslant\sigma\leqslant\sigma^*<\infty$. Then we have the following estimate*

$$\begin{aligned}
\max_{1\leqslant j\leqslant n}\|\nabla\times\delta\boldsymbol{e}_j\|^2 + \max_{1\leqslant j\leqslant n}\|\nabla\times\nabla\times\boldsymbol{e}_j\|^2 + \sum_{i=1}^{n}\|\nabla\times\delta\boldsymbol{e}_i - \nabla\times\delta\boldsymbol{e}_{i-1}\|^2&\\
+\sum_{i=1}^{n}\|\nabla\times\nabla\times\boldsymbol{e}_i - \nabla\times\nabla\times\boldsymbol{e}_{i-1}\|^2 &\lesssim \left(1+\sum_{i=1}^{n} h_i^2\tau\right).
\end{aligned}$$

Proof. We start by applying the curl operator to the first equation in (DPi), then we multiply it with $\nabla\times\delta\boldsymbol{e}_i\tau$, integrate over Ω, use Green's theorem (Lemma 1 guarantees $\nabla\times\nabla\times\boldsymbol{e}_i\times\boldsymbol{n}=\mathbf{0}$ on Γ) and sum up for $i=1,\ldots,j$

$$\begin{aligned}
\sum_{i=1}^{j}\left(\nabla\times\delta^2\boldsymbol{e}_i, \nabla\times\delta\boldsymbol{e}_i\right)\tau + \sum_{i=1}^{j}(\nabla\times\nabla\times\boldsymbol{e}_i, \nabla\times\nabla\times\delta\boldsymbol{e}_i)\,\tau + \sum_{i=1}^{j}((\sigma*\nabla\times\delta\boldsymbol{e})_i, \nabla\times\delta\boldsymbol{e}_i)\,\tau&\\
= \sum_{i=1}^{j}(\nabla\times\boldsymbol{N}(\boldsymbol{e}_{i-1}), \nabla\times\delta\boldsymbol{e}_i)\,\tau + \sum_{i=1}^{j} h_i\,(\nabla\times\boldsymbol{f}, \nabla\times\delta\boldsymbol{e}_i)\,\tau - \sum_{i=1}^{j}(\sigma_i\nabla\times\boldsymbol{E}_0, \nabla\times\delta\boldsymbol{e}_i)\,\tau.&
\end{aligned}$$

To bound the first two terms on the left hand side, we employ Abel's summation rule once again, i.e.

$$\sum_{i=1}^{j}\left(\nabla\times\delta^2\boldsymbol{e}_i,\nabla\times\delta\boldsymbol{e}_i\right)\tau=\frac{\|\nabla\times\delta\boldsymbol{e}_j\|^2}{2}-\frac{\|\nabla\times\boldsymbol{W}_0\|^2}{2}+\frac{1}{2}\sum_{i=1}^{j}\|\nabla\times\delta\boldsymbol{e}_i-\nabla\times\delta\boldsymbol{e}_{i-1}\|^2,$$

$$\sum_{i=1}^{j}(\nabla\times\nabla\times\boldsymbol{e}_i,\nabla\times\nabla\times\delta\boldsymbol{e}_i)\,\tau=\frac{\|\nabla\times\nabla\times\boldsymbol{e}_j\|^2}{2}-\frac{\|\nabla\times\nabla\times\boldsymbol{E}_0\|^2}{2}+\frac{1}{2}\sum_{i=1}^{j}\|\nabla\times\nabla\times\boldsymbol{e}_i-\nabla\times\nabla\times\boldsymbol{e}_{i-1}\|^2.$$

Last term on the left hand side can be bounded as follows

$$\left|\sum_{i=1}^{j}((\sigma*\nabla\times\delta\boldsymbol{e})_i,\nabla\times\delta\boldsymbol{e}_i)\,\tau\right|\lesssim\left(1+\sum_{i=1}^{j}\|\nabla\times\delta\boldsymbol{e}_i\|^2\,\tau\right).$$

We continue with estimates for the right hand side. Starting with the first term, we obtain

$$\begin{aligned}\sum_{i=1}^{j}(\nabla\times\boldsymbol{N}(\boldsymbol{e}_{i-1}),\nabla\times\delta\boldsymbol{e}_i)\,\tau&\lesssim\left(\|\boldsymbol{E}_0\|^2_{\mathbf{H}^1(\Omega)}+\sum_{i=1}^{j}\|\boldsymbol{e}_i\|^2_{\mathbf{H}^1(\Omega)}\,\tau\right)+\sum_{i=1}^{j}\|\nabla\times\delta\boldsymbol{e}_i\|^2\,\tau\\&\lesssim\left(\|\boldsymbol{E}_0\|^2_{\boldsymbol{X}_N}+\sum_{i=1}^{j}\|\boldsymbol{e}_i\|^2_{\boldsymbol{X}_N}\,\tau\right)+\sum_{i=1}^{j}\|\nabla\times\delta\boldsymbol{e}_i\|^2\,\tau\\&\overset{Lemma\,4,5}{\lesssim}\left(1+\sum_{i=1}^{j}h_i^2\tau\right)+\sum_{i=1}^{j}\|\nabla\times\delta\boldsymbol{e}_i\|^2\,\tau.\end{aligned}$$

The rest of the right hand side terms can be bounded via Cauchy's and Young's inequalities

$$\sum_{i=1}^{j}h_i\,(\nabla\times\boldsymbol{f},\nabla\times\delta\boldsymbol{e}_i)\,\tau\lesssim\sum_{i=1}^{j}h_i^2\,\|\nabla\times\boldsymbol{f}\|^2\,\tau+\sum_{i=1}^{j}\|\nabla\times\delta\boldsymbol{e}_i\|^2\,\tau\lesssim\sum_{i=1}^{j}h_i^2\tau+\sum_{i=1}^{j}\|\nabla\times\delta\boldsymbol{e}_i\|^2\,\tau,$$

$$\left|\sum_{i=1}^{j}(\sigma_i\nabla\times\boldsymbol{E}_0,\nabla\times\delta\boldsymbol{e}_i)\,\tau\right|\lesssim\|\nabla\times\boldsymbol{E}_0\|^2+\sum_{i=1}^{j}\|\nabla\times\delta\boldsymbol{e}_i\|^2\,\tau\lesssim\left(1+\sum_{i=1}^{j}\|\nabla\times\delta\boldsymbol{e}_i\|^2\,\tau\right).$$

Now, we can congregate all partial results above to see that

$$\begin{aligned}\|\nabla\times\delta\boldsymbol{e}_j\|^2+\sum_{i=1}^{j}\|\nabla\times\delta\boldsymbol{e}_i-\nabla\times\delta\boldsymbol{e}_{i-1}\|^2+\|\nabla\times\nabla\times\boldsymbol{e}_j\|^2+\sum_{i=1}^{j}\|\nabla\times\nabla\times\boldsymbol{e}_i-\nabla\times\nabla\times\boldsymbol{e}_{i-1}\|^2\\\lesssim\left(1+\sum_{i=1}^{j}h_i^2\tau\right)+\sum_{i=1}^{j}\|\nabla\times\delta\boldsymbol{e}_i\|^2\,\tau.\end{aligned}$$

Using Grönwall's Lemma 2 and taking maximum over $1\leqslant j\leqslant n$, we conclude the proof. □

Lemma 7. *Let $\boldsymbol{N}$ be a global Lipschitz continuous function and suppose $\boldsymbol{f}\in\boldsymbol{X}_N,\boldsymbol{W}_0\in\boldsymbol{X}_N,\boldsymbol{E}_0\in\boldsymbol{X}_N,\nabla\times\nabla\times\boldsymbol{E}_0\in\mathbf{L}^2(\Omega),m\in C^2([0,\mathcal{T}]),\int_\Gamma\boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{n}\phi\,d\Gamma\neq0$ and $0<\sigma_*\leqslant\sigma\leqslant\sigma^*<\infty$. Then we have the following estimates*

(i) $\max_{1\leqslant j\leqslant n}\|\boldsymbol{e}_j\|^2_{\boldsymbol{X}_N}+\max_{1\leqslant j\leqslant n}\|\nabla\times\nabla\times\boldsymbol{e}_j\|^2\lesssim1$

(ii) $\max_{1\leqslant j\leqslant n}|h_j|^2\lesssim1$

(iii) $\max_{1\leqslant j\leqslant n}\left\|\delta^2\boldsymbol{e}_j\right\|_{(\mathbf{H}_0(\mathbf{curl};\Omega))^*}\lesssim1.$

Proof. (i) According to the proof of Lemma 1 and (DMPi), we have

$$h_i^2 \lesssim (1 + \|\nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 + \|\boldsymbol{e}_{i-1}\|^2_{\boldsymbol{X}_N}) \implies \sum_{i=1}^{j} h_i^2 \tau \lesssim \left(1 + \sum_{i=1}^{j} \|\nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 \tau + \sum_{i=1}^{j} \|\boldsymbol{e}_{i-1}\|^2_{\boldsymbol{X}_N} \tau \right).$$

Lemmas 4,5 and 6 together with the bound above yield

$$\|\boldsymbol{e}_j\|^2_{\boldsymbol{X}_N} + \|\nabla \times \nabla \times \boldsymbol{e}_j\|^2 \lesssim \left(1 + \sum_{i=1}^{j} h_i^2 \tau \right) \lesssim \left(1 + \sum_{i=1}^{j} \|\boldsymbol{e}_{i-1}\|^2_{\boldsymbol{X}_N} \tau + \sum_{i=1}^{j} \|\nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 \tau \right).$$

Thus, employing Grönwall's Lemma 2 once again and taking maximum over $1 \leqslant j \leqslant n$ the first statement of Lemma 7 is proven.

