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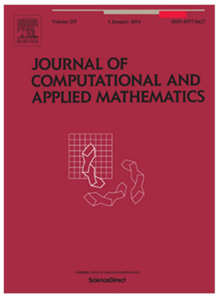

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Matthew M. Lin, Chun-Yueh Chiang

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An accelerated technique for solving one type of discrete-time algebraic Riccati equations

Matthew M. Lin[1,]

Department of Mathematics, National Cheng Kung University, Tainan 701, Taiwan.

Chun-Yueh Chiang[2,*]

Center for General Education, National Formosa University, Huwei 632, Taiwan.

Abstract

Algebraic Riccati equations are encountered in many applications of control and engineering problems, e.g., LQG problems and H^{∞} control theory. In this work, we study the properties of one type of discrete-time algebraic Riccati equations. Our contribution is twofold. First, we present sufficient conditions for the existence of a unique positive definite solution. Second, we propose an accelerated algorithm to obtain the positive definite solution with the rate of convergence of any desired order. Numerical experiments strongly support that our approach performs extremely well even in the almost critical case. As a byproduct, we provide show that this method is capable of computing the unique negative definite solution, once it exists.

Keywords: algebraic Riccati equations, Sherman Morrison Woodbury formula, positive definite solution, semigroup property, doubling algorithm, r-superlinear with order r
2000 MSC: 39B12, 39B42, 47J22, 65H05, 15A24

1. Introduction

Originated from the study of control theory, the discrete-time algebraic Riccati equation (DARE) of the compact form:

$$X = H + A^H X (I + GX)^{-1} A \tag{1}$$

*Corresponding author
Email addresses: mhlin@mail.ncku.edu.tw (Matthew M. Lin), chiang@nfu.edu.tw (Chun-Yueh Chiang)

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has been extensively investigated; see [20, 13, 9, 16, 23, 10, 17, 6, 15, 5, 22, 24, 19] and the references therein. Theoretically, the DARE is highly related to the so-called nonlinear matrix equation (NME) with plus sign [8, 3]:

$$X + B^H X^{-1} B = Q \tag{2a}$$

or the one with minor sign [1, 18]:

$$X - B^H X^{-1} B = Q, \tag{2b}$$

where $B \in \mathbb{C}^{n\times n}$ and Q is a $n \times n$ (Hermitian) positive definite matrix. This is because when G is a positive definite matrix and X is a Hermitian solution of Eq. (1), let $\widehat{X} := G + GXG$, $\widehat{B} := AG$ and $\widehat{Q} := G + GHG + \widehat{B}^H G^{-1}\widehat{B}$. The form of Eq. (1) becomes

$$\begin{aligned} G + GXG &= G + G(H + A^H X(I + GX)^{-1}A)G \\ &= G + G(H + A^H (X + G^{-1})(I + GX)^{-1}A - A^H G^{-1}(I + GX)^{-1}A)G \\ &= G + GHG + (AG)^H G^{-1}(AG) - (AG)^H (G + GXG)^{-1}(AG), \end{aligned}$$

or equivalently, $\widehat{X} + \widehat{B}^H \widehat{X}^{-1}\widehat{B} = \widehat{Q}$, a form of NME with plus sign (2a). Alternatively, if B is a nonsingular matrix and X is a positive definite solution of Eq. (2b), let $\widehat{A} := B^{-H}B$, $\widehat{G} := B^{-H}QB^{-1}$ and $\widehat{H} := Q$. The form of Eq. (2b) becomes

$$\begin{aligned} X &= Q + B^H X^{-1} B = Q + B^H (Q + B^H X^{-1} B)^{-1} B \\ &= Q + \widehat{A}^H (X^{-1} + \widehat{G})\widehat{A} = \widehat{H} + \widehat{A}^H X(I + \widehat{G}X)\widehat{A}, \end{aligned}$$

a form of DARE (1). Only recently has the conjugate NME $X + A^H \bar{X}^{-1} A = Q$ received considerable attention; see [14, 25, 11, 4]. It can be said that one of its main application of the conjugate NME is the study of modern quantum theory by means of consimilarity [25]. A parallel study of this conjugate NME is to investigate the conjugate discrete-time algebraic Riccati equation (CDARE) in the form with the plus sign:

$$X = H + A^H \overline{X}(I + G\overline{X})^{-1} A \tag{3a}$$

or in the form with the minus sign:

$$X = H - A^H \overline{X}(I + G\overline{X})^{-1} A, \tag{3b}$$

where $A \in \mathbb{C}^{n\times n}$, matrices G and H are two positive definite matrices of size $n \times n$, and the n-square matrix X is an unknown Hermitian matrix and to be determined.

In the paper, we derive some sufficient conditions for the existence of the unique positive solution. Moreover, we present a numerical procedure, based on the fixed point iteration, to solve CDAREs, and show that the speed of convergence can be of any desired order.

An immediate question is whether this conjugate formulae (3) could be equivalently transformed to the compact form (1). To this end, we use the notations:

$$\mathcal{F}_{\pm}(X) = H \pm A^H \overline{X} \Delta_{G,\overline{X}} A, \tag{4}$$

with $\Delta_{G,X} := (I + GX)^{-1}$ to simplify our discussion, that is, (3) can also be represented by

$$X = \mathcal{F}_{\pm}(X).$$

Following from the fact that

$$\Delta_{G,\overline{\mathcal{F}}_{\pm}(X)} = \Delta_{G,\overline{H}} \mp \Delta_{G,\overline{H}} G \overline{A}^H X \Delta_{G_1,X} \overline{A} \Delta_{G,\overline{H}}, \tag{5}$$

it can be seen that

$$\begin{aligned} \mathcal{F}_{\pm}^{(2)}(X) &:= \mathcal{F}_{\pm}(\mathcal{F}_{\pm}(X)) = H \pm A^H \overline{\mathcal{F}}_{\pm}(X) \Delta_{G,\overline{\mathcal{F}}_{\pm}(X)} A \\ &= H_1 \pm (\Pi_1 + \Pi_2 + \Pi_3), \end{aligned} \tag{6}$$

where

$$\begin{aligned} \Pi_1 &= \pm A^H \overline{A}^H X \Delta_{\overline{G},X} \overline{A} \Delta_{G,\overline{H}} A, \\ \Pi_2 &= \mp A^H \overline{H} \Delta_{G,\overline{H}} G \overline{A}^H X \Delta_{G_1,X} \overline{A} \Delta_{G,\overline{H}} A, \\ \Pi_3 &= -A^H \overline{A}^H X \Delta_{\overline{G},X} \overline{A} \Delta_{G,\overline{H}} G \overline{A}^H X \Delta_{G_1,X} \overline{A} \Delta_{G,\overline{H}} A, \\ G_1 &= \overline{G} \pm \overline{A} \Delta_{G,\overline{H}} G \overline{A}^H, && (7) \\ H_1 &= H \pm A^H \overline{H} \Delta_{G,\overline{H}} A. && (8) \end{aligned}$$

Note that (5) is an application of the well-known Sherman Morrison Woodbury formula, which can be stated as follows.

Lemma 1.1. *[2] Let A and B be two arbitrary matrices of size n, and let X and Y be two $n \times n$ nonsingular matrices. Assume that $Y^{-1} \pm BX^{-1}A$ is nonsingular. Then, $X \pm AYB$ is invertible and*

$$(X \pm AYB)^{-1} = X^{-1} \mp X^{-1}A(Y^{-1} \pm BX^{-1}A)^{-1}BX^{-1}.$$

We further observe that

$$\begin{aligned} \Pi_1 + \Pi_3 &= \pm A^H \overline{A}^H X \Delta_{\overline{G},X} \left(I_n \mp \overline{A} \Delta_{G,\overline{H}} G \overline{A}^H X \Delta_{G_1,X} \right) \overline{A} \Delta_{G,\overline{H}} A \\ &= \pm A^H \overline{A}^H X \Delta_{\overline{G},X} \Delta_{\overline{G},X}^{-1} \Delta_{G_1,X} \overline{A} \Delta_{G,\overline{H}} A \\ &= \pm A^H \overline{A}^H X \Delta_{G_1,X} \overline{A} \Delta_{G,\overline{H}} A. \end{aligned}$$

Thus, we have

$$\begin{aligned} \Pi_1 + \Pi_2 + \Pi_3 &= A^H \left(\pm I_n \mp \overline{H} \Delta_{G,\overline{H}} G \right) \overline{A}^H X \Delta_{G_1,X} \overline{A} \Delta_{G,\overline{H}} A \\ &= \pm A_1^H X \Delta_{G_1,X} A_1, \end{aligned}$$

where

$$A_1 := \overline{A}\Delta_{G,\overline{H}}A. \tag{9}$$

This concludes that (3) can be transformed into the standard DAREs

$$X = H_1 + A_1^H X \Delta_{G_1,X} A_1. \tag{10}$$

Starting with a fixed point iteration, we propose a 3-term iterative method in Section 3. We show that this method has a semigroup property and is equivalent to the structured doubling algorithm (SDA), i.e.,

$$\begin{aligned} A_{k+1} &= A_k(I + G_k H_k)^{-1} A_k, \\ G_{k+1} &= G_k + A_k(I + G_k H_k)^{-1} G_k A_k^H, \\ H_{k+1} &= H_k + A_k^H H_k (I + G_k H_k)^{-1} A_k, \end{aligned}$$

under a specific transformation. Though the SDA is known for its efficiency of computing the solution of DARE [15] with quadratic convergence, we use this semigroup property to build up an accelerated iterative method with the rate of convergence of any desired order.

The paper is organized as follows. In Section 2 and Section 3, we propose, respectively, sufficient conditions for the existence of unique positive definite solutions of (3) by means of the solvable analysis of (1). Based on the fixed point iteration, we construct a way to solve the unique positive definite solutions of (3). We show in Theorem 3.1 that this way satisfies a semigroup property. In Section 4, we apply this property to build up an accelerated approach to compute the positive definite solution with r-superlinear convergence of order r, for any integer $r > 1$. In Section 5, we examine two examples to illustrate the capacity and efficiency of our proposed accelerated technique. In Section 6, we make our concluding remarks.

In the subsequent discussion, the symbols $\mathbb{C}^{n\times n}$ and $\mathbb{P}_n$ stand for the set of $n \times n$ complex matrices and positive definite matrices, respectively. We denote the $m \times m$ identity matrix by I_m, the conjugate matrix of A by $\overline{A}$, the conjugate transpose matrix of A by A^H, the spectrum of A by $\sigma(A)$ and use $\rho(A)$ to denote the spectral radius of a square matrix A. We use the symbol $A > 0$ (or $A \geq 0$) to represent that A is a Hermitian positive definite matrix (or a Hermitian positive semidefinite matrix) and the Loewner order $A > B$ (or $A \geq B$) if $A - B > 0$ (or $A - B \geq 0$). A matrix operator f is order preserving on $\mathbb{P}_n$ if $f(A) \geq f(B)$ when $A \geq B$ and $A, B \in \mathbb{P}_n$.

2. Solvability properties

In this section, we present sufficient conditions for unique existence of the positive definite solutions of (3). To this end, we start by investigating the solvability of the standard conjugate Stein matrix equation:

$$X = Q + A^H \overline{X} A, \tag{11}$$

where $A \in \mathbb{C}^{n\times n}$ and $Q \in \mathbb{P}_n$.

Its proof is based on the following well-known fact.