(ii) The second statement is directly implied by (i).

(iii) We take $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$ and make a scalar multiplication with (DPi). Then we integrate in Ω and use Green's theorem to observe

$$\left(\delta^2 \boldsymbol{e}_i, \boldsymbol{\varphi}\right) = (\boldsymbol{N}(\boldsymbol{e}_{i-1}), \boldsymbol{\varphi}) + h_i (\boldsymbol{f}, \boldsymbol{\varphi}) - (\sigma_i \boldsymbol{E}_0, \boldsymbol{\varphi}) - ((\sigma * \delta \boldsymbol{e})_i, \boldsymbol{\varphi}) - (\nabla \times \boldsymbol{e}_i, \nabla \times \boldsymbol{\varphi}).$$

Using statements (i), (ii) and Lemma 4, we conclude

$$\left|\left(\delta^2 \boldsymbol{e}_i, \boldsymbol{\varphi}\right)\right| \lesssim \|\boldsymbol{\varphi}\| + \|\nabla \times \boldsymbol{\varphi}\| \lesssim \|\boldsymbol{\varphi}\|_{\mathbf{H}_0(\mathbf{curl}\,;\Omega)}.$$

Therefore,

$$\left\|\delta^2 \boldsymbol{e}_j\right\|_{(\mathbf{H}_0(\mathbf{curl}\,;\Omega))*} = \sup_{\boldsymbol{\varphi} \neq \mathbf{0},\, \boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)} \frac{\left|\left(\delta^2 \boldsymbol{e}_j, \boldsymbol{\varphi}\right)\right|}{\|\boldsymbol{\varphi}\|_{\mathbf{H}_0(\mathbf{curl}\,;\Omega)}} \lesssim 1.$$

□

5. Existence of a solution

We construct piece-wise constant and piece-wise linear in time functions and show the convergence of subsequences of these functions to a weak solution $\{\boldsymbol{E}, h\}$ which satisfies (9) and (7). These functions are also called Rothe's functions and are created in the following manner

$$\begin{array}{ll}
\overline{\boldsymbol{E}}_n(t) = \boldsymbol{e}_i & t \in (t_{i-1}, t], \\
\boldsymbol{E}_n(t) = \boldsymbol{e}_{i-1} + (t - t_{i-1}) \delta \boldsymbol{e}_i & t \in (t_{i-1}, t], \\
\overline{\boldsymbol{E}}_n(0) = \boldsymbol{E}_n(0) = \boldsymbol{E}_0, & \\
\overline{\boldsymbol{W}}_n(t) = \delta \boldsymbol{e}_i & t \in (t_{i-1}, t], \\
\boldsymbol{W}_n(t) = \delta \boldsymbol{e}_{i-1} + (t - t_{i-1}) \delta \boldsymbol{e}_i & t \in (t_{i-1}, t], \\
\overline{\boldsymbol{W}}_n(0) = \boldsymbol{W}_n(0) = \boldsymbol{W}_0, & \\
\bar{h}_n(t) = h_i & t \in (t_{i-1}, t], \\
\overline{m_n}(t) = m_i,\ \overline{m'_n}(t) = m'_i,\ \overline{m''_n}(t) = m''_i & t \in (t_{i-1}, t], \\
\overline{\sigma_n}(t) = \sigma_i & t \in (t_{i-1}, t].
\end{array}$$

Now, we can rewrite (DPi) and (DMPi) in a continuous form (for $t \in (t_{i-1}, t_i]$)

$$\begin{array}{rll}
\partial_t \boldsymbol{W}_n(t) + (\overline{\sigma_n} * \overline{\boldsymbol{W}}_n)(t_i) + \nabla \times \nabla \times \overline{\boldsymbol{E}}_n(t) & = \boldsymbol{N}(\overline{\boldsymbol{E}}_n(t - \tau)) + \bar{h}_n(t) \boldsymbol{f} - \bar{\sigma}(t) \boldsymbol{E}_0 & \text{in } \Omega \\
\overline{\boldsymbol{E}}_n(t) \times \boldsymbol{n} & = 0 & \text{on } \Gamma \\
\boldsymbol{E}_n(0) & = \boldsymbol{E}_0 & \\
\boldsymbol{W}_n(0) & = \boldsymbol{W}_0, &
\end{array} \qquad \text{(DP)}$$

$$\bar{h}_n(t) = \frac{\overline{m''_n}(t) + (\overline{\sigma_n} * \overline{m'_n})(t_i) + \big(\nabla\times\nabla\times\overline{\boldsymbol{E}_n}(t-\tau), \nabla\phi\big) - \int_\Gamma \boldsymbol{N}(\overline{\boldsymbol{E}_n}(t-\tau))\cdot\boldsymbol{n}\phi\,\mathrm{d}\Gamma}{\int_\Gamma \boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{n}\phi\,\mathrm{d}\Gamma}. \tag{DMP}$$

Then the variational formulation of (DP) has the following structure for any $t \in (t_{i-1}, t_i]$ and $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$

$$(\partial_t \boldsymbol{W}_n(t), \boldsymbol{\varphi}) + \big((\overline{\sigma_n} * \overline{\boldsymbol{W}}_n)(t_i) + \overline{\sigma_n}(t)\boldsymbol{E}_0, \boldsymbol{\varphi}\big) + \big(\nabla\times\overline{\boldsymbol{E}_n}(t), \nabla\times\boldsymbol{\varphi}\big) = \big(\boldsymbol{N}(\overline{\boldsymbol{E}_n}(t-\tau)), \boldsymbol{\varphi}\big) + \bar{h}_n(t)\,(\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{\varphi})\,. \tag{11}$$

The abbreviations DP and DMP stands for the Discretized Problem and the Discretized Measured Problem, respectively.

Theorem 1. *Let $\Omega \in C^{1,1}$ or Ω be convex. Assume that $\boldsymbol{N}$ and σ are global Lipschitz continuous functions and $\boldsymbol{f} \in \boldsymbol{X}_N, \boldsymbol{E}_0 \in \boldsymbol{X}_N, \boldsymbol{W}_0 \in \boldsymbol{X}_N, \nabla\times\nabla\times\boldsymbol{E}_0 \in \mathbf{L}^2(\Omega), m \in C^2([0,\mathcal{T}]), \int_\Gamma \boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{n}\phi\, d\Gamma \neq 0, \phi \in C^\infty(\bar{\Omega})$ with $meas(supp(\phi) \cap \Gamma) > 0$ and $0 < \sigma_* \leqslant \sigma \leqslant \sigma^* < \infty$.*

Then there exists a weak solution $\{\boldsymbol{E}, h\}$ which satisfies (9) *and* (7)*. Furthermore, we have $h \in L^2((0,\mathcal{T}))$, $\boldsymbol{E} \in C([0,\mathcal{T}];\mathbf{L}^2(\Omega)) \cap L^\infty((0,\mathcal{T});\boldsymbol{X}_N)$ with $\partial_t\boldsymbol{E} \in L^2((0,\mathcal{T});\mathbf{L}^2(\Omega)) \cap C([0,\mathcal{T}];\boldsymbol{X}_N^*)$, $\partial_t^2\boldsymbol{E} \in L^2((0,\mathcal{T});(\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*)$ and $\nabla\times\nabla\times\boldsymbol{E} \in L^\infty((0,\mathcal{T});\mathbf{L}^2(\Omega))$.*

Proof. The Lipschitz continuity of σ implies

$$\overline{\sigma_n} \to \sigma \ \text{ in } \ L^2([0,\mathcal{T}]). \tag{12}$$

Lemma 7 says that $\int_0^{\mathcal{T}} \left|\bar{h}_n(s)\right|^2 \,\mathrm{d}s \lesssim 1$. From the reflexivity of the space $L^2((0,\mathcal{T}))$, we have a subsequence of $\bar{h}_n$ (still denoted as $\bar{h}_n$ for the sake of clarity), which converges weakly to h in this space, i.e.

$$\bar{h}_n(t) \rightharpoonup h(t) \quad \text{in} \quad L^2((0,\mathcal{T})). \tag{13}$$

Remark 5. *In the sequel, we will frequently use the convergence of subsequences. To enhance the readability and clarity of our paper, we will not distinguish between the original sequence and a subsequence and continue to denote the subsequence with the same denotation as the original sequence.*

From [24, Theorem 2.8], we have a compact embedding for any Lipschitz domain Ω.

$$\boldsymbol{X}_N \Subset \mathbf{L}^2(\Omega).$$

Deducing from Lemmas 4,5,6, and 7, we obtain

$$\int_0^{\mathcal{T}} \|\partial_t\boldsymbol{E}_n(t)\|^2 \,\mathrm{d}t \lesssim 1, \qquad \|\boldsymbol{E}_n(t)\|_{\boldsymbol{X}_N} \lesssim 1 \quad \forall t \in [0,\mathcal{T}].$$

Applying [23, Lemma 1.3.13], we obtain an existence of a vector field $\boldsymbol{E} \in C([0,\mathcal{T}];\mathbf{L}^2(\Omega)) \cap L^\infty((0,\mathcal{T});\boldsymbol{X}_N)$ with $\partial_t\boldsymbol{E} \in L^2((0,\mathcal{T});\mathbf{L}^2(\Omega))$ and a subsequence of $\boldsymbol{E}_n$ for which the following convergence results hold

$$\begin{array}{lll} \boldsymbol{E}_n \to \boldsymbol{E} & \text{in} & C([0,\mathcal{T}];\mathbf{L}^2(\Omega)), \\ \overline{\boldsymbol{E}_n} \to \boldsymbol{E} & \text{in} & L^2((0,\mathcal{T};\mathbf{L}^2(\Omega))), \\ \boldsymbol{E}_n(t) \rightharpoonup \boldsymbol{E}(t) & \text{in} & \boldsymbol{X}_N, \quad \forall t \in [0,\mathcal{T}], \\ \overline{\boldsymbol{E}_n}(t) \rightharpoonup \boldsymbol{E}(t) & \text{in} & \boldsymbol{X}_N, \quad \forall t \in [0,\mathcal{T}], \\ \overline{\boldsymbol{W}}_n = \partial_t\boldsymbol{E}_n \rightharpoonup \partial_t\boldsymbol{E} & \text{in} & L^2((0,\mathcal{T});\mathbf{L}^2(\Omega))). \end{array} \tag{14}$$

Lemma 7 together with $\boldsymbol{X}_N \subset \mathbf{H}_0(\mathbf{curl}\,;\Omega) \subset (\mathbf{H}_0(\mathbf{curl}\,;\Omega))^* \subset \boldsymbol{X}_N^*$ implies

$$\|\partial_t\boldsymbol{W}_n\|_{\boldsymbol{X}_N^*} \lesssim \|\partial_t\boldsymbol{W}_n\|_{(\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*} \lesssim 1.$$

Now, thanks to the embedding $\boldsymbol{X}_N \Subset \mathbf{L}^2(\Omega)$, which implies $\mathbf{L}^2(\Omega) \Subset \boldsymbol{X}_N^*$ and Lemma 4, we also have for $t \in (t_{i-1}, t_i]$

$$\|\boldsymbol{W}_n\|_{\boldsymbol{X}_N^*} \lesssim \|\boldsymbol{W}_n\| = \left\|\delta \boldsymbol{e}_{i-1} + (t - t_{i-1})\delta^2 \boldsymbol{e}_i\right\| \lesssim (\|\delta \boldsymbol{e}_i\| + \|\delta \boldsymbol{e}_{i-1}\|) \lesssim 1.$$

Hence, the sequence $\boldsymbol{W}_n$ is equi-bounded in $C([0,\mathcal{T}]; \boldsymbol{X}_N^*)$. Moreover, for any $t \neq s, t, s \in [0, \mathcal{T}]$ and any $\boldsymbol{\varphi} \in \boldsymbol{X}_N$, we have

$$|(\boldsymbol{W}_n(t) - \boldsymbol{W}_n(s), \boldsymbol{\varphi})| = \left|\int_s^t (\partial_t \boldsymbol{W}_n(z), \boldsymbol{\varphi})\ \mathrm{d}z\right| \leqslant |t-s|\, \|\partial_t \boldsymbol{W}_n\|_{\boldsymbol{X}_N^*} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N} \lesssim |t-s|\, \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N}.$$

Thus the sequence $\boldsymbol{W}_n$ is also equi-continuous in $C([0,\mathcal{T}]; \boldsymbol{X}_N^*)$ and so applying a modification of Arzelà-Ascoli theorem (see [23, Lemma 1.3.10]), we conclude that the sequence is compact in there, i.e.

$$\boldsymbol{W}_n \to \boldsymbol{W} \ \text{ in } \ C([0,\mathcal{T}]; \boldsymbol{X}_N^*). \tag{15}$$

Now, for any $t \in (t_{i-1}, t_i]$ and any $\boldsymbol{\varphi} \in \boldsymbol{X}_N$, we have

$$\left|\left(\boldsymbol{W}_n - \overline{\boldsymbol{W}}_n, \boldsymbol{\varphi}\right)\right| = \left|\int_t^{t_i} (\partial_t \boldsymbol{W}_n(s), \boldsymbol{\varphi})\ \mathrm{d}s\right| \leqslant \tau\, \|\partial_t \boldsymbol{W}_n\|_{\boldsymbol{X}_N^*} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N} \lesssim \tau\, \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N} \stackrel{n\to\infty}{\longrightarrow} 0.$$

Therefore,

$$\overline{\boldsymbol{W}}_n \to \boldsymbol{W} \ \text{ in } \ C([0,\mathcal{T}]; \boldsymbol{X}_N^*). \tag{16}$$

Using this and (14), we conclude the following for any $\boldsymbol{\varphi} \in \boldsymbol{X}_N$

$$\int_0^{\mathcal{T}} (\partial_t \boldsymbol{E}, \boldsymbol{\varphi})\ \mathrm{d}t = \lim_{n\to\infty} \int_0^{\mathcal{T}} \left(\overline{\boldsymbol{W}}_n(t), \boldsymbol{\varphi}\right)\ \mathrm{d}t = \int_0^{\mathcal{T}} (\boldsymbol{W}, \boldsymbol{\varphi})\ \mathrm{d}t \implies \partial_t \boldsymbol{E} = \boldsymbol{W}.$$

Lemma 7 implies $\partial_t \boldsymbol{W}_n \in L^2((0,\mathcal{T}); (\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*)$. Since this space is reflexive, there exists a $\boldsymbol{z}$ from the space $L^2((0,\mathcal{T}); (\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*)$ such that $\partial_t \boldsymbol{W}_n \rightharpoonup \boldsymbol{z}$ in this space. Using the previous results for the sequence $\boldsymbol{W}_n$, we conclude for any $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$

$$\begin{aligned}\int_0^t \left(\partial_t^2 \boldsymbol{E}(s), \boldsymbol{\varphi}\right)\ \mathrm{d}s = (\boldsymbol{W}(t) - \boldsymbol{W}(0), \boldsymbol{\varphi}) &= \lim_{n\to\infty} (\boldsymbol{W}_n(t) - \boldsymbol{W}_n(0), \boldsymbol{\varphi}) \\ &= \lim_{n\to\infty} \int_0^t (\partial_t \boldsymbol{W}_n(s), \boldsymbol{\varphi})\ \mathrm{d}s = \int_0^t (\boldsymbol{z}(s), \boldsymbol{\varphi})\ \mathrm{d}s.\end{aligned}$$

Therefore, $\partial_t^2 \boldsymbol{E} = \boldsymbol{z}$, i.e. $\partial_t \boldsymbol{W}_n \rightharpoonup \partial_t^2 \boldsymbol{E}$ in $L^2((0,\mathcal{T}); (\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*)$. For the convolution term, we have the following estimate for any $\boldsymbol{\varphi} \in \boldsymbol{X}_N$ and $t \in (t_{i-1}, t_i]$

$$\begin{aligned}&\left|(\overline{\sigma_n} * (\overline{\boldsymbol{W}}_n, \boldsymbol{\varphi}))(t_i) - (\overline{\sigma_n} * (\overline{\boldsymbol{W}}_n, \boldsymbol{\varphi}))(t)\right| \\ &= \left|\int_0^{t_i} \left(\overline{\boldsymbol{W}}_n(t_i - s), \boldsymbol{\varphi}\right) \overline{\sigma_n}(s)\ \mathrm{d}s - \int_0^t \left(\overline{\boldsymbol{W}}_n(t - s), \boldsymbol{\varphi}\right) \overline{\sigma_n}(s)\ \mathrm{d}s\right| \\ &= \left|\int_0^t \left(\overline{\boldsymbol{W}}_n(t_i - s) - \overline{\boldsymbol{W}}_n(t - s), \boldsymbol{\varphi}\right) \overline{\sigma_n}(s)\ \mathrm{d}s - \int_t^{t_i} \left(\overline{\boldsymbol{W}}_n(t_i - s), \boldsymbol{\varphi}\right) \overline{\sigma_n}(s)\ \mathrm{d}s\right| \\ &\lesssim \tau \left\|\overline{\boldsymbol{W}}_n(t_i)\right\|_{\boldsymbol{X}_N^*} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N} + \int_0^t \left\|\overline{\boldsymbol{W}}_n(t_i - s) - \overline{\boldsymbol{W}}_n(t - s)\right\|_{\boldsymbol{X}_N^*} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N}\ \mathrm{d}s.\end{aligned}$$