Lemma 2.1. *[2, Proposition 8.6,3.] Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of positive semidefinite matrices satisfying $A_j \geq A_i \geq 0$ if $j \geq i$, and assume that B is another positive semidefinite matrix satisfying $B \geq A_i$ for all $i > 0$. Then, $A := \lim_{i\to\infty} A_i$ exists and $B \geq A \geq 0$.*

Upon using Lemma 2.1, our next result is to propose a necessary and sufficient condition for the existence of a unique positive definite solution of (11).

Lemma 2.2. *The equation* (11) *has a unique positive definite solution if and only if $\rho(\overline{A}A) < 1$.*

Proof. Assume that X_p is the unique positive definite solution of (11). Thus, X_p is a solution of the equation:

$$X = Q + A^H\overline{Q}A + (\overline{A}A)^H X(\overline{A}A). \tag{12}$$

This implies that for any integer $k > 0$,

$$X_p = \sum_{i=0}^{k} \left((\overline{A}A)^i\right)^H (Q + A^H\overline{Q}A)(\overline{A}A)^i + \left((\overline{A}A)^{k+1}\right)^H X_p(\overline{A}A)^{k+1} > 0. \tag{13}$$

Since $Q + A^H\overline{Q}A > 0$ and X_p is positive definite, we see that

$$\sum_{i=0}^{\infty} \left((\overline{A}A)^i\right)^H (Q + A^H\overline{Q}A)(\overline{A}A)^i$$

converges, and hence $\rho(\overline{A}A) < 1$.

Conversely, assume that $\rho(\overline{A}A) < 1$. Let $A \otimes B$ be the Kronecker product of matrices A an B. Observe from (12) that

$$\left(I - (\overline{A}A)^\top \otimes (\overline{A}A)^H\right) \mathrm{vec(X)} = \mathrm{vec(Q + A^H\overline{Q}A)},$$

where $\mathrm{vec}(\cdot)$ is the column stretching function defined as

$$\mathrm{vec(A)} = [\mathrm{a_{11}}, \cdots, \mathrm{a_{m1}}, \cdots, \mathrm{a_{1n}}, \cdots, \mathrm{a_{mn}}]^\top$$

for any $m \times n$ matrix $A = [a_{ij}]$. This implies that the solution, say X_p, of (12) exists. Also, from [26, Lemma 12], we know that (11) has a solution if $\rho(\overline{A}A) < 1$. Following from (13), we have

$$X_p = \sum_{i=0}^{\infty} \left((\overline{A}A)^i\right)^H (Q + A^H\overline{Q}A)(\overline{A}A)^i,$$

which is positive definite. Once (12) has a unique positive definite solution, this solution is also the unique positive definite solution of (11). This completes the proof.

□

Note that Lemma 2.2 enables us to discuss the solvability of (3). To make our discussion more clearly and explicitly, the rest of this section is divided into two parts, respectively: One is for (3a) and the other is for (3b).

2.1. The solvability of (3a)

Using the formula in (4), let P_1 be a set defined by

$$P_1 = \{X > 0 | X \geq \mathcal{F}_+(X)\}. \tag{14}$$

Consider the fixed point iteration

$$X_{k+1} = \mathcal{F}_+(X_k) \tag{15}$$

with $X_1 = H$. It is easy to see that $\{X_k\}$ is a monotone increasing matrix sequence with respect to the Loewner order. Once P_1 is nonempty, choose a matrix X_{P_1} in P_1. It can be shown by induction that for any integer $k > 0$, $X_k \leq X_{P_1}$. This is because for $k = 1$, it is true that $X_{P_1} \geq H = X_1$. Assume that this statement is true for $k = n$. Then,

$$\begin{aligned} X_{P_1} &\geq H + A^H \overline{X}_{P_1} \Delta_{G,\overline{X}_{P_1}} A \\ &\geq X_{n+1} + A^H \left((I + \overline{X}_{P_1} G)^{-1} \overline{X}_{P_1} - (I + \overline{X}_n G)^{-1} \overline{X}_n \right) A \\ &= X_{n+1} + A^H \left((\overline{X}_{P_1}^{-1} + G)^{-1} - (\overline{X}_n^{-1} + G)^{-1} \right) A \geq X_{n+1}. \end{aligned}$$

Hence, the sequence $\{X_k\}$ converges, i.e.,

$$X_* := \lim_{k \to \infty} X_k \tag{16}$$

exists and satisfies (3a).

In addition, let

$$T_X = \Delta_{G,\overline{X}} A, \tag{17}$$

and

$$\widehat{T}_X = \overline{T}_X T_X. \tag{18}$$

It can be seen that for this $X_{P_1} \in P_1$, we know that

$$\begin{aligned} X_{P_1} - T_{X_{P_1}}^H \overline{X}_{P_1} T_{X_{P_1}} &= X_{P_1} - A^H (I - \overline{X}_{P_1} G (I + \overline{X}_{P_1} G)^{-1}) \overline{X}_{P_1} \Delta_{G,\overline{X}_{P_1}} A \\ &= X_{P_1} - A^H \overline{X}_{P_1} T_{X_{P_1}} + A^H \overline{X}_{P_1} G \Delta_{\overline{X}_{P_1},G} \overline{X}_{P_1} \Delta_{G,\overline{X}_{P_1}} A \\ &\geq H + (\Delta_{\overline{X}_{P_1},G} \overline{X}_{P_1} A)^H G (\Delta_{\overline{X}_{P_1},G} \overline{X}_{P_1} A), \end{aligned}$$

which yields

$$X_{P_1} \geq \widehat{T}_{X_{P_1}}^H X_{P_1} \widehat{T}_{X_{P_1}} + H,$$

or, equivalently,

$$X_{P_1} \geq \sum_{k=0}^{m} (\widehat{T}_{X_{P_1}}^H)^k H \widehat{T}_{X_{P_1}}^k$$

for any integer $m > 0$. This implies that the specific matrix computation $\widehat{T}_{X_{P_1}}$ satisfying

$$\rho(\widehat{T}_{X_{P_1}}) < 1.$$

In particular, it can be seen that if X solves (3a),

$$\begin{aligned}
\widehat{T}_X &= \Delta_{\overline{G},X}\overline{A}\Delta_{G,\overline{X}}A \\
&= \Delta_{\overline{G},X}\overline{A}\left(I + G\left(\overline{H} + \overline{A}^H X\Delta_{\overline{G},X}\overline{A}\right)\right)^{-1} A \\
&= \Delta_{\overline{G},X}\overline{A}\left(\Delta_{G,\overline{H}} - \Delta_{G,\overline{H}}G\overline{A}^H X\left(I + (\overline{G} + \overline{A}\Delta_{G,\overline{H}}G\overline{A}^H)X\right)^{-1}\overline{A}\Delta_{G,\overline{H}}\right) A \\
&= \Delta_{\overline{G},X}\overline{A}(\Delta_{G,\overline{H}} - \Delta_{G,\overline{H}}G\overline{A}^H X\Delta_{G_1,X}\overline{A}\Delta_{G,\overline{H}})A \\
&= \Delta_{\overline{G},X}A_1 - \Delta_{\overline{G},X}\overline{A}\left(\Delta_{G,\overline{H}}G\overline{A}^H X\Delta_{G_1,X}\overline{A}\Delta_{G,\overline{H}}\right) A \\
&= \Delta_{\overline{G},X}\left(I + G_1 X - \overline{A}\Delta_{G,\overline{H}}G\overline{A}^H X\right)\Delta_{G_1,X}A_1 \\
&= \Delta_{G_1,X}A_1.
\end{aligned}$$

To make it clearly, we summarize results as follows.

theorem 2.1. *Let P_1, T_X, and $\widehat{T}_X$ be the notation defined in* (14), (17), *and* (18), *respectively.*

(a) *If P_1 is nonempty, then there exists a positive definite solution of* (3a).

(b) *If $X \in P_1$, then $\rho(\widehat{T}_X) < 1$.*

(c) *If X solves Eq.* (3a), *then $\widehat{T}_X = \Delta_{G_1,X}A_1$.*

Inspired by our above findings, we now propose a necessary and sufficient condition for the existence and uniqueness of the positive definite solution of (3a).

theorem 2.2. *The set P_1 is nonempty if and only if there exists a unique positive definite solution of* (3a).

Proof. If P_1 is nonempty, Theorem (2.1) implies that there exists a positive definite solution of (3a). Next, we show that the positive definite solution of (3a) is unique. To this end, let X_1 and X_2 be two positive definite solutions of (3a). It follows that

$$\begin{aligned}
X_1 - X_2 &= A^H(I + \overline{X}_1 G)^{-1}(\overline{X}_1(I + G\overline{X}_2) - (I + \overline{X}_1 G)\overline{X}_2)(I + G\overline{X}_2)^{-1}A \\
&= T_{X_1}^H(\overline{X}_1 - \overline{X}_2)T_{X_2}.
\end{aligned}$$

Subsequently, we have

$$X_1 - X_2 = (\widehat{T}_{X_1}^H)^k(X_1 - X_2)\widehat{T}_{X_2}^k$$

for any integer $k > 0$, which gives rise to the fact that

$$X_1 - X_2 = \lim_{k\to\infty}(\widehat{T}_{X_1}^H)^k(X_1 - X_2)\widehat{T}_{X_2}^k = 0.$$

This is because X_1 and X_2 are in P_1 and from Theorem 2.1 (b), we know that $\rho(\widehat{T}_{X_1}) < 1$ and $\rho(\widehat{T}_{X_2}) < 1$.

Conversely, if there exists a unique solution of (3a), it is trivial that P_1 is nonempty. □

Note that Theorem 2.2 provides a necessary and sufficient condition for the existence of a unique positive definite solution of (3a). However, the assumption $P_1 \neq \phi$ is not easy to check. A useful sufficient condition for the existence of a unique positive definite solution of (3a) can be written as follows.

corollary 2.1. *Assume that the coefficient matrix A in* (3a) *satisfies $\rho(\overline{A}A) < 1$. Then, there exists a unique positive definite solution to* (3a).

Proof. Since $\rho(\overline{A}A) < 1$, it follows from Lemma 2.2 that there exists a positive definite matrix X_1 such that

$$X_1 = H + A^H \overline{X}_1 A \geq H + A^H \overline{X}_1 A - (\overline{X}_1 A)^H (G^{-1} + \overline{X}_1)^{-1} (\overline{X}_1 A) = \mathcal{F}_+(X_1).$$

Thus, P_1 is nonempty. From Theorem 2.1, there exists a positive definite solution of (3a).

□

2.2. The solvability of (3b)

In this section, we discuss a counterpart of (3a). To start with, we let P_2 be a set defined by

$$P_2 := \{X > 0 | H \geq X \geq \mathcal{F}_-(X)\}, \tag{19}$$

and let H_1, G_1, and A_1 be matrices defined in (8), (7) and (9) with minus signs. Note that the set P_2 is nonempty, since $H \in P_2$.

Our purpose in this section is to show that there exists one and only one positive matrix $X \in P_2$, and X satisfies (3b) and $\rho(\widehat{T}_X) < 1$. To prove these facts and make this work self-contained, we recall the result for nonlinear matrix equations in [7, Lemma 5.5] and [7, Theorem 5.6].

theorem 2.3. *Let $\mathcal{F}(X) = -X + X\mathcal{H}(X)X$ be an order preserving mapping of $\mathbb{P}_n$ into $n \times n$ negative definite matrices. Assume that $\mathcal{H}$ satisfies the following two properties:*

$$\begin{aligned} \mathcal{H}(X)X\mathcal{H}(X) &\leq \mathcal{H}(X), \\ \mathcal{H}(Y) - \mathcal{H}(X) &= \mathcal{H}(X)(X - Y)\mathcal{H}(Y). \end{aligned}$$

Then, there is a unique positive definite solution X to the equation

$$X - A^H X A + A^H X \mathcal{H}(X) X A = H,$$

where $A, H \in \mathbb{C}^{n\times n}$ and $H \geq 0$. Moreover, for this solution X, the spectrum radius of the matrix $\widehat{T}_X$ defined by

$$\widehat{T}_X = A - \mathcal{H}(X) X A$$

satisfying $\rho(\widehat{T}_X) < 1$.

Corresponding to (3), we consider the case that $\mathcal{F}(X) = -X + X\mathcal{H}(X)X$, where $\mathcal{H}(X) = \Delta_{G_1,X}G_1$ and show that this $\mathcal{F}(X)$ satisfies the requirement of Theorem 2.3.

corollary 2.2. *Let $\mathcal{F}(X) = -X + X\mathcal{H}(X)X$ be a mapping of $\mathbb{P}_n$, where $\mathcal{H}(X) = \Delta_{G_1,X}G_1$ and $G_1 > 0$. Then,*

(a) *$\mathcal{F}(X) = -X\Delta_{G_1,X}$, i.e., $0 \leq X \leq Y$ implies that $\mathcal{F}(X) \geq \mathcal{F}(Y)$.*

(b) *$\mathcal{H}(X)X\mathcal{H}(X) \leq \mathcal{H}(X)$ and $\mathcal{H}(Y) - \mathcal{H}(X) = \mathcal{H}(X)(X - Y)\mathcal{H}(Y)$.*

(c) *There is a unique positive definite solution X to the DARE*

$$X - A_1^H X\Delta_{G_1,X}A_1 = H_1,$$

where $H_1 > 0$. Moreover, for this solution X, $\rho(\widehat{T}_X) < 1$ with the matrix $\widehat{T}_X$ defined by $\widehat{T}_X = \Delta_{G_1,X}A_1$.

Proof. Clearly, $\mathcal{F}(X) = -X + X\Delta_{G_1,X}G_1X = -X\Delta_{G_1,X}$. Following from a direct computation, we see that $\mathcal{H}(X)$ satisfies the following two properties:

$$\begin{aligned}
\mathcal{H}(X)X\mathcal{H}(X) &= (I + G_1X)^{-1}G_1X(I + G_1X)^{-1}G_1 \\
&= (I + G_1X)^{-1}(I - (I + G_1X)^{-1})G_1 \\
&= \mathcal{H}(X) - \Delta_{X,G_1}^H G_1\Delta_{X,G_1} \leq \mathcal{H}(X), \\
\mathcal{H}(Y) - \mathcal{H}(X) &= \mathcal{H}(X)(X - Y)\mathcal{H}(Y).
\end{aligned}$$

Note that

$$\begin{aligned}
H_1 &= X - A_1^H X\Delta_{G_1,X}A_1 \\
&= X - A_1^H XA_1 + A_1^H XG_1\Delta_{X,G_1}XA_1,
\end{aligned}$$

and

$$\widehat{T}_X = (I - (I + G_1X)^{-1}G_1X)A_1 = A_1 - \mathcal{H}(X)XA_1.$$

Thus, part (c) follows directly from Remark 2.3, which completes the proof. $\square$

Based on Theorem 2.3, we have the condition of the existence of a unique positive definite solution of (3b).

theorem 2.4. *Let G_1 and H_1 be two matrices defined by* (7) *and* (8) *with minus signs, and let P_2 be the set in* (19).

(a) *If $H_1 > 0$, then there exists a positive definite matrix X in P_2 such that X is also a solution of* (3b).

(b) *If $G_1 > 0$ and $H_1 > 0$, then the positive definite solution of* (3b) *exists uniquely. In particular, $\widehat{T}_X := \Delta_{\overline{G},X}\overline{A}\Delta_{G,\overline{X}}A = \Delta_{G_1,X}A_1$ and $\rho(\widehat{T}_X) < 1$.*

Proof. It is true that the set $[H_1, H] = \{X \in \mathbb{P}_n | H_1 \leq X \leq H\}$ is a compact convex subset of the Banach space $\mathbb{C}^{n\times n}$ with an unitarily invariant matrix norm. Also, the operator $\mathcal{F}_-$ maps $[H_1, H]$ into itself, since

$$H_1 = \mathcal{F}_-(H) \leq \mathcal{F}_-(X) \leq H$$

for $H_1 \leq X \leq H$. It then follows from the Schauder fixed point theorem (see, e.g. [21]) that $\mathcal{F}_-$ has a fixed point X in $[H_1, H]$. This implies that there exists a element $X \in P_2$ and X solves (3b).

Considering this solution X of (3b), it follows that X is a solution of the equation

$$X = F_-^{(2)}(X) = H_1 + A_1^H X \Delta_{G_1,X} A_1. \tag{20}$$

Note that the uniqueness of the solution of (3b) is guaranteed, once the solution of (20) is unique. By Corollary 2.2, this is immediately true, since $G_1 > 0$ and $H_1 > 0$. Also,

$$\begin{aligned}
\widehat{T}_X &= \Delta_{\overline{G},X}\overline{A}\Delta_{G,\overline{X}}A \\
&= \Delta_{\overline{G},X}\overline{A}\left(I + G\left(\overline{H} - \overline{A}^H X \Delta_{\overline{G},X}\overline{A}\right)\right)^{-1} A \\
&= \Delta_{\overline{G},X}\overline{A}\left(\Delta_{G,\overline{H}} + \Delta_{G,\overline{H}} G\overline{A}^H X \left(I + (\overline{G} - \overline{A}\Delta_{G,\overline{H}}G\overline{A}^H)X\right)^{-1} \overline{A}\Delta_{G,\overline{H}}\right) A \\
&= \Delta_{\overline{G},X}\overline{A}(\Delta_{G,\overline{H}} + \Delta_{G,\overline{H}}G\overline{A}^H X \Delta_{G_1,X}\overline{A}\Delta_{G,\overline{H}})A \\
&= \Delta_{\overline{G},X}A_1 + \Delta_{\overline{G},X}\overline{A}\left(\Delta_{G,\overline{H}}G\overline{A}^H X\Delta_{G_1,X}\overline{A}\Delta_{G,\overline{H}}\right) A \\
&= \Delta_{\overline{G},X}\left(I + G_1 X + \overline{A}\Delta_{G,\overline{H}}G\overline{A}^H X\right)\Delta_{G_1,X}A_1 \\
&= \Delta_{G_1,X}A_1.
\end{aligned}$$

By Corollay 2.2, $\rho(\widehat{T}_X) < 1$, since

$$\widehat{T}_X = \Delta_{G_1,X}A_1 = (I - (I + G_1 X)^{-1} G_1 X)A_1 = A_1 - \mathcal{H}(X)XA_1,$$

which completes the proof.

□

3. Iterative method and convergence analysis

In this section, a method originated from the fixed point iteration will be presented to solve (3) indirectly. A direct method to solve (3) is referred to appendix 6.1 for the details. We show that our proposed approach can give rise to an accelerated way with the rate of r-superlinear convergence up to any desired order in Section 4.

Let $\mathcal{R}(X) = H_1 + A_1^H X \Delta_{G_1,X} A_1$ represent the computation of the right-hand side of (10), and let X_d be a solution of (10), that is,

$$X_d = \mathcal{R}(X_d).$$

Following from a similar derivation for (10), it can be seen that

$$X_d = \mathcal{R}(\mathcal{R}(X_d)) = H_2 + A_2^H X_d \Delta_{G_2,X_d} A_2,$$

where $A_2 = A_1 \Delta_{G_1,H_1} A_1$, $G_2 = G_1 + A_1 \Delta_{G_1,H_1} G_1 A_1^H$, and $H_2 = H_1 + A_1^H H_1 \Delta_{G_1,H_1} A_1$. Continually, we have

$$X_d = \mathcal{R}^{(k-1)}(\mathcal{R}(X_d)) = H_k + A_k^H X_d \Delta_{G_k,X_d} A_k,$$

where A_k, G_k, and H_k for $k = 1, 2, \ldots$, be three matrices denoted by

$$A_k = A_1 \Delta_{G_{k-1},H_1} A_{k-1}, \tag{21a}$$
$$G_k = G_1 + A_1 \Delta_{G_{k-1},H_1} G_{k-1} A_1^H, \tag{21b}$$
$$H_k = H_{k-1} + A_{k-1}^H H_1 \Delta_{G_{k-1},H_1} A_{k-1}, \tag{21c}$$

with initial matrices G_1, H_1, and A_1 defined by (7), (8), and (9), respectively. Note that the iterative method given by (21) provide a direct way to solve (10) and an indirect way to solve (3). We show in the next result that (21) has a semigroup property. Its proof is quite lengthy, though it is done by mathematical induction. To the reader's interest, we put the proof in the appendix 6.2.

theorem 3.1. *If all sequences of matrices generated by* (21) *are well-defined, then the sequence* (A_k, G_k, H_k) *satisfies the following property:*

$$A_{i+j} = A_j (I + G_i H_j)^{-1} A_i, \tag{22a}$$
$$G_{i+j} = G_j + A_j (I + G_i H_j)^{-1} G_i (A_j)^H, \tag{22b}$$
$$H_{i+j} = H_i + (A_i)^H H_j (I + G_i H_j)^{-1} A_i, \tag{22c}$$

for all integers $i, j \geq 1$.

Based on Theorem 3.1, we have $H_k = H_1 + A_1^H H_{k-1} \Delta_{G_1,H_{k-1}} A_1$. Hence, the iteration in (21) is called the fixed point iteration, since its purpose is to construct a convergent sequence H_k to solve (10).

From Theorem 2.1, we know that if the coefficient matrix A satisfies $\rho(\overline{A}A) < 1$, then the set $P_1$is nonempty and the positive definite solution of (3a) uniquely exists. Our next result is to prove that the sequence of (A_k, G_k, H_k) in (21) is well-defined, and H_k tends to this positive definite solution.

Lemma 3.1. *Let* $A, G, H \in \mathbb{C}^{n \times n}$ *and* $G, H > 0$ *be coefficient matrices in* (3a). *Then,*

(a) (A_k, Q_k, H_k) *is well-defined for all integers* $k \geq 1$.

(b) *If* $X \in P_1$, *then* X *is an upper bound of* $\{H_k\}$. *In particular,*

$$X \geq H_k \geq H_{k-1} \geq \cdots \geq H_1 \geq H.$$

(c) *If* $\rho(\overline{A}A) < 1$, H_k *converges to the unique positive definite solution of* (3a) *as* $k \to \infty$.

Proof. First, the proof of part (a) is completed, once the matrix Δ_{G_{k-1},H_1} exists for any integer $k \geq 2$. This suffices to show that the product of any eigenvalue of G_{k-1} and H_1 is not equal to -1. From (7) and (8), it can be seen that

$$G_1 = \overline{G}^H + \overline{A}G^H \Delta_{\overline{H}^H,G^H} \overline{A}^H = {G_1}^H > 0, \tag{23}$$

$$H_1 = H^H + A^H \Delta_{\overline{H}^H,G^H} \overline{H}^H A = H_1^H > 0, \tag{24}$$

since $G, H > 0$. Similarly, we have $G_k = G_k^H > 0$ and $H_k = H_k^H > 0$ for any integer $k \geq 2$. This implies that $\sigma(G_{k-1}H_1) \subseteq \mathbb{R}^+$, since $G_{k-1} > 0$ and $H_1 > 0$, which completes the proof of part (a). Here $\mathbb{R}^+$ is the positive real line.