The first term on the right hand side can be estimated as $\tau\, \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N}$ since $\left\|\overline{\boldsymbol{W}}_n\right\|_{\boldsymbol{X}_N^*} \lesssim 1$. We can bound the second term in the following manner

$$\begin{aligned}
\left\|\overline{\boldsymbol{W}}_n(t_i-s)-\overline{\boldsymbol{W}}_n(t-s)\right\|_{\boldsymbol{X}_N^*} &= \left\|\overline{\boldsymbol{W}}_n(t_i-s)-\boldsymbol{W}_n(t-s)+\boldsymbol{W}_n(t-s)-\overline{\boldsymbol{W}}_n(t-s)\right\|_{\boldsymbol{X}_N^*}\\
&\leqslant \left\|\overline{\boldsymbol{W}}_n(t_i-s)-\boldsymbol{W}_n(t-s)\right\|_{\boldsymbol{X}_N^*}+\left\|\boldsymbol{W}_n(t-s)-\overline{\boldsymbol{W}}_n(t-s)\right\|_{\boldsymbol{X}_N^*}\\
&\lesssim \tau\left\|\partial_t\boldsymbol{W}_n(t_i)\right\|_{\boldsymbol{X}_N^*}\\
&= \tau\left\|\delta^2\boldsymbol{e}_i\right\|_{\boldsymbol{X}_N^*}\\
&\lesssim \tau\left\|\delta^2\boldsymbol{e}_i\right\|_{(\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*}\\
&\lesssim \tau.
\end{aligned}$$

This implies

$$\int_0^t \left\|\overline{\boldsymbol{W}}_n(t_i-s)-\overline{\boldsymbol{W}}_n(t-s)\right\|_{\boldsymbol{X}_N^*}\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N}\ \mathrm{d}s\lesssim\tau\,\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N}.$$

Therefore,

$$\left|(\overline{\sigma_n}*\left(\overline{\boldsymbol{W}}_n,\boldsymbol{\varphi}\right))(t_i)-(\overline{\sigma_n}*\left(\overline{\boldsymbol{W}}_n,\boldsymbol{\varphi}\right))(t)\right|\lesssim\tau\,\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N}\overset{n\to\infty}{\longrightarrow}0.$$

Now, using (12),(16), and Lebesgue dominated convergence theorem, we conclude for any $\boldsymbol{\varphi}\in\boldsymbol{X}_N$ and $t\in(t_{i-1},t_i]$

$$\lim_{n\to\infty}(\overline{\sigma_n}*\left(\overline{\boldsymbol{W}}_n,\boldsymbol{\varphi}\right))(t_i)=\lim_{n\to\infty}(\overline{\sigma_n}*\left(\overline{\boldsymbol{W}}_n,\boldsymbol{\varphi}\right))(t)=(\sigma*(\partial_t\boldsymbol{E},\boldsymbol{\varphi}))(t).$$

Thanks to Lemma 4 and the Lipschitz continuity of $\boldsymbol{N}$, we have for any $t\in(t_{i-1},t_i]$ and $\boldsymbol{\varphi}\in\boldsymbol{X}_N$

$$\left|\left(\boldsymbol{N}(\overline{\boldsymbol{E}_n}(t-\tau))-\boldsymbol{N}(\overline{\boldsymbol{E}_n}(t)),\boldsymbol{\varphi}\right)\right|\lesssim\left\|\overline{\boldsymbol{E}_n}(t-\tau)-\overline{\boldsymbol{E}_n}(t)\right\|\|\boldsymbol{\varphi}\|=\|\boldsymbol{e}_i-\boldsymbol{e}_{i-1}\|\|\boldsymbol{\varphi}\|=\|\delta\boldsymbol{e}_i\|\|\boldsymbol{\varphi}\|\tau\lesssim\tau\,\|\boldsymbol{\varphi}\|.$$

Since $\overline{\boldsymbol{E}_n}\to\boldsymbol{E}$ in $L^2((0,\mathcal{T});\mathbf{L}^2(\Omega))$, also $\boldsymbol{N}(\overline{\boldsymbol{E}_n}(t-\tau))\to\boldsymbol{N}(\boldsymbol{E})$ in this space.
Now, we can integrate (11) in time over $t\in[0,\xi]\subset[0,\mathcal{T}]$ and according to the results above, we can pass to the limit for $n\to\infty$ and $\boldsymbol{\varphi}\in\boldsymbol{X}_N$ to obtain

$$\begin{aligned}
(\partial_t\boldsymbol{E}(\xi),\boldsymbol{\varphi})-(\boldsymbol{W}_0,\boldsymbol{\varphi})+\int_0^\xi((\sigma*\partial_t\boldsymbol{E})(t)+\sigma(t)\boldsymbol{E}_0,\boldsymbol{\varphi})\ \mathrm{d}t+\int_0^\xi(\nabla\times\boldsymbol{E}(t),\nabla\times\boldsymbol{\varphi})\ \mathrm{d}t&\\
=\int_0^\xi(\boldsymbol{N}(\boldsymbol{E}(t)),\boldsymbol{\varphi})\ \mathrm{d}t+\int_0^\xi h(t)\,(\boldsymbol{f},\boldsymbol{\varphi})\ \mathrm{d}t.&
\end{aligned}$$

Then differentiating with respect to the time variable ξ yields

$$\left(\partial_t^2\boldsymbol{E},\boldsymbol{\varphi}\right)+((\sigma*\partial_t\boldsymbol{E})+\sigma\boldsymbol{E}_0,\boldsymbol{\varphi})+(\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{\varphi})=(\boldsymbol{N}(\boldsymbol{E}),\boldsymbol{\varphi})+h(t)\,(\boldsymbol{f},\boldsymbol{\varphi}).$$

The equation above is true a.e. in $[0,\mathcal{T}]$ and for any $\boldsymbol{\varphi}\in\boldsymbol{X}_N$. The space $\boldsymbol{X}_N$ is dense in $\mathbf{H}_0(\mathbf{curl}\,;\Omega)$, therefore, (9) is valid for any $\boldsymbol{\varphi}\in\mathbf{H}_0(\mathbf{curl}\,;\Omega)$ and $\partial_t^2\boldsymbol{E}\in(\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*$ a.e. in $[0,\mathcal{T}]$.

Next step is to pass to the limit for $n \to \infty$ in (DMP). Since $m \in C^2([0,\mathcal{T}])$ and σ is bounded, we deduce

$$\begin{aligned}
\left|(\overline{\sigma_n} * \overline{m_n'})(t_i) - (\overline{\sigma_n} * \overline{m_n'})(t)\right| &= \left|\int_0^{t_i} \overline{m_n'}(t_i - s)\overline{\sigma_n}(s)\,\mathrm{d}s - \int_0^t \overline{m_n'}(t-s)\overline{\sigma_n}(s)\,\mathrm{d}s\right| \\
&= \left|\int_0^t (\overline{m_n'}(t_i - s) - \overline{m_n'}(t-s))\overline{\sigma_n}(s)\,\mathrm{d}s - \int_t^{t_i} \overline{m_n'}(t_i - s)\overline{\sigma_n}(s)\,\mathrm{d}s\right| \\
&\leqslant O(\tau) + \int_t^{t_i} \left|(\overline{m_n'}(t_i - s) - \overline{m_n'}(t-s))\overline{\sigma_n}(s)\right|\,\mathrm{d}s \\
&\overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Taking into account (12), we observe the following

$$(\overline{\sigma_n} * \overline{m_n'})(t) \overset{n\to\infty}{\longrightarrow} (\sigma * m')(t).$$

Thanks to Lemma 7 and (14), we have for any $\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(\Omega)$ and $t \in [0,\mathcal{T}]$

$$\begin{aligned}
\left(\nabla \times \nabla \times \overline{\boldsymbol{E}_n}(t), \boldsymbol{\varphi}\right) &\overset{Green's\ theorem}{=} \left(\nabla \times \overline{\boldsymbol{E}_n}(t), \nabla \times \boldsymbol{\varphi}\right) \\
&\overset{Green's\ theorem}{=} \left(\overline{\boldsymbol{E}_n}(t), \nabla \times \nabla \times \boldsymbol{\varphi}\right) \\
&\overset{n\to\infty}{\longrightarrow} (\boldsymbol{E}(t), \nabla \times \nabla \times \boldsymbol{\varphi}) = (\nabla \times \nabla \times \boldsymbol{E}(t), \boldsymbol{\varphi}).
\end{aligned}$$