Second, if there exists $X \in P_1$, then $X \geq H$. Note that

$$H_k = \mathcal{F}_+^{(2)}(H_{k-1}) = \mathcal{F}_+^{(2(k-1))}(H_1) = \mathcal{F}_+^{(2k-1)}(H), \tag{25}$$

for all integers $k \geq 2$. Thus, we have

$$X - H_k \geq \mathcal{F}_+^{(2k-1)}(X) - \mathcal{F}_+^{(2k-1)}(H) \geq 0, \tag{26}$$

since $\mathcal{F}_+$ is an order preserving operator. It follows from (25) and (26) that

$$X \geq H_k \geq H_{k-1} \geq \cdots \geq H_1 \geq H,$$

which completes the proof of part (b).

Third, since $\rho(\overline{A}A) < 1$, Theorem 2.1 implies that there exists $X \in P_1$. It follows from Lemma 2.1 that the sequence $\{H_k\}$ converges, i.e.

$$H_* := \lim_{k\to\infty} H_k$$

exists and satisfies (10). By Theorem 2.1, there exists a unique positive definite solution to (3a). Since the solution of (3a) is also a solution of (10). Provided $G_1 > 0$, Corollary 2.2 implies that (10) can have only one positive definite solution, which completes the proof.

□

For (3b), a similar result can be derived as follows. Since the proof is similar to Lemma 3.1, we omit our proof here.

Lemma 3.2. *For* (3b)*, let* G_1 *and* H_1 *be matrices defined by* (7) *and* (8) *with minus signs, and let* P_2 *be the set in* (19)*. Suppose that* H_1 *and* $G_1 > 0$*. Then,*

(a) (A_k, Q_k, H_k) *is well-defined for all integers* $k \geq 1$.
(b) *If* $X \in P_2$*, then* X *is an upper bound of* $\{H_k\}$*. In particular,*

$$H \geq X \geq H_{k+1} \geq H_k \geq \cdots \geq H_1.$$

(c) H_k *tends to the unique positive definite solution of Eq.* (3b) *as* $k \to \infty$.

From Lemma 3.1 and Lemma 3.2, we have the numerical behavior of the sequence $\{H_k\}$. To our interest, we would like to predict the behavior of the sequence $\{G_k\}$ in (21). We thus consider the following dual matrix equations

$$X = \overline{G} + \overline{A}\,\overline{X}(I + \overline{H}\,\overline{X})^{-1}\overline{A}^H, \tag{27a}$$
$$X = \overline{G} - \overline{A}\,\overline{X}(I + \overline{H}\,\overline{X})^{-1}\overline{A}^H. \tag{27b}$$

It has been shown in Theorem 2.1 and Theorem 2.4 that there exists a unique positive definite solution X of (3), once certain conditions are satisfied. Here, we assume that the coefficient matrix A is nonsingular and define $Y = -X^{-1}$, where X is the solution of (27). Following from (27), we have

$$A^{-1}(\overline{X} - G)A^{-H} = \pm(X^{-1} + H)^{-1}.$$

This implies that

$$X^{-1} + H = \pm A^H(\overline{X} - G)^{-1}A,$$

That is,

$$Y = H \pm A^H\overline{Y}(I + G\overline{Y})^{-1}A,$$

which is exactly equivalent to the matrix equation (3). Like Theorem 2.1 and Theorem 2.4, we thus have the following result.

theorem 3.2. *Assume that A is nonsingular. Then,*

1. *There exists a unique negative definite solution to* (3a) *if* $\rho(A\overline{A}) < 1$.
2. *There exists a unique negative definite solution to* (3b) *if* $G_1 > 0$ *and* $H_1 > 0$.

Now, we would like to investigate the relationship between the sequence $\{G_k\}$ and the dual equations (27). For the sake of simplicity, let $\mathcal{G}_\pm(X)$ be the matrix operator defined by

$$\mathcal{G}_\pm(X) = \overline{G} \pm \overline{A}\,\overline{X}\Delta_{\overline{H},\overline{X}}\overline{A}^H.$$

Then, the dual equations (27) can be rewritten as

$$X = \mathcal{G}_\pm(X).$$

Analogous to the case of operator $\mathcal{F}_\pm$, we have the following formula

$$X = \mathcal{G}_\pm^{(2)}(X) = \widetilde{H}_1 + \widetilde{A}_1^H X \Delta_{\widetilde{G}_1,X}\widetilde{A}_1,$$

where

$$\widetilde{A}_1 = \overline{\overline{A}^H}\Delta_{\overline{H},G}\overline{A}^H = A_1^H, \tag{28a}$$
$$\widetilde{G}_1 = H \pm A^H\Delta_{\overline{H},G}\overline{H}A = H_1, \tag{28b}$$
$$\widetilde{H}_1 = \overline{G} \pm \overline{A}G\Delta_{\overline{H},G}\overline{A}^H = G_1, \tag{28c}$$

or even more,

$$X = \mathcal{G}_{\pm}^{(2k)}(X) = \widetilde{H}_k + \widetilde{A}_k^H X \Delta_{\widetilde{G}_k, X} \widetilde{A}_k,$$

where

$$\widetilde{A}_k = \widetilde{A}_1 \Delta_{\widetilde{G}_{k-1}, \widetilde{H}_1} \widetilde{A}_{k-1}, \tag{29a}$$
$$\widetilde{G}_k = \widetilde{G}_1 + \widetilde{A}_1 \Delta_{\widetilde{G}_{s-1}, \widetilde{H}_1} \widetilde{G}_{s-1} \widetilde{A}_1^H, \tag{29b}$$
$$\widetilde{H}_k = \widetilde{H}_{k-1} + \widetilde{A}_{k-1}^H \widetilde{H}_1 \Delta_{\widetilde{G}_{k-1}, \widetilde{H}_1} \widetilde{A}_{k-1}. \tag{29c}$$

By induction on k, it is true that

$$\widetilde{A}_k = A_k^H,\ \widetilde{G}_k = H_k,\ \widetilde{H}_k = G_k. \tag{30}$$

Thus, the sequence of matrices $(\widetilde{A}_k, \widetilde{G}_k, \widetilde{H}_k)$ generated by the iterations (21) with initial matrices $(\widetilde{A}_1, \widetilde{G}_1, \widetilde{H}_1) = (A_1^H, H_1, G_1)$ is well-defined, once the sequence of matrices (A_k, G_k, H_k) is well-defined. Let D_1 and D_2 be two sets defined by

$$D_1 = \{Y > 0 | Y \geq \mathcal{G}_+(Y)\}, \tag{31}$$
$$D_2 = \{Y > 0 | \overline{G} \geq Y \geq \mathcal{G}_-(Y)\}, \tag{32}$$

respectively. By (29), we have the following result. Its proof is similar to Lemma 3.1 and Lemma 3.2 and is omitted here.

Lemma 3.3. *Let $A, G, H \in \mathbb{C}^{n\times n}$ be the coefficient matrices of* (3) *such that $G, H > 0$. Then,*

1. *For* (3a),
   (a) *$(\widetilde{A}_k, \widetilde{G}_k, \widetilde{H}_k)$ is well-defined for all integers $k \geq 1$.*
   (b) *If $Y \in D_1$, then Y is an upper bound of $\{G_k\}$. In particular,*
   $$Y \geq G_k \geq G_{k-1} \geq \cdots \geq G_1 \geq \overline{G}.$$
   (c) *If $\rho(A\overline{A}) < 1$, G_k converges to the unique positive definite solution of* (27a) *as $k \to \infty$.*
2. *Assume that $G_1 > 0$ and $H_1 > 0$. For* (3b),
   (a) *$(\widetilde{A}_k, \widetilde{G}_k, \widetilde{H}_k)$ is well-defined for all integers $k \geq 1$.*
   (b) *If $Y \in D_2$, then Y is an upper bound of $\{G_k\}$. In particular,*
   $$\overline{G} \geq Y \geq G_{k+1} \geq G_k \geq \cdots \geq G_1.$$
   (c) *G_k tends to the unique positive definite solution of* (27b) *as $k \to \infty$.*

In summary, following from Lemmas 3.1, 3.2, and 3.3, we have the following main result of this section.

theorem 3.3. *Let $A, G, H \in \mathbb{C}^{n\times n}$ be the coefficient matrices of* (3) *such that $G, H > 0$. Consider the sequence of matrices (A_k, G_k, H_k) generated by iterations* (21) *with a given initial matrices (A_1, G_1, H_1) defined by* (9), (7), *and* (8), *respectively. Let $H_\infty = \lim_{\ell\to\infty} H_\ell$ and $G_\infty = \lim_{\ell\to\infty} G_\ell$. Then,*

1. *Assume that $\rho(A\overline{A}) < 1$. For* (3a),
   (a) *H_∞ is the unique positive definite solution to* (3a).
   (b) *$-G_\infty^{-1}$ is the unique negative definite solution to* (3a) *if A is nonsingular.*
2. *Assume that $H_1 > 0$ and $G_1 > 0$. For* (3b),
   (a) *H_∞ is the unique positive definite solution to* (3b).
   (b) *$-G_\infty^{-1}$ is the unique negative definite solution to* (3b) *if A is nonsingular.*

Remark 3.1. *It is interesting to ask whether the matrix $Y = -X^{-1}$, where X is the solution of* (27), *is still a negative positive solution of* (3) *if A is singular. To answer this question, we see that*

$$\begin{aligned} I + G\overline{Y} &= I - G(G \pm AX(I + HX)^{-1}A^H)^{-1} \\ &= I - G(G^{-1} \mp G^{-1}AX(I + HX \pm A^H G^{-1}AX)^{-1}A^H G^{-1}) \\ &= \pm AX(I + (H \pm A^H G^{-1}A)X)^{-1}A^H G^{-1}. \end{aligned}$$

Namely, rank$(I + G\overline{Y})$=rank(A). We conclude that the matrix $Y = -X^{-1}$ is not a solution of (3) *when A is singular, since $I + G\overline{Y}$ is not invertible.*

4. An acceleration of iterative method

Let $\{A_k, G_k, H_k\}$ be the sequence of matrices generated by (21). It has been shown in Theorem 3.1 that matrices A_k, G_k, and H_k, for each k, depend only on the subscripts in A_i, A_j, G_i, G_j, H_i, and H_j, once $i + j = k$. Our next algorithm is to fully take advantage of this invariance to design an algorithm with speed of convergence of any desired order.

Algorithm 4.1. (An accelerated iteration method to solve (3))

1. *Given a positive integer $r > 1$, let $(\widehat{A}_0, \widehat{G}_0, \widehat{H}_0) = (A_1, G_1, H_1)$ with initial matrices G_1, H_1, and A_1 defined by* (7), (8), *and* (9), *respectively;*
2. *For $k = 1, 2, \ldots$, iterate*

$$\begin{aligned} \widehat{A}_k &:= A_{k-1}^{(r-1)}(I_n + \widehat{G}_{k-1}H_{k-1}^{(r-1)})^{-1}\widehat{A}_{k-1}, \\ \widehat{G}_k &:= G_{k-1}^{(r-1)} + A_{k-1}^{(r-1)}(I_n + \widehat{G}_{k-1}H_{k-1}^{(r-1)})^{-1}\widehat{G}_{k-1}(A_{k-1}^{(r-1)})^H, \\ \widehat{H}_k &:= \widehat{H}_{k-1} + \widehat{A}_{k-1}^H H_{k-1}^{(r-1)}(I_n + \widehat{G}_{k-1}H_{k-1}^{(r-1)})^{-1}\widehat{A}_{k-1}, \end{aligned}$$

   until convergence (see Section 5 for example), where the sequence $(A_{k-1}^{(r-1)}, G_{k-1}^{(r-1)}, H_{k-1}^{(r-1)})$ is defined in step 3.