Since $\mathbf{C}_0^\infty(\Omega)$ is dense in $\mathbf{H}_0(\mathbf{curl}\,;\Omega)$ and in $\mathbf{L}^2(\Omega)$, we conclude $\left(\nabla \times \nabla \times \overline{\boldsymbol{E}_n}(t), \boldsymbol{\varphi}\right) \to (\nabla \times \nabla \times \boldsymbol{E}(t), \boldsymbol{\varphi})$ for any $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$ and for any $t \in [0,\mathcal{T}]$. Again, thanks to Lemma 6 and Lemma 7, we have

$$\int_0^\xi \left\|\nabla \times \nabla \times \overline{\boldsymbol{E}_n}(t) - \nabla \times \nabla \times \overline{\boldsymbol{E}_n}(t-\tau)\right\|^2\,\mathrm{d}t \leqslant \sum_{i=1}^n \left\|\nabla \times \nabla \times \boldsymbol{e}_i - \nabla \times \nabla \times \boldsymbol{e}_{i-1}\right\|^2 \tau \lesssim \tau \overset{n\to\infty}{\longrightarrow} 0.$$

Using this and the fact that $\phi \in H^1(\Omega)$ for any $\phi \in C_0^\infty(\overline{\Omega})$, we obtain the following convergence result

$$\lim_{n\to\infty} \int_0^\xi \left(\nabla \times \nabla \times \bar{\boldsymbol{E}}_n(t-\tau), \nabla\phi\right)\,\mathrm{d}t = \int_0^\xi \left(\nabla \times \nabla \times \boldsymbol{E}(t), \nabla\phi\right)\,\mathrm{d}t.$$

Lemma 4, Lemma 5, and the embedding $\boldsymbol{X}_N \subset \mathbf{H}^1(\Omega)$ gives us an estimate for $\overline{\boldsymbol{E}_n}$, i.e.

$$\left\|\overline{\boldsymbol{E}_n}(t)\right\|^2_{\mathbf{H}^1(\Omega)} \lesssim 1.$$

We recall the Nečas inequality cf. [27] or [28, (7.116)]

$$\|w\|^2_{L^2(\Gamma)} \leqslant \varepsilon \|\nabla w\|^2 + C_\varepsilon \|w\|^2, \quad \forall w \in H^1(\Omega), \tag{17}$$

where $0 < \varepsilon < \varepsilon_0$ and $C_\varepsilon := \frac{C}{\varepsilon}$ for some $C > 0$. Strong convergence of $\overline{\boldsymbol{E}_n}(t)$ towards $\boldsymbol{E}(t)$ in $\mathbf{L}^2(\Omega)$ for any $t \in [0,\mathcal{T}]$, cf. (14), and the inequality above imply

$$\left\|\overline{\boldsymbol{E}_n}(t) - \boldsymbol{E}(t)\right\|^2_{\mathbf{L}^2(\Gamma)} \lesssim \varepsilon \left\|\overline{\boldsymbol{E}_n}(t) - \boldsymbol{E}(t)\right\|^2_{\mathbf{H}^1(\Omega)} + C_\varepsilon \left\|\overline{\boldsymbol{E}_n}(t) - \boldsymbol{E}(t)\right\|^2 \lesssim \varepsilon + C_\varepsilon \left\|\overline{\boldsymbol{E}_n}(t) - \boldsymbol{E}(t)\right\|^2.$$

Thus,

$$\lim_{n\to\infty} \left\|\overline{\boldsymbol{E}_n}(t) - \boldsymbol{E}(t)\right\|^2_{\mathbf{L}^2(\Gamma)} \lesssim \varepsilon \overset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{i.e.} \quad \overline{\boldsymbol{E}_n}(t) \to \boldsymbol{E}(t) \quad \text{in} \quad \mathbf{L}^2(\Gamma), \quad \forall t \in [0,\mathcal{T}].$$

Using Lemma 4 and same technique as above, we conclude the following for $t \in [t_{i-1}, t_i]$

$$\begin{aligned}\left\|\overline{\boldsymbol{E}_n}(t) - \overline{\boldsymbol{E}_n}(t-\tau)\right\|^2_{\mathbf{L}^2(\Gamma)} \lesssim & \varepsilon \left\|\overline{\boldsymbol{E}_n}(t) - \overline{\boldsymbol{E}_n}(t-\tau)\right\|^2_{\mathbf{H}^1(\Omega)} + C_\varepsilon \left\|\overline{\boldsymbol{E}_n}(t) - \overline{\boldsymbol{E}_n}(t-\tau)\right\|^2 \\ \lesssim & \varepsilon + C_\varepsilon \left\|\delta \boldsymbol{e}_i\right\|^2 \tau^2 \quad \overset{\tau \to 0}{\longrightarrow} \quad \varepsilon \quad \overset{\varepsilon \to 0}{\longrightarrow} \quad 0.\end{aligned}$$

Now, we integrate (DMP) in time over $t \in [0, \xi] \subset [0, \mathcal{T}]$, consider all convergence results above, take into account that $m \in C^2([0, \mathcal{T}])$ and the Lipschitz continuity of $\boldsymbol{N}$ and pass to the limit for $n \to \infty$ to obtain

$$\lim_{n\to\infty} \int_0^\xi \bar{h}_n = \frac{\int_0^\xi m''(t) + \int_0^\xi [(\sigma * m')(t) + \sigma(t)m(0)] + \int_0^\xi \int_\Omega \nabla \times \nabla \times \boldsymbol{E} \cdot \nabla\phi \,\mathrm{d}\boldsymbol{x} - \int_0^\xi \int_\Gamma \boldsymbol{N}(\boldsymbol{E}) \cdot \boldsymbol{n}\phi \,\mathrm{d}\Gamma}{\int_\Gamma \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{n}\phi \,\mathrm{d}\Gamma}.$$

Differentiation with respect to the time variable ξ yields (7), which also concludes our proof. □

6. Uniqueness

Due to the nonlinear term $\boldsymbol{N}$, we are not able to provide an uniqueness proof without any further regularity assumptions on the solution $\boldsymbol{E}$. Thus, we assume $\boldsymbol{E} \in \mathbf{H}^{1,\infty}(\Omega)$. Taking this into account and also presume that $\boldsymbol{N}$ is supposedly smooth, i.e. $\boldsymbol{N} \in \mathbf{C^2}$, we conclude the following for any vector fields $\boldsymbol{u}, \boldsymbol{v} : \mathbb{R}^3 \to \mathbb{R}^3$ and some $\xi_1, \xi_2, \xi_3 \in [0, 1]$

$$\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v}) = \begin{pmatrix} \boldsymbol{N}_1(\boldsymbol{u}) - \boldsymbol{N}_1(\boldsymbol{v}) \\ \boldsymbol{N}_2(\boldsymbol{u}) - \boldsymbol{N}_2(\boldsymbol{v}) \\ \boldsymbol{N}_3(\boldsymbol{u}) - \boldsymbol{N}_3(\boldsymbol{v}) \end{pmatrix} = \begin{pmatrix} \nabla\boldsymbol{N}_1(\boldsymbol{v} + \xi_1(\boldsymbol{u} - \boldsymbol{v})) \cdot (\boldsymbol{u} - \boldsymbol{v}) \\ \nabla\boldsymbol{N}_2(\boldsymbol{v} + \xi_2(\boldsymbol{u} - \boldsymbol{v})) \cdot (\boldsymbol{u} - \boldsymbol{v}) \\ \nabla\boldsymbol{N}_3(\boldsymbol{v} + \xi_3(\boldsymbol{u} - \boldsymbol{v})) \cdot (\boldsymbol{u} - \boldsymbol{v}) \end{pmatrix}.$$

Assuming that $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}^{1,\infty}(\Omega)$ and using the Cauchy-Schwarz inequality, we obtain an estimate for derivatives in the following form

$$\begin{aligned}\left|\partial x_j(\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v}))\right| \leqslant & \sum_{i=1}^{3} \Big[\left|\partial x_j \nabla\boldsymbol{N}_i(\boldsymbol{v} + \xi_i(\boldsymbol{u} - \boldsymbol{v}))\right| \left|\partial x_j(\boldsymbol{v} + \xi_i(\boldsymbol{u} - \boldsymbol{v}))\right| \left|\boldsymbol{u} - \boldsymbol{v}\right| \\ & + \left|\nabla\boldsymbol{N}_i(\boldsymbol{v} + \xi_i(\boldsymbol{u} - \boldsymbol{v}))\right| \left|\partial x_j(\boldsymbol{u} - \boldsymbol{v})\right| \Big] \\ \lesssim & (\left|\boldsymbol{u} - \boldsymbol{v}\right| + \left|\partial x_j(\boldsymbol{u} - \boldsymbol{v})\right|).\end{aligned}$$

Now, we can provide some further estimates which are obtained in the similar manner as estimate above

$$\left\|\nabla \times (\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v}))\right\| \lesssim \left\|\boldsymbol{u} - \boldsymbol{v}\right\|_{\mathbf{H}^1(\Omega)} \quad \text{if } \boldsymbol{N} \in \mathbf{C^2}, \quad \boldsymbol{u}, \boldsymbol{v} \in \mathbf{H^{1,\infty}(\Omega)} \tag{18}$$

and

$$\left\|\nabla \cdot (\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v}))\right\| \lesssim \left\|\boldsymbol{u} - \boldsymbol{v}\right\|_{\mathbf{H}^1(\Omega)} \quad \text{if } \boldsymbol{N} \in \mathbf{C^2}, \quad \boldsymbol{u}, \boldsymbol{v} \in \mathbf{H^{1,\infty}(\Omega)}. \tag{19}$$

We continue with the uniqueness theorem.

Theorem 2. *Let the assumptions of Theorem 1 be satisfied. Moreover, presume that $\boldsymbol{N} \in \mathbf{C^2}$. Then there exists at most one weak solution $\{\boldsymbol{E}, h\}$ to the problem (5),(3), (4) and (7) fulfilling $h \in L^\infty((0, \mathcal{T})), \boldsymbol{E} \in C([0, \mathcal{T}]; \mathbf{L}^2(\Omega)) \cap L^\infty((0, \mathcal{T}); \mathbf{H}^{1,\infty}(\Omega))$ with $\partial_t \boldsymbol{E} \in L^2((0, \mathcal{T}); \mathbf{L}^2(\Omega)) \cap C([0, \mathcal{T}]; \boldsymbol{X}_N^*), \partial_t^2 \boldsymbol{E} \in L^2((0, \mathcal{T}); (\mathbf{H}_0(\mathbf{curl}; \Omega))^*)$ and $\nabla \times \nabla \times \boldsymbol{E} \in L^\infty((0, \mathcal{T}); \mathbf{L}^2(\Omega))$.*

Proof. Let us have two solutions $\{\boldsymbol{E}, h\}$ and $\{\boldsymbol{G}, g\}$ to the problem (5),(3), (4), (7) and denote

$$\boldsymbol{E} - \boldsymbol{G} = \boldsymbol{P}, \qquad h(t) - g(t) = p(t).$$

Our goal is to show that $\boldsymbol{P} = \mathbf{0}$ a.e. in $\Omega \times (0, \mathcal{T})$ and $p = 0$ a.e. in $(0, \mathcal{T})$.The measurements for both solutions are the same, therefore, we have

$$p(t) = \frac{(\nabla \times \nabla \times \boldsymbol{P}(t), \nabla\phi) - \int_\Gamma (\boldsymbol{N}(\boldsymbol{E}(t)) - \boldsymbol{N}(\boldsymbol{G}(t))) \cdot \boldsymbol{n}\phi \,\mathrm{d}\Gamma}{\int_\Gamma \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{n}\phi \,\mathrm{d}\Gamma}. \tag{20}$$

Subtracting (5) for $\boldsymbol{E}$ and $\boldsymbol{G}$ yields

$$\partial_t^2 \boldsymbol{P} + (\sigma * \partial_t \boldsymbol{P}) + \nabla \times \nabla \times \boldsymbol{P} = \boldsymbol{f} p + \boldsymbol{N}(\boldsymbol{E}) - \boldsymbol{N}(\boldsymbol{G}). \tag{21}$$

Remark 6. *$\boldsymbol{P} \times \boldsymbol{n} = \mathbf{0}$ on the boundary Γ and $\boldsymbol{P}(\boldsymbol{x}, 0) = \mathbf{0}$, $\partial_t \boldsymbol{P}(\boldsymbol{x}, 0) = \mathbf{0}$.*

We continue with several energy estimates implied by (20) and (21).
Part (A): Looking at (20), taking into account the embedding $\mathbf{H}^1(\Omega) \subset \mathbf{L}^2(\Gamma)$ and the Lipschitz continuity of $\boldsymbol{N}$, we obtain

$$|p(\xi)|^2 \lesssim \|\boldsymbol{P}(\xi)\|^2_{\mathbf{H}^1(\Omega)} + \|\nabla \times \nabla \times \boldsymbol{P}(\xi)\|^2 . \tag{A}$$

Part (B): Multiply (21) with $\partial_t \boldsymbol{P}$, integrate over Ω, use Green's theorem and then integrate in time to deduce

$$\frac{1}{2}\|\partial_t \boldsymbol{P}(\xi)\|^2 + \frac{1}{2}\|\nabla \times \boldsymbol{P}(\xi)\|^2 \leqslant \int_0^\xi |p| \|\boldsymbol{f}\| \|\partial_t \boldsymbol{P}\| + \int_0^\xi \|\boldsymbol{N}(\boldsymbol{E}) - \boldsymbol{N}(\boldsymbol{G})\| \|\partial_t \boldsymbol{P}\| + \int_0^\xi \|(\sigma * \partial_t \boldsymbol{P})\| \|\partial_t \boldsymbol{P}\| .$$

We also have the following bounds for the terms on the right hand side

$$\begin{aligned}
&\int_0^\xi |p| \|\boldsymbol{f}\| \|\partial_t \boldsymbol{P}\| \lesssim \int_0^\xi |p|^2 + \int_0^\xi \|\partial_t \boldsymbol{P}\|^2 ,\\
&\int_0^\xi \|(\sigma * \partial_t \boldsymbol{P})\| \|\partial_t \boldsymbol{P}\| \lesssim \int_0^\xi \|\partial_t \boldsymbol{P}\|^2 ,\\
&\int_0^\xi \|\boldsymbol{N}(\boldsymbol{E}) - \boldsymbol{N}(\boldsymbol{G})\| \|\partial_t \boldsymbol{P}\| \lesssim \int_0^\xi \|\boldsymbol{P}\| \|\partial_t \boldsymbol{P}\| \lesssim \int_0^\xi \|\partial_t \boldsymbol{P}\|^2 .
\end{aligned}$$

Here, we used the traditional Cauchy and Young inequalities, boundedness of σ, the Lipschitz continuity of $\boldsymbol{N}$ and $\|\boldsymbol{P}(t)\| = \left\|\int_0^t \partial_t \boldsymbol{P}(s)\right\| \leqslant \int_0^t \|\partial_t \boldsymbol{P}(s)\|$. Collecting all partial results and applying Grönwall's Lemma 3, we conclude that

$$\|\partial_t \boldsymbol{P}(\xi)\|^2 + \|\nabla \times \boldsymbol{P}(\xi)\|^2 \lesssim \int_0^\xi |p|^2 . \tag{B}$$

Remark 7. *This result also implies $\|\boldsymbol{P}(\xi)\|^2 \lesssim \int_0^\xi |p|^2$.*

Part (C): Apply the divergence operator to (21), then multiply it by $\nabla \cdot \partial_t \boldsymbol{P}$ and integrate in space and time to obtain

$$\begin{aligned}
\frac{1}{2}\|\nabla \cdot \partial_t \boldsymbol{P}(\xi)\|^2 \leqslant & \int_0^\xi |p| \|\nabla \cdot \boldsymbol{f}\| \|\nabla \cdot \partial_t \boldsymbol{P}\| + \int_0^\xi \|(\sigma * \nabla \cdot \partial_t \boldsymbol{P})\| \|\nabla \cdot \partial_t \boldsymbol{P}\| \\
& + \int_0^\xi \|\nabla \cdot (\boldsymbol{N}(\boldsymbol{E}) - \boldsymbol{N}(\boldsymbol{G}))\| \|\nabla \cdot \partial_t \boldsymbol{P}\| .
\end{aligned}$$

First two terms on the right hand side are estimated via the Young's inequality, i.e.

$$\int_0^\xi |p| \, \|\nabla \cdot \boldsymbol{f}\| \, \|\nabla \cdot \partial_t \boldsymbol{P}\| + \int_0^\xi \|(\sigma * \nabla \cdot \partial_t \boldsymbol{P})\| \, \|\nabla \cdot \partial_t \boldsymbol{P}\| \lesssim \int_0^\xi |p|^2 + \int_0^\xi \|\nabla \cdot \partial_t \boldsymbol{P}\|^2 .$$

For the last term we use (19), (B), the embedding $\boldsymbol{X}_N \subset \mathbf{H}^1(\Omega)$, and the inequality $\|\nabla \cdot \boldsymbol{P}(t)\| = \left\| \int_0^t \nabla \cdot \partial_s \boldsymbol{P}(s) \, \mathrm{d}s \right\| \leqslant \int_0^t \|\nabla \cdot \partial_s \boldsymbol{P}(s)\| \, \mathrm{d}s$

$$\begin{aligned}
\int_0^\xi \|\nabla \cdot (\boldsymbol{N}(\boldsymbol{E}) - \boldsymbol{N}(\boldsymbol{G}))\| \, \|\nabla \cdot \partial_t \boldsymbol{P}\| &\lesssim \int_0^\xi \|\boldsymbol{P}\|_{\mathbf{H}^1(\Omega)} \|\nabla \cdot \partial_t \boldsymbol{P}\| \\
&\lesssim \int_0^\xi \|\boldsymbol{P}\|_{\boldsymbol{X}_N} \|\nabla \cdot \partial_t \boldsymbol{P}\| \\
&\lesssim \int_0^\xi (\|\boldsymbol{P}\| + \|\nabla \times \boldsymbol{P}\|) \, \|\nabla \cdot \partial_t \boldsymbol{P}\| + \int_0^\xi \|\nabla \cdot \boldsymbol{P}\| \, \|\nabla \cdot \partial_t \boldsymbol{P}\| \\
&\lesssim \int_0^\xi \left(\|\boldsymbol{P}\|^2 + \|\nabla \times \boldsymbol{P}\|^2 \right) + \int_0^\xi \|\nabla \cdot \partial_t \boldsymbol{P}\|^2 \\
&\lesssim \int_0^\xi |p|^2 + \int_0^\xi \|\nabla \cdot \partial_t \boldsymbol{P}\|^2 .
\end{aligned}$$