3. *For $\ell = 1, \cdots, r-2$, iterate*

$$
\begin{aligned}
A_{k-1}^{(\ell+1)} &:= A_{k-1}^{(\ell)}(I_n + \widehat{G}_{k-1}H_{k-1}^{(\ell)})^{-1}\widehat{A}_{k-1},\\
G_{k-1}^{(\ell+1)} &:= G_{k-1}^{(\ell)} + A_{k-1}^{(\ell)}(I_n + \widehat{G}_{k-1}H_{k-1}^{(\ell)})^{-1}\widehat{G}_{k-1}(A_{k-1}^{(\ell)})^H,\\
H_{k-1}^{(\ell+1)} &:= \widehat{H}_{k-1} + \widehat{A}_{k-1}^H H_{k-1}^{(\ell)}(I_n + \widehat{G}_{k-1}H_{k-1}^{(\ell)})^{-1}\widehat{A}_{k-1},
\end{aligned}
$$

with $(A_{k-1}^{(1)}, G_{k-1}^{(1)}, H_{k-1}^{(1)}) = (\widehat{A}_{k-1}, \widehat{G}_{k-1}, \widehat{H}_{k-1})$.

By Theorem 3.1, we have the following result. Its proof is straightforwardly done by induction. We thus omit the proof here.

Remark 4.1. *If (A_k, G_k, H_k) for all integers $k \geq 1$ is well-defined, that*

$$(\widehat{A}_k, \widehat{G}_k, \widehat{H}_k) = (A_{r^k}, G_{r^k}, H_{r^k})$$

for all integers $k \geq 1$.

The convergence analysis of Algorithm 4.1 can be done by means of the following properties. Since the proof is long and tedious, we put it in Appendix 6.3.

Lemma 4.1. *Assume that (A_k, G_k, H_k) is a well-defined sequence of matrices from (21) and this sequence is convergent. Let*

$$
\begin{aligned}
H_\infty &= \lim_{k\to\infty} H_k, \quad G_\infty = \lim_{k\to\infty} G_k,\\
T_k &= \Delta_{G_k, H_\infty} A_k, \quad S_k = A_k \Delta_{G_\infty, H_k},
\end{aligned} \tag{33}
$$

for all integers $k \geq 1$. Then, the following three conditions are satisfied.

1. *$T_k = T_1^k$ and $S_k = S_1^k$.*
2. *$H_\infty - H_k = T_k^H H_\infty A_k = T_k^H(H_\infty^{-1} + G_k)T_k$ and $G_\infty - G_k = S_k G_\infty A_k^H = S_k(G_\infty^{-1} + H_k)S_k^H$.*
3. *$\sigma(T_1) = \sigma(S_1^H)$.*

Let all the sequences in Algorithm 4.1 be well-defined. Our next result is to show that once $\rho(T_1) < 1$, the convergence speed of $(\widehat{A}_k, \widehat{G}_k, \widehat{H}_k)$ is r-superlinearly with order r, for any integer $r > 0$. The definition of r-superlinear convergence is referred to [12, Definition 4.1.3.].

theorem 4.1. *Suppose that $\{\widehat{A}_k, \widehat{G}_k, \widehat{H}_k\}$ is the sequence of matrices generated by iterations (21) and be well-defined and convergent. Let H_∞, G_∞ and T_k, S_k, for all integers $k \geq 1$, be matrices defined by (33). Then,*

$$
\begin{aligned}
&\limsup_{k\to\infty} \sqrt[r^k]{\|\widehat{A}_k\|} \leq \rho(T_1), \quad \limsup_{k\to\infty} \sqrt[r^k]{\|G_\infty - \widehat{G}_k\|} \leq \rho(T_1)^2,\\
&\limsup_{k\to\infty} \sqrt[r^k]{\|H_\infty - \widehat{H}_k\|} \leq \rho(T_1)^2.
\end{aligned}
$$

Proof. From Lemma 4.1, we know that $\widehat{A}_k = A_{r^k} = (I + G_{r^k}H_\infty)T_1^{r^k}$, $G_\infty - \widehat{G}_k = S_{r^k}(G_\infty^{-1} + H_{r^k})S_{r^k}^H$, and $H_\infty - \widehat{H}_k = T_{r^k}^H(H_\infty^{-1} + G_{r^k})T_{r^k}$. It follows that

$$\begin{aligned}
&\limsup_{k\to\infty} \sqrt[r^k]{\|\widehat{A}_k\|} = \limsup_{k\to\infty} \sqrt[r^k]{G_{r^k}(G_{r^k}^{-1} + H_\infty)T_1^{r^k}} \\
&\leq \limsup_{k\to\infty} \sqrt[r^k]{\|G_\infty\|} \cdot \limsup_{k\to\infty} \sqrt[r^k]{\|G_1^{-1} + H_\infty\|} \cdot \limsup_{k\to\infty} \sqrt[r^k]{\|T_1^{r^k}\|} = \rho(T_1), \\
&\limsup_{k\to\infty} \sqrt[r^k]{\|G_\infty - \widehat{G}_k\|} \leq \limsup_{k\to\infty} \sqrt[r^k]{\|S_1^{r^k}\|} \cdot \limsup_{k\to\infty} \sqrt[r^k]{\|(G_1^{-1} + H_\infty)\|} \cdot \\
&\limsup_{k\to\infty} \sqrt[r^k]{\|S_1^{r^k}\|} = \rho(T_1)^2, \\
&\limsup_{k\to\infty} \sqrt[r^k]{\|H_\infty - \widehat{H}_k\|} \leq \limsup_{k\to\infty} \sqrt[r^k]{\|T_1^{r^k}\|} \cdot \limsup_{k\to\infty} \sqrt[r^k]{\|(H_1^{-1} + G_\infty)\|} \cdot \\
&\limsup_{k\to\infty} \sqrt[r^k]{\|T_1^{r^k}\|} = \rho(T_1)^2.
\end{aligned}$$

Here, the last equalities follow from the well-known Gelfand's formula such that for any matrix norm $\|\cdot\|$, we have $\rho(A) = \limsup_{k\to\infty} \|A^k\|^{1/k}$.

□

5. Numerical experiments

Under the assumptions of Theorem 3.3, two numerical examples are used in this section to demonstrate the application of accelerated techniques given by Algorithm 4.1. We compare Algorithm 4.1 with the standard fixed point iterations:

$$X_{k+1} = \mathcal{F}_\pm(X_k), \quad \text{with } X_1 = H. \tag{34}$$

It can be shown that the convergence speed of (34) is r-linearly if $\rho(T_1) < 1$. The details for the convergence analysis can be found in Appendix 6.1. For clarity, two things should be emphasized here. First, the unique negative definite solution of (3) can be obtained by Algorithm 4.1 when A is nonsingular. That is, Algorithm 4.1 enable us to solve the unique positive and negative definite solutions, simultaneously. Second, when $\rho(T_1) \approx 1$, then iteration (34) could be very slow. However, this disadvantage can be overcome without any difficulty by Algorithm 4.1. While solving (3), we show that the use of Algorithm 4.1 tends to has less computational time and higher accuracy than the fixed point methods given by (34).

All computations were performed using MATLAB/version 2016b on MacBook Air with a 2.2 GHZ Intel Core i7 processor and 8 GB of memory. To gauge the effectiveness of our algorithm, we employ the parameters, residual

(Res) and the normalized residual (NRes) with definitions defined below:

$$\begin{aligned}\mathrm{Res} &:= \|X - \mathcal{F}_{\pm}(X)\|_F,\\ \mathrm{NRes} &:= \frac{\|X - \mathcal{F}_{\pm}(X)\|_F}{\|H\|_F + \|A\|_F^2 \|X\|_F \|\Delta_{G,X}\|_F},\end{aligned}$$

where X is an approximate maximum positive solution to (3). All iterations are terminated whenever Res or NRes is less than or equal to $n\mathbf{u}$, where $\mathbf{u} = 2^{-52} \cong 2.22 \times 10^{-16}$ is the machine zero.

Example 5.1. *Let $n = 100$ and $\widehat{G}, \widehat{H} \in \mathbb{R}^{n\times n}$ be two real diagonal matrices with given positive diagonal elements between 0 and 1. They are then reshuffled by the unitary matrix $Q \in \mathbb{C}^{n\times n}$ to form*

$$(G, H) = (Q^H \widehat{G} Q, Q^H \widehat{H} Q), \tag{35}$$

that is, in `MATLAB` *commands, we define*

$$\begin{aligned}\widehat{G} &= 1e2 * diag(rand(n)),\ \widehat{H} = 1e2 * diag(rand(n)),\\ Q &= orth(crandn(n)).\end{aligned}$$

For (3a)*, Theorem 2.1 implies that a unique positive definite solution exists, if $\rho(\overline{A}A) < 1$. To satisfy this constraint, let $\widehat{A}$ be a randomly generated square complex matrix, let a be a random number lying in the interval $(0, 1)$, and let $temp$ be the spectral radius of $\widehat{A}^H \widehat{A}$, namely,*

$$\begin{aligned}\widehat{A} &= crandn(n),\ a = rand,\\ temp &= max(abs(eig(conj(\widehat{A}) * \widehat{A}))).\end{aligned}$$

We then have a matrix

$$A = \sqrt{a} * \widehat{A}/\sqrt{temp} \tag{36}$$

satisfying $\rho(\overline{A}A) < 1$ so that the unique positive definite solution to (3a) *exists.*

For (3b)*, we have shown that the unique positive definite solution exists if $G_1 > 0$ and $H_1 > 0$. To this end, we repeatedly generate matrices A, G, and H by* (35) *and* (36) *until G_1 and $H_1 > 0$ are satisfied. We record numerical results in Table 2.*

Note that in Tables 1 and 2, the values in the second row are the results obtained using the standard fixed point method given in (34)*, and the values in the other rows are results obtained using Algorithm 4.1 with $r = 2, 3, 4, 5$, respectively. The minimal number of iterations (*MinIt*), the maximal number of iterations (*MaxIt*), the average number of iterations (*AveIt*), and the average elapsed times of iterations (*AveTime*) performed by the fixed point method and our algorithm are recorded by choosing 100 initial matrices (G, H, A) randomly, as are described above. Let N_1 and N_r, with $r = 2, 3, 4, 5$, be the least integer numbers satisfying*

$$\rho(T_1)^{N_1} < n \cdot \mathbf{u}, \quad \left(\rho(T_1)^2\right)^{r^{N_r}} < n \cdot \mathbf{u}, \quad r = 2, 3, 4, 5,$$

respectively. That is, N_1 and N_r, with $r = 2, 3, 4, 5$, are integer numbers defined by

$$N_1 = \left[\frac{\log_{10}(n \cdot \mathbf{u})}{\log_{10}(\rho(T_1))} \right] + 1 \tag{37}$$

and

$$N_r = \left[\log_r \left(\frac{\log_{10}(n \cdot \mathbf{u})}{\log_{10}(\rho(T_1)^2)} \right) \right] + 1. \tag{38}$$

*Here, the symbol $[x]$ denotes the floor of x, i.e., the largest integer less than or equal to x and $T_1 = \Delta_{G_1,H_\infty} A_1$. We then record in the fifth column of Tables 1 and 2 the number of iterations (*TheIt*) estimated by means of* (37) *and* (38)*. The records show that the estimated numbers* TheIt *are highly correlated to the numerical iterative numbers* AveIt*. This implies that in practice,* TheIt *can be served as a priori prediction of the possible iterative numbers. Also, we can see from the records in the columns of* AveIt *and* TheIt *that our algorithm outperform the fixed point method not only in the number of required iterations, but also in the elapsed times.*

Table 1: Numerical experiments by means of the fixed point method (F.P.) given in (34) and accelerated methods originated from Algorithm (Alg.) 4.1 ($r = 2, 3, 4, 5$) to solve (3a).