We employ Grönwall's Lemma 3 to get

$$\|\nabla \cdot \partial_t \boldsymbol{P}(\xi)\|^2 \lesssim \int_0^\xi |p|^2 \qquad \text{and} \qquad \|\nabla \cdot \boldsymbol{P}(\xi)\|^2 \lesssim \int_0^\xi |p|^2 . \tag{C}$$

Part (D): Apply the curl operator to (21), multiply it by $\nabla \times \partial_t \boldsymbol{P}$, then integrate in Ω and use Green's theorem and then integrate in time to obtain

$$\begin{aligned}
\frac{1}{2} \|\nabla \times \partial_t \boldsymbol{P}(\xi)\|^2 + \frac{1}{2} \|\nabla \times \nabla \times \boldsymbol{P}(\xi)\|^2 &\leqslant \int_0^\xi |p| \, \|\nabla \times \boldsymbol{f}\| \, \|\nabla \times \partial_t \boldsymbol{P}\| + \int_0^\xi \|(\sigma * \nabla \times \partial_t \boldsymbol{P})\| \, \|\nabla \times \partial_t \boldsymbol{P}\| \\
&\quad + \int_0^\xi \|\nabla \times (\boldsymbol{N}(\boldsymbol{E}) - \boldsymbol{N}(\boldsymbol{G}))\| \, \|\nabla \times \partial_t \boldsymbol{P}\| .
\end{aligned}$$

Again, we use Young's inequality to handle the first two terms on the right hand side

$$\int_0^\xi |p| \, \|\nabla \times \boldsymbol{f}\| \, \|\nabla \times \partial_t \boldsymbol{P}\| + \int_0^\xi \|(\sigma * \nabla \times \partial_t \boldsymbol{P})\| \, \|\nabla \times \partial_t \boldsymbol{P}\| \lesssim \int_0^\xi |p|^2 + \int_0^\xi \|\nabla \times \partial_t \boldsymbol{P}\|^2 .$$

The last term on the right hand side is estimated with the help of (18), the embedding $\boldsymbol{X}_N \subset \mathbf{H}^1(\Omega)$, (B) and (C)

$$\begin{aligned}
\int_0^\xi \|\nabla \times (\boldsymbol{N}(\boldsymbol{E}) - \boldsymbol{N}(\boldsymbol{G}))\| \, \|\nabla \times \partial_t \boldsymbol{P}\| &\lesssim \int_0^\xi \|\boldsymbol{P}\|_{\mathbf{H}^1(\Omega)} \|\nabla \times \partial_t \boldsymbol{P}\| \lesssim \int_0^\xi \|\boldsymbol{P}\|_{\boldsymbol{X}_N} \|\nabla \times \partial_t \boldsymbol{P}\| \\
&\lesssim \int_0^\xi \left(\|\boldsymbol{P}\|^2 + \|\nabla \cdot \boldsymbol{P}\|^2 + \|\nabla \times \boldsymbol{P}\|^2 \right) + \int_0^\xi \|\nabla \times \partial_t \boldsymbol{P}\|^2 \\
&\lesssim \int_0^\xi |p|^2 + \int_0^\xi \|\nabla \times \partial_t \boldsymbol{P}\|^2 .
\end{aligned}$$

Utilizing the Grönwall Lemma 3, we achieve the following estimate

$$\|\nabla \times \partial_t \boldsymbol{P}(\xi)\|^2 + \|\nabla \times \nabla \times \boldsymbol{P}(\xi)\|^2 \lesssim \int_0^\xi |p|^2 . \tag{D}$$

Summary: Taking into account the embedding $\boldsymbol{X}_N \subset \mathbf{H}^1(\Omega)$ and gathering the results from (A), (B), (C) and (D), we have

$$\|\boldsymbol{P}(\xi)\|^2_{\mathbf{H}^1(\Omega)} + \|\nabla \times \nabla \times \boldsymbol{P}(\xi)\|^2 \lesssim \int_0^\xi |p|^2 \lesssim \int_0^\xi \left(\|\boldsymbol{P}\|^2_{\mathbf{H}^1(\Omega)} + \|\nabla \times \nabla \times \boldsymbol{P}\|^2 \right).$$

Thus, employing the Grönwall Lemma 3 one more time, we see that $\boldsymbol{P} = \mathbf{0}$ a.e. in $\Omega \times (0, \mathcal{T})$ and from (A), we conclude that $p = 0$ a.e. in $(0, \mathcal{T})$. □

7. Numerical experiment

The main goal of this section is to support theoretical results stated above. We want to demonstrate the convergence of numerical scheme proposed in Section 3. Since Rothe's method is semi-discrete, we only analyze the time dependent part of the error of the numerical solution. Consider the following test problem. Find $\{\boldsymbol{E}(\boldsymbol{x}, t), h(t)\}$ such that [2]

$$\begin{aligned} \partial_t^2 \boldsymbol{E}(\boldsymbol{x}, t) + (\sigma * \partial_t \boldsymbol{E})(t) + \sigma(t)\boldsymbol{E}(\boldsymbol{x}, 0) + \nabla \times \nabla \times \boldsymbol{E}(\boldsymbol{x}, t) &= \boldsymbol{f}(\boldsymbol{x})h(t) + \boldsymbol{N}(\boldsymbol{E}(\boldsymbol{x}, t)) + \boldsymbol{F}(\boldsymbol{x}, t) \quad \text{in } \Omega \times (0, \mathcal{T}) \\ \boldsymbol{E}(\boldsymbol{x}, t) \times \boldsymbol{n} &= \mathbf{0} \quad \text{in } \Gamma \times (0, \mathcal{T}) \\ \boldsymbol{f}(\boldsymbol{x}) \times \boldsymbol{n} &= \mathbf{0} \quad \text{on } \Gamma. \end{aligned} \tag{22}$$

With initial data prescribed as

$$\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0(\boldsymbol{x}), \qquad \partial_t \boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{W}_0(\boldsymbol{x})$$

and additional measurement in the form of (6).

7.1. Setting of the experiment

Let Ω be a sphere in $\mathbb{R}^3$ with radius $r = 1$ i.e. $\Omega = \left\{\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 < 1\right\}$ and $t \in (0, \mathcal{T})$ with $\mathcal{T} = 1$. To show the convergence of our scheme, we need an exact solution $\{\boldsymbol{E}(\boldsymbol{x}, t), h(t)\}$, so we can compute the error of the numerical solution. For that reason, we define the exact solution as

$$\boldsymbol{E}(\boldsymbol{x}, t) = e^t \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad h(t) = e^t.$$

The remaining functions are determined accordingly

$$\begin{aligned} &\sigma(t) = 4t^3 + 8t^2 + 16t + 32, \\ &\boldsymbol{N}(\boldsymbol{E}(\boldsymbol{x}, t)) = |\boldsymbol{E}(\boldsymbol{x}, t)|^{-1/2} \boldsymbol{E}(\boldsymbol{x}, t) + \boldsymbol{E}(\boldsymbol{x}, t), \\ &\boldsymbol{f}(\boldsymbol{x}) = 88 \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \boldsymbol{F}(\boldsymbol{x}, t) = -\left(\frac{1}{(x^2 + y^2 + z^2)^{1/4}} + 12t^2 + 40t + 56 \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{aligned}$$

[2] If the vector field $\boldsymbol{F}(\boldsymbol{x}, t)$ is sufficiently smooth then all theoretical results achieved in previous Sections remain true. Hence, we can add this term to the right hand side in (22).

In order to examine the nature of error (whether it is diminishing with the decreasing time step) of our numerical solution $\{\boldsymbol{E}_{numerical}, h_{numerical}\}$, we compute multiple solutions for time steps $\tau = 0.1, 0.05, 0.025, 0.0125, 0.00625$. The spatial part of our time-space domain is then divided into 553 cells (tetrahedra) with diameters ranging from 0.38513 to 0.65231. We use Lagrange finite elements of order 2 at each time step to provide a numerical solution. This leads to a system with 16590 degrees of freedom.