Method	MinIt	MaxIt	AveIt	TheIt	AveTime
F.P. with "+"	1	7	3.43	3.95	3.8526e-02
Alg. 4.1 with $r = 2$	1	2	1.41	1.23	2.1181e-02
Alg. 4.1 with $r = 3$	1	2	1.04	1.02	1.9381e-02
Alg. 4.1 with $r = 4$	1	1	1	1	2.1875e-02
Alg. 4.1 with $r = 5$	1	1	1	1	2.4443e-02

Table 2: Numerical experiments by means of the fixed point method (F.P.) given in (34) and accelerated methods originated from Algorithm (Alg.) 4.1 ($r = 2, 3, 4, 5$) to solve (3b).

Method	MinIt	MaxIt	AveIt	TheIt	AveTime
F.P. with "-"	1	10	3.8	4.35	4.2394e-02
Alg. 4.1 with $r = 2$	1	3	1.58	1.38	2.3283e-02
Alg. 4.1 with $r = 3$	1	2	1.08	1.05	1.9851e-02
Alg. 4.1 with $r = 4$	1	2	1.02	1.01	2.1808e-02
Alg. 4.1 with $r = 5$	1	1	1	1	2.4341e-02

In the next example, we show that as the value of $\rho(T_x)$ come closer to 1, the fixed point method will fail to converge, but our algorithm can converge with no difficulty.

Example 5.2. *If $n = 1$, the corresponding equations of* (3) *become to*

$$x \quad = \quad h \pm \frac{|a|^2 \overline{x}}{1 + g\overline{x}}, \tag{39}$$

where $a \in \mathbb{C}$ and the real numbers $g, h > 0$. To measure performance of different methods, four cases, i.e., $\rho(T_1) = \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, \sqrt{0.9999}$ with different parameters will be taken into account; namely, we set $g = 1$ and $a = \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, \sqrt{0.9999}$, and $\sqrt{0.99999}$ corresponding to $\rho(T_1) = \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}$ and $\sqrt{0.9999}$, respectively. For these parameters a and $\rho(T_1)$, there exists a unique positive definite solution x of (39) decided by

$$x = \frac{a}{\rho(T_1)} - 1 > 0,$$

since $\rho(T_1) = (1+x)^{-1}a$ and $|a|^2 < 1$. Thus, the resulting parameter

$$h = x + \frac{|a|^2\overline{x}}{1+g\overline{x}} > 0 \quad (or\ h = x - \frac{|a|^2\overline{x}}{1+g\overline{x}} > 0)$$

satisfies the constraint for (3). Also, for the minus case, i.e., $x = h - \frac{|a|^2\overline{x}}{1+g\overline{x}}$, we have

$$h_1 = h - \frac{|a|^2\overline{h}}{1+g\overline{h}} > 0,$$
$$g_1 = \overline{g} - \frac{|a|^2 g}{1+g\overline{h}} > 0.$$

Under conditions of Theorems 2.1 and 2.4 we see that there only exists a unique positive definite solution for both cases of (39).

*In Tables 3 and 4, the values in the second row, $r = 1$, are the results obtained using the fixed point method, and the values in the other rows are results obtained using Algorithm 4.1 with $r = 2, 3, 4, 5$, respectively. The number of iterations (*Its*), the output residual (*Res*), and the elapsed times of iterations (*Time*) performed by the fixed point method and our algorithm are recorded correspondingly.*

Table 3 shows that even with 10000 steps, the solution obtained from the fixed point method can only have accuracy up to 10^{-13}. What is worse, Table 4 shows that the fixed point method can hardly solve (39) with minus sign, even after 10000 steps. The residuals and elapsed times in Table 3 show that our accelerated technique can solve (39) more accurately and efficiently. Also, the number of iterations by the fixed point method increase dramatically, while those by our accelerated techniques only has a small increase. This implies that our algorithm could provide a more reliable way to obtain numerical solutions, even if the extreme case, i.e., $\rho(T_1) \approx 1$, is encountered.

6. Conclusion

In this paper, we propose sufficient conditions for the existence of a unique positive definite solution of (3). Note that an intuitive way to solve (3) is to

Table 3: The ITs, Res and Time for the problem $x = h + \frac{|a|^2\overline{x}}{1+\overline{x}}$.

	$\rho(T_1)$	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	$\sqrt{0.9999}$
F.P. with "+"	Its	25	49	116	*(>10000)
	Res	8.3267e-17	1.6653e-16	1.9429e-16	3.7124e-13
	Time	1.3828e-02	9.2210-03	1.1571e-02	7.9136
Alg. 4.1 with $r = 2$	Its	4	5	6	17
	Res	2.7756e-17	2.7756-17	0	2.7105e-20
	Time	1.1086e-02	7.7436e-03	8.9338e-03	5.7342e-03
Alg. 4.1 with $r = 3$	Its	3	3	4	11
	Res	5.5511e-17	2.7756e-17	0	0
	Time	4.6519e-03	2.8196e-03	4.3291e-03	3.2251e-03
Alg. 4.1 with $r = 4$	Its	2	3	3	9
	Res	5.5511e-17	2.7756e-17	0	0
	Time	5.4670e-04	4.6242e-04	6.5906e-04	4.3089e-04
Alg. 4.1 with $r = 5$	Its	2	2	3	8
	Res	2.7756e-17	1.6653e-16	0	0
	Time	3.7408-04	3.1476e-04	5.8860e-04	3.7914e-04

Table 4: The ITs, Res and Time for the problem $x = h - \frac{|a|^2\overline{x}}{1+\overline{x}}$.

	$\rho(T_1)$	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	$\sqrt{0.9999}$
F.P. with "-"	Its	25	50	120	*(>100000)
	Res	1.3878e-16	1.9429e-16	2.2204e-16	4.0837e-09
	Time	8.6382e-03	6.8811e-03	8.9200e-03	8.3106
Alg. 4.1 with $r = 2$	Its	4	5	6	18
	Res	2.7756e-17	5.5511e-17	8.3267e-17	1.5491e-17
	Time	4.4193e-03	9.6181e-03	6.4900e-03	7.6470e-03
Alg. 4.1 with $r = 3$	Its	3	3	4	11
	Res	5.5511e-17	5.5511e-17	2.7756e-17	1.7171e-17
	Time	4.7336e-03	3.9281e-03	2.9638e-03	3.3373e-03
Alg. 4.1 with $r = 4$	Its	2	3	3	9
	Res	2.7756e-17	0	8.3267e-17	1.7362e-17
	Time	4.3161e-04	5.1107e-04	4.7924e-04	4.2425e-04
Alg. 4.1 with $r = 5$	Its	2	3	3	8
	Res	5.5511e-17	0	8.3267e-17	4.1064e-18
	Time	3.2604e-04	5.4106e-04	3.9956e-04	3.9606e-04

apply the fixed point method. Though this method is guaranteed to converge, the convergence rate tends to be slow. Numerically, we provide an accelerated way to speed up the entire iteration. This way is based on the discovery of the semigroup property property, i.e., (22). We show that our accelerated method converge rapidly with the rate of convergence of any desired order. Additionally, this method can be used to solve the unique negative definite solution of (3), once it exists. The investigation of sufficient conditions for the existence of the negative definite solution of (3) is also included in this work.

Appendix

6.1. Convergence analysis of the fixed point iteration: $X = \mathcal{F}_{\pm}(X)$

We start our analysis by discussing the convergence property of the DARE. From Corollary (2.2), we know that the DARE (1) has a unique positive definite solution Z_* if $H_1 > 0$ and $G_1 > 0$. Let

$$Z_{k+1} = H_1 + A_1^H Z_k \Delta_{G_1,Z_k} A_1$$

be the fixed point iteration of (1) with an initial positive definite matrix Z_1. Like the discussion in Section 2.1, we immediately have the following two results:

(a) The sequence $\{Z_k\}$ is monotone increasing if and only if $Z_1 \leq Z_2$; the sequence $\{Z_k\}$ is monotone decreasing if and only if $Z_1 \geq Z_2$.

(b) If $Z_* \geq Z_1$, then Z_* is an upper bounded of the sequence$\{Z_k\}$; if $Z_* \leq Z_1$, then Z_* is a lower bounded of the sequence $\{Z_k\}$. Moreover, we have $\lim_{k\to\infty} Z_k = Z_*$ in either case.

Taking $0 < Z_1 \leq H_1$, for example, we see that the sequence $\{Z_k\}$ is monotone increasing, $Z_* \geq Z_k$ for all k, and $\lim_{k\to\infty} Z_k = Z_*$. Moreover,

$$\begin{aligned}
&Z_* - Z_{k+1} \\
&= A_1^H \Delta_{Z_*,G_1}(Z_*(I + G_1 Z_k) - (I + Z_* G_1) Z_k)\Delta_{G_1,Z_k} A_1 \\
&= A_1^H \Delta_{Z_*,G_1}(Z_* - Z_k)\Delta_{G_1,Z_k} A_1 = T_{Z_*}^H (Z_* - Z_k) T_{Z_k} \\
&= T_{Z_*}^H (Z_* - Z_k) T_{Z_*} + T_{Z_*}^H (Z_* - Z_k)(T_{Z_k} - T_{Z_*}) \\
&= T_{Z_*}^H (Z_* - Z_k) T_{Z_*} + T_{Z_*}^H (Z_* - Z_k)\Delta_{G_1,Z_k}(I + G_1 Z_* - I - G_1 Z_k) T_{Z_*} \\
&= T_{Z_*}^H (Z_* - Z_k) T_{Z_*} + T_{Z_*}^H \left[(Z_* - Z_k) G_1 \Delta_{Z_k,G_1}(Z_* - Z_k)\right] T_{Z_*},
\end{aligned} \tag{40}$$

where $T_{Z_*} = \Delta_{G_1,Z_\infty} A_1$. Given a positive number $\epsilon > 0$, there exists a positive integer k_0 such that

$$Z_* - Z_k \leq \epsilon I,$$

for any positive integer $k \geq k_0$. Since $G\Delta_{Z_k,G} \leq G \leq mI$ for a sufficiently large m, it follows from (40) that for this $k_0 > 0$ and $k \geq k_0$,

$$Z_* - Z_k \leq (1+\epsilon m) T_{Z_*}^H (Z_* - Z_{k-1}) T_{Z_*} \leq (1+\epsilon m)^{k-k_0} (T_{Z_*}^H)^{k-k_0} (Z_* - Z_{k_0}) T_{Z_*}^{k-k_0},$$

or, equivalently,

$$\limsup_{k\to\infty} \sqrt[k]{\|Z_* - Z_k\|} \leq (1+\epsilon m)\rho(T_{Z_*})^2. \tag{41}$$