Remark 8. *Sometimes, when computing the electromagnetic solutions, we obtain a non-physical (spurious) solution. This is when the discretized space V_h (spatial discretization) does not belong in $V = \lim_{h\to 0} V_h$. If we require a divergence-free constraint on the discrete test functions we suppress the non-physical solutions. In this case the use of Nédélec (edge) finite elements is the obvious choice. However, in our case, the solution $\boldsymbol{E}(t) \in \boldsymbol{X}_N$ does not have to be divergence-free, even though its divergence is controlled. Therefore, we choose the Lagrange finite elements for our numerical computations.*

The part of the boundary where the measurement (6) was done is displayed in Fig. 2. The errors for the numerical

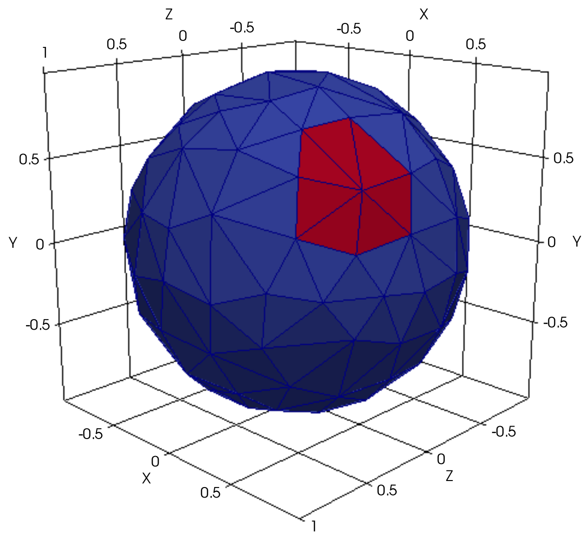

Figure 2. The boundary measurement

solutions are computed in the following manner

$$error_{\boldsymbol{E}} = \frac{\|\boldsymbol{E}_{numerical} - \boldsymbol{E}_{exact}\|_{L^2((0,\mathcal{T});\mathbf{L}^2(\Omega))}}{\|\boldsymbol{E}_{exact}\|_{L^2((0,\mathcal{T});\mathbf{L}^2(\Omega))}}, \quad error_h = \sqrt{\int_0^T |h_{numerical} - h_{exact}|^2 \, \mathrm{d}t}.$$

We can see the time step dependency of these errors in Fig. 3 and Fig. 4 and in the Table 7.1 below. The quality of

τ	0.1	0.05	0.025	0.0125	0.00625
$error_{\boldsymbol{E}}$	0.1387535	0.0648619	0.030508687	0.0141785	0.006821868
$error_h$	0.07026298	0.03500252	0.0187912	0.01103294	0.00724477

Table 1. Values of $error_{\boldsymbol{E}}$ and $error_h$ for different time steps τ

the numerical reconstruction of the source term $h(t)$ can be seen in Fig. 5.

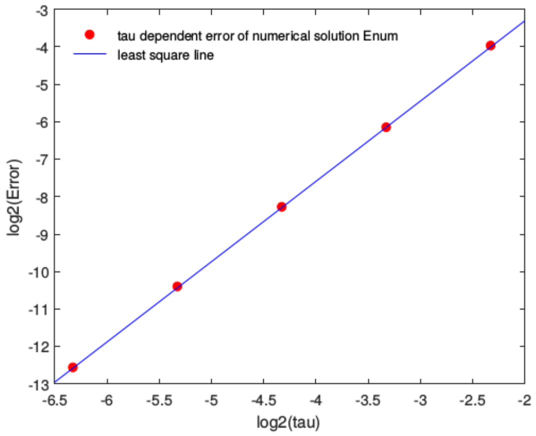

Figure 3. τ dependency of error of $\boldsymbol{E}_{numerical}$

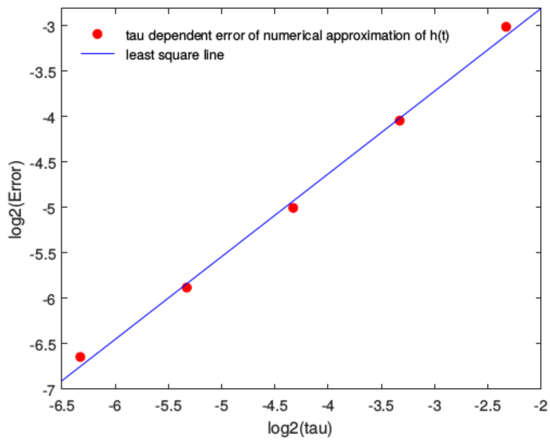

Figure 4. τ dependency of error of $h_{numerical}$

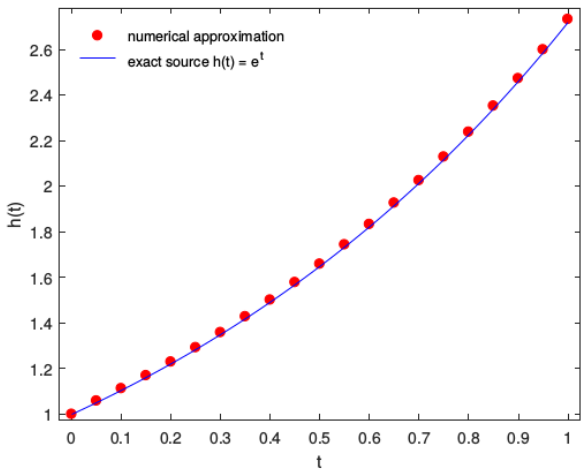

Figure 5. Reconstruction of the source term $h(t) = e^t$ using the time step $\tau = 0.05$

The performance of our algorithm with noise in the measurement is pictured in Fig. 6. As we can see, the reconstruction of the source term was quite good for 1% and 5% noise in the data. However, when 15% of noise was present, our reconstruction was slightly off. To reconstruct the source, we also need the information about the first and second order time derivatives of the function $m(t)$ (measurement). Therefore, if the noise in the data is too high (15%), the smoothness of $m(t)$ is not sufficient. This causes the errors in the reconstruction.

If the error of a given numerical solution $\boldsymbol{E}_\tau$ from the exact solution $\boldsymbol{E}_{exact}$ depends smoothly on a time step τ then there exist an error coefficient A such that

$$\boldsymbol{E}_\tau - \boldsymbol{E}_{exact} = A\tau^p + O(\tau^{p+1}),$$

where p represents the order of convergence. Using the formula above, we can estimate the order of convergence for our numerical solutions $\boldsymbol{E}_\tau$ and h_τ, i.e.

$$\frac{\boldsymbol{E}_\tau - \boldsymbol{E}_{exact}}{\boldsymbol{E}_{\tau/2} - \boldsymbol{E}_{exact}} = \frac{A\tau^p + O(\tau^{p+1})}{A(\tau/2)^p + O(\tau^{p+1})} = 2^p + O(\tau).$$

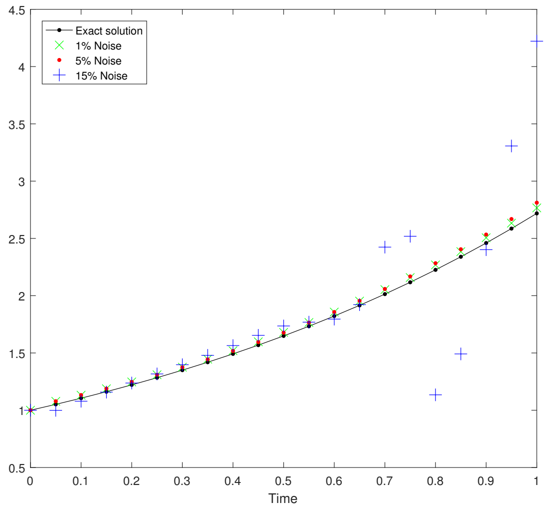

Figure 6. Source reconstruction using noisy data.

Which gives us (analogously for h_τ)

$$\log_2\left(\frac{\boldsymbol{E}_\tau - \boldsymbol{E}_{exact}}{\boldsymbol{E}_{\tau/2} - \boldsymbol{E}_{exact}}\right) = p + O(\tau).$$

Applying this method, we obtain an estimation for the order of convergence of our numerical solutions.

	$\log_2 \frac{\boldsymbol{E}_{0.01}}{\boldsymbol{E}_{0.05}}$	$\log_2 \frac{\boldsymbol{E}_{0.05}}{\boldsymbol{E}_{0.025}}$	$\log_2 \frac{\boldsymbol{E}_{0.025}}{\boldsymbol{E}_{0.0125}}$	$\log_2 \frac{\boldsymbol{E}_{0.0125}}{\boldsymbol{E}_{0.00625}}$		$\log_2 \frac{h_{0.01}}{h_{0.05}}$	$\log_2 \frac{h_{0.05}}{h_{0.025}}$	$\log_2 \frac{h_{0.025}}{h_{0.0125}}$	$\log_2 \frac{h_{0.0125}}{h_{0.00625}}$
$p_{\boldsymbol{E}}$	1.097	1.088	1.105	1.055	p_h	1.0053	0.897	0.768	0.6068

Table 2. Estimation of the order of convergence for $\boldsymbol{E}_\tau$ and h_τ

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