Since ϵ is arbitrary, (41) induces that

$$\limsup_{k\to\infty} \sqrt[k]{\|Z_* - Z_k\|} \leq \rho(T_{Z_*})^2. \tag{42}$$

When the sequence $\{Z_k\}$ is monotone decreasing and bounded below. A similar argument yields for the estimation (42). Thus, by (42), our discussion to the convergence analysis of the fixed point method $X = \mathcal{F}_\pm(X)$ is divided into two scenarios:

1. Consider the fixed-point iteration $X_{k+1} = \mathcal{F}_+(X_k)$ with $X_1 = H$. As is discussed in Section 2.1, we know that the sequence $\{X_k\}$ is a monotone increasing matrix sequence. In particular, if the solution X_* of (3a) exists, it can be shown that

$$\limsup_{k\to\infty} \sqrt[k]{\|X_* - X_{2k}\|} \leq \rho(T_1)^2,$$
$$\limsup_{k\to\infty} \sqrt[k]{\|X_* - X_{2k+1}\|} \leq \rho(T_1)^2,$$

   since $X_1 = H$, $X_2 = H_1$, $F_+^{(2)}(X) = H_1 + A_1^H X \Delta_{G_1,X} A_1$, and $T_1 = \Delta_{G_1,X_\infty} A_1$. Thus, we have

$$\limsup_{k\to\infty} \sqrt[k]{\|X_k - X_*\|} \leq \rho(T_1).$$

2. Consider the fixed-point iteration $X_{k+1} = \mathcal{F}_-(X_k)$ with $X_1 = H$. Note that if $X_i \geq X_j$ for any integer $i, j \geq 1$, then

$$\begin{aligned} X_{i+1} - X_{j+1} &= A^H \overline{X}_j \Delta_{G,\overline{X}_j} A - A^H \overline{X}_i \Delta_{G,\overline{X}_i} A \\ &= A^H[(\overline{X}_j^{-1} + G)^{-1} - (\overline{X}_i^{-1} + G)^{-1}]A \leq 0, \end{aligned}$$

   i.e.,

$$X_{i+1} \leq X_{j+1}, \tag{43}$$

   if $X_i \geq X_j$ for any integer $i, j \geq 1$. Also, if H_1 and $G_1 > 0$, then by Lemma 3.2, we have

$$0 < X_2 \leq X_4 \leq \cdots \leq H, \tag{44}$$

   since $X_{k+2} = \mathcal{F}_-^{(2)}(X_k) = \mathcal{F}_-^{(k-1)}(H)$ for any even number $k > 0$. By (43) and (44), it can be seen that

$$H = X_1 \geq X_3 \geq X_5 \geq \cdots > 0.$$

Here, the first and last inequality follows from the fact that

$$X_3 = F_-(X_2) = H - A^H \overline{X}_2 \Delta_{G,\overline{X}_2} A \leq H = X_1,$$
$$X_1 \geq X_2 > 0, \quad X_3 \geq X_4 > 0,$$

and so on. Upon using the fact that the positive definite solution of

$$X = F_-^{(2)}(X)$$

is unique once H_1 and $G_1 > 0$, we know that $\lim_{k\to\infty} X_{2k} = \lim_{k\to\infty} X_{2k+1} := X_*$, where X_* is the unique positive definite solution of (3b). Furthermore,

$$\limsup_{k\to\infty} \sqrt[k]{\|X_* - X_{2k}\|} \leq \rho(T_1)^2,$$

since $X_2 = H_1$ and $F_-^{(2)}(X) = H_1 + A_1^H X \Delta_{G_1,X} A_1$. Note that

$$\begin{aligned} X_{2k+1} - X_* = {} & T_1^H (X_{2k-1} - X_*) T_1 \\ & + T_1^H \left[(X_{2k-1} - X_*) G \Delta_{X_{2k-1},G} (X_{2k-1} - X_*) \right] T_1. \end{aligned}$$

for any positive integer $k \geq 1$. Like the discussion of (42), we have

$$\limsup_{k\to\infty} \sqrt[k]{\|X_* - X_{2k+1}\|} \leq \rho(T_1)^2.$$

This implies that We conclude that

$$\limsup_{k\to\infty} \sqrt[k]{\|X_k - X_*\|} \leq \rho(T_1).$$

6.2. The proof of Theorem 3.1

Proof. To simply our discussion, let $\Delta_{i,j} := (I + G_i H_j)^{-1}$ for all $i, j \in \mathbb{N}$. Then, we have

$$\begin{aligned} H_j \Delta_{i,j} &= H_j^H (I + G_i^H H_j^H)^{-1} = (I + H_j^H G_i^H)^{-1} H_j^H = \Delta_{i,j}^H H_j, \\ \Delta_{i,j} G_i &= (I + G_i^H H_j^H)^{-1} G_i^H = G_i^H (I + H_j^H G_i^H)^{-1} = G_i \Delta_{i,j}^H, \\ I - H_j \Delta_{i,j} G_i &= I - H_j G_i (I + G_i H_j)^{-1} = (I + H_j G_i)^{-1} = \Delta_{i,j}^H. \end{aligned}$$

For each i, we will prove (22) by induction with respect to j. The proof is divided into two parts. First, for $i = 1$, we show that

$$\begin{aligned} A_{1+j} &= A_j (I + G_1 H_j)^{-1} A_1, \\ G_{1+j} &= G_j + A_j (I + G_1 H_j)^{-1} G_1 A_j^H, \\ H_{1+j} &= H_1 + A_1^H H_j (I + G_1 H_j)^{-1} A_1. \end{aligned}$$

by induction. Note that for $j = 1$, it is trivial from the definition of A_2, G_2 and H_2. Now suppose that it is true for $j = s$. It follows from Lemma 1.1 and (21) that

$$\begin{aligned}
\Delta_{1,s+1} &= \left(I + G_1\left(H_s + A_s^H H_1 (I + G_s H_1)^{-1} A_s\right)\right)^{-1} \\
&= \Delta_{1,s} - \Delta_{1,s}(G_1 A_s^H H_1)\left((I + G_s H_1) + A_s \Delta_{1,s}(G_1 A_s^H H_1)\right)^{-1} A_s \Delta_{1,s} \\
&= \Delta_{1,s} - \Delta_{1,s} G_1 A_s^H H_1 \Delta_{s+1,1} A_s \Delta_{1,s}, \qquad (45) \\
\Delta_{s+1,1} &= \left(I + (G_s + A_s (I + G_1 H_s)^{-1} G_1 A_s^H) H_1\right)^{-1} \\
&= \Delta_{s,1} - \Delta_{s,1} A_s \left((I + G_1 H_s) + (G_1 A_s^H H_1)\Delta_{s,1} A_s\right)^{-1} G_1 A_s^H H_1 \Delta_{s,1} \\
&= \Delta_{s,1} - \Delta_{s,1} A_s \Delta_{1,s+1} G_1 A_s^H H_1 \Delta_{s,1}, \qquad (46) \\
\Delta_{s+1,1} &= \Delta_{s,1}(I + G_s H_1)\Delta_{s+1,1} \\
&= \Delta_{s,1}(I + (G_{s+1} - A_s \Delta_{1,s} G_1 A_s^H) H_1)\Delta_{s+1,1} \\
&= \Delta_{s,1}\left(I + G_{s+1} H_1 - A_s \Delta_{1,s} G_1 A_s^H H_1\right)\Delta_{s+1,1} \\
&= \Delta_{s,1} - \Delta_{s,1} A_s \Delta_{1,s} G_1 A_s^H H_1 \Delta_{s+1,1}. \qquad (47)
\end{aligned}$$

Then, by induction hypothesis, we have

$$\begin{aligned}
A_{1+(s+1)} &= A_1 \Delta_{s+1,1} A_{s+1}, \\
&= A_1 \Delta_{s,1}\left((I + G_{s+1} H_1) - A_s \Delta_{1,s} G_1 A_s^H H_1\right)\Delta_{s+1,1} A_{s+1} \\
&= A_1 \Delta_{s,1}\left(I - A_s \Delta_{1,s} G_1 A_s^H H_1 \Delta_{s+1,1}\right) A_s \Delta_{1,s} A_1 \\
&= A_1 \Delta_{s,1} A_s \left(\Delta_{1,s} - \Delta_{1,s} G_1 A_s^H H_1 \Delta_{s+1,1} A_s \Delta_{1,s}\right) A_1 \text{ (by (45))} \\
&= A_{s+1} \Delta_{1,s+1} A_1, \\
G_{1+(s+1)} &= G_1 + A_1 \Delta_{s+1,1} G_{s+1} A_1^H, \\
&= G_1 + A_1(\Delta_{s,1} - \Delta_{s,1} A_s \Delta_{1,s+1} G_1 A_s^H H_1 \Delta_{s,1})(G_s + A_s \Delta_{1,s} G_1 A_s^H) A_1^H \text{ (by (46))} \\
&= G_1 + A_1 \Delta_{s,1} G_s A_1^H \\
&\quad - A_1 \Delta_{s,1} A_s \left(\Delta_{1,s+1} G_1 A_s^H H_1 \Delta_{s,1} G_s\right) A_1^H \\
&\quad + A_1 \Delta_{s,1} A_s \left(\Delta_{1,s} G_1 A_s^H\right) A_1^H \\
&\quad - A_1 \Delta_{s,1} A_s \left(\Delta_{1,s+1} G_1 A_s^H H_1 \Delta_{s,1} A_s \Delta_{1,s} G_1 A_s^H\right) A_1^H \\
&= G_{s+1} - A_{s+1} \Delta_{1,s+1} G_1 A_s^H H_1 \Delta_{s,1} G_s A_1^H \\
&\quad + A_{s+1}\left(I - \Delta_{1,s+1} G_1 A_s^H H_1 \Delta_{s,1} A_s\right)\Delta_{1,s} G_1 A_s^H A_1^H \\
&= G_{s+1} + A_{s+1} \Delta_{1,s+1} G_1 A_s^H \left(I - H_1 \Delta_{s,1} G_s\right) A_1^H \\
&= G_{s+1} + A_{s+1} \Delta_{1,s+1} G_1 A_s^H (I + H_1 G_s)^{-1} A_1^H \\
&= G_{s+1} + A_{s+1} \Delta_{1,s+1} G_1 A_{s+1}^H,
\end{aligned}$$

where $I - \Delta_{1,s+1} G_1 A_s^H H_1 \Delta_{s,1} A_s = \Delta_{1,s+1}\Delta_{1,s}^{-1}$, and finally,

$$\begin{aligned}
H_{1+(s+1)} &= H_{s+1} + A_{s+1}^H H_1 \Delta_{s+1,1} A_{s+1}, \\
&= H_{s+1} + \left(A_1^H \Delta_{1,s}^H A_s^H\right) H_1 \Delta_{s+1,1} \left(A_s \Delta_{1,s} A_1\right) \\
&= H_{s+1} + A_1^H (I + H_s G_1)^{-1} A_s^H H_1 \Delta_{s+1,1} A_s \Delta_{1,s} A_1 \\
&= H_{s+1} + A_1^H \left(I - H_s \Delta_{1,s} G_1\right) A_s^H H_1 \Delta_{s+1,1} A_s \Delta_{1,s} A_1 \\
&= H_1 + A_1^H H_s \Delta_{1,s} A_1 \\
&- A_1^H H_s \Delta_{1,s} G_1 A_s^H H_1 \Delta_{s+1,1} A_s \Delta_{1,s} A_1 \\
&+ A_1^H A_s^H H_1 \Delta_{s,1} \left(I - A_s \Delta_{1,s} G_1 A_s^H H_1 \Delta_{s+1,1}\right) A_s \Delta_{1,s} A_1 \\
&= H_1 + A_1^H H_s \Delta_{1,s} A_1 \\
&- A_1^H H_s \Delta_{1,s} G_1 A_s^H H_1 \Delta_{s+1,1} A_s \Delta_{1,s} A_1 \\
&+ A_1^H A_s^H H_1 \Delta_{s,1} A_s \Delta_{1,s} A_1 \\
&- A_1^H A_s^H H_1 \Delta_{s,1} A_s \Delta_{1,s} G_1 A_s^H H_1 \Delta_{s+1,1} A_s \Delta_{1,s} A_1 \text{ (by (47))} \\
&= H_1 + A_1^H H_{s+1} \Delta_{1,s+1} A_1, \text{ (by (45))}
\end{aligned}$$

where $I - A_s \Delta_{1,s} G_1 A_s^H H_1 \Delta_{s+1,1} = \Delta_{s,1}^{-1} \Delta_{s+1,1}$, which completes the proof for $i = 1$.

Assume that (22) is true for $i = s$ and any integer $j > 0$. Then, for any integer $j > 0$, we have

$$\begin{aligned}
\Delta_{s+1,j} &= ((I + G_s H_j) + A_s (I + G_1 H_s)^{-1} G_1 A_s^H H_j)^{-1} \\
&= \Delta_{s,j} - \Delta_{s,j} A_s [(I + G_1 H_s) + G_1 A_s^H H_j (I + G_s H_j) A_s]^{-1} G_1 A_s^H H_j \Delta_{s,j} \\
&= \Delta_{s,j} - \Delta_{s,j} A_s \Delta_{1,s+j} G_1 A_s^H H_j \Delta_{s,j}, \qquad (48) \\
\Delta_{1,s+j} &= ((I + G_1 H_s) + G_1 A_s^H H_j (I + G_s H_j)^{-1} A_s)^{-1} \\
&= \Delta_{1,s} - \Delta_{1,s} G_1 A_s^H H_j [(I + G_s H_j) + A_s \Delta_{1,s} G_1 A_s^H H_j]^{-1} A_s \Delta_{1,s} \\
&= \Delta_{1,s} - \Delta_{1,s} G_1 A_s^H H_j \Delta_{s+1,j} A_s \Delta_{1,s}, \qquad (49) \\
\Delta_{s,j+1} &= (I + G_s H_{j+1})^{-1} = \left(I + G_s (H_1 + A_1^H H_j \Delta_{1,j} A_1)\right)^{-1} \\
&= \Delta_{s,1} - \Delta_{s,1} G_s A_1^H H_j (I + G_1 H_j + A_1 \Delta_{s,1} G_s A_1^H H_j)^{-1} A_1 \Delta_{s,1} \\
&= \Delta_{s,1} - \Delta_{s,1} G_s A_1^H H_j \Delta_{s+1,j} A_1 \Delta_{s,1}, \qquad (50)
\end{aligned}$$

and

$$
\begin{aligned}
&\Delta_{s+1,j}G_{s+1} - \Delta_{s,j}G_s \\
&= (\Delta_{s,j} - \Delta_{s,j}A_s\Delta_{1,s+j}G_1A_s^H H_j\Delta_{s,j})(G_s + A_s\Delta_{1,s}G_1A_s^H) - \Delta_{s,j}G_s \text{ (by (48))} \\
&= \Delta_{s,j}A_s\Delta_{1,s}G_1A_s^H - \Delta_{s,j}A_s\Delta_{1,s+j}G_1A_s^H H_j\Delta_{s,j}G_s - \Delta_{s,j}A_s\Delta_{1,s+j}G_1A_s^H H_j\Delta_{s,j}A_s\Delta_{1,s}G_1A_s^H \\
&= \Delta_{s,j}A_s\left(I - \Delta_{1,s+j}G_1A_s^H H_j\Delta_{s,j}A_s\right)\Delta_{1,s}G_1A_s^H - \Delta_{s,j}A_s\Delta_{1,s+j}G_1A_s^H H_j\Delta_{s,j}G_s \\
&= \Delta_{s,j}A_s\Delta_{1,s+j}\Delta_{1,s}^{-1}\Delta_{1,s}G_1A_s^H - \Delta_{s,j}A_s\Delta_{1,s+j}G_1A_s^H H_j\Delta_{s,j}G_s \\
&= \Delta_{s,j}A_s\Delta_{1,s+j}G_1A_s^H\left(I - H_j\Delta_{s,j}G_s\right) \\
&= \Delta_{s,j}A_s\left(\Delta_{1,s+j}G_1\right)\left(\Delta_{s,j}A_s\right)^H, \qquad (51)
\end{aligned}
$$

$$
\begin{aligned}
&H_{s+j}\Delta_{1,s+j} - H_s\Delta_{1,s} \\
&= (H_s + A_s^H H_j\Delta_{s,j}A_s)(\Delta_{1,s} - \Delta_{1,s}G_1A_s^H H_j\Delta_{s+1,j}A_s\Delta_{1,s}) - H_s\Delta_{1,s} \text{ (by (49))} \\
&= A_s^H H_j\Delta_{s,j}A_s\Delta_{1,s} - H_s\Delta_{1,s}G_1A_s^H H_j\Delta_{s+1,j}A_s\Delta_{1,s} \\
&- A_s^H H_j\Delta_{s,j}A_s\Delta_{1,s}G_1A_s^H H_j\Delta_{s+1,j}A_s\Delta_{1,s} \\
&= A_s^H H_j\Delta_{s,j}\left(I - A_s\Delta_{1,s}G_1A_s^H H_j\Delta_{s+1,j}\right)A_s\Delta_{1,s} - H_s\Delta_{1,s}G_1A_s^H H_j\Delta_{s+1,j}A_s\Delta_{1,s} \\
&= A_s^H H_j\Delta_{s,j}\Delta_{s,j}^{-1}\Delta_{s+1,j}A_s\Delta_{1,s} - H_s\Delta_{1,s}G_1A_s^H H_j\Delta_{s+1,j}A_s\Delta_{1,s} \\
&= \left(I - H_s\Delta_{1,s}G_1\right)A_s^H H_j\Delta_{s+1,j}A_s\Delta_{1,s} \\
&= \left(A_s\Delta_{1,s}\right)^H\left(H_j\Delta_{s+1,j}\right)A_s\Delta_{1,s}. \qquad (52)
\end{aligned}
$$

Thus, it follows from Lemma 1.1 and induction hypothesis that the following result holds for $i = s+1$ and any integer $j > 0$.

$$
\begin{aligned}
A_{(s+1)+j} &= A_{s+(j+1)} = A_{1+j}\Delta_{s,j+1}A_s \\
&= A_j\Delta_{1,j}A_1\left(\Delta_{s,1} - \Delta_{s,1}G_sA_1^H H_j\Delta_{s+1,j}A_1\Delta_{s,1}\right)A_s \text{ (by (50))} \\
&= A_j\left(\Delta_{1,j} - \Delta_{1,j}A_1\Delta_{s,1}G_sA_1^H H_j\Delta_{s+1,j}\right)A_1\Delta_{s,1}A_s \\
&= A_j\left(\Delta_{1,j} - \Delta_{1,j}A_1\Delta_{s,1}G_sA_1^H H_j\Delta_{s+1,j}\right)A_{s+1} \\
&= A_j\Delta_{1,j}\left(I + G_{s+1}H_j - A_1\Delta_{s,1}G_sA_1^H H_j\right)\Delta_{s+1,j}A_{s+1} \\
&= A_j\Delta_{1,j}\left(I + (G_{s+1} - A_1\Delta_{s,1}G_sA_1^H)H_j\right)\Delta_{s+1,j}A_{s+1} \\
&= A_j\Delta_{1,j}\left(I + G_1H_j\right)\Delta_{s+1,j}A_{s+1} \\
&= A_j\Delta_{s+1,j}A_{s+1}, \\
G_{s+1+j} &= G_{1+(s+j)} = G_{s+j} + A_{s+j}\Delta_{1,s+j}G_1A_{s+j}^H \\
&= (G_j + A_j\Delta_{s,j}G_sA_j^H) + A_j\left((\Delta_{s,j}A_s)(\Delta_{1,s+j}G_1)(\Delta_{s,j}A_s)^H\right)A_j^H \\
&= G_j + A_j\Delta_{s+1,j}G_{s+1}A_j^H. \text{ (by (51))} \\
H_{s+1+j} &= H_{1+(s+j)} = H_1 + A_1^H H_{s+j}\Delta_{1,s+j}A_1 \\
&= H_1 + A_1^H H_s\Delta_{1,s}A_1 + A_1^H(H_{s+j}\Delta_{1,s+j} - H_s\Delta_{1,s})A_1 \\
&= (H_1 + A_1^H H_s\Delta_{1,s}A_1) + A_1^H\left((A_s\Delta_{1,s})^H(H_j\Delta_{s+1,j})A_s\Delta_{1,s}\right)A_1 \text{ (by (52))} \\
&= H_{s+1} + A_{s+1}^H H_j\Delta_{s+1,j}A_{s+1}.
\end{aligned}
$$

Now, the induction process is completed and thus the result is followed. $\square$

6.3. The proof of Lemma 4.1

Proof. Observe that $T_k = T_1^k$ is definitely true for $k = 1$. Suppose T_k is true for some $k \geq 1$. Then, by using the fact that

$$H_\infty = H_1 + A_1^H H_\infty \Delta_{G_1,H_\infty} A_1,\ A_{k+1} = A_1 \Delta_{G_k,H_1} A_k,\ \text{and}\ G_{k+1} = G_1 + A_1 \Delta_{G_k,H_1} G_k A_1^H,$$

we have

$$\begin{aligned}
&T_1^{k+1} = \Delta_{G_1,H_\infty}(A_1 \Delta_{G_k,H_\infty} A_k) \\
&= \Delta_{G_1,H_\infty}(A_1(I + G_k H_1 + G_k A_1^H H_\infty \Delta_{G_1,H_\infty} A_1)^{-1} A_k) \\
&= \Delta_{G_1,H_\infty} A_1[\Delta_{G_k,H_1} - \Delta_{G_k,H_1} G_k A_1^H H_\infty \left((I + G_1 H_\infty) + A_1 \Delta_{G_k,H_1} G_k A_1^H H_\infty\right)^{-1} A_1 \Delta_{G_k,H_1}] A_k \\
&= \Delta_{G_1,H_\infty} A_{k+1} - \Delta_{G_1,H_\infty} A_1 \left(\Delta_{G_k,H_1} G_k A_1^H H_\infty \Delta_{G_{k+1},H_\infty} A_1 \Delta_{G_k,H_1}\right) A_k \\
&= \Delta_{G_1,H_\infty}\left(I + G_{k+1} H_\infty - A_1 \Delta_{G_k,H_1} G_k A_1^H H_\infty\right) \Delta_{G_{k+1},H_\infty} A_{k+1} \\
&= \Delta_{G_{k+1},H_\infty} A_{k+1} = T_{k+1},
\end{aligned}$$

which concludes that T_k holds for all $k \geq 1$.

Observe that $S_k = S_1^k$ is definitely true for $k = 1$. Suppose S_k is true for some $k \geq 1$. Then, by using the fact that

$$G_\infty = G_1 + A_1 G_\infty \Delta_{H_1,G_\infty} A_1^H,\ A_{k+1} = A_k \Delta_{G_1,H_k} A_1,\ \text{and}\ H_{k+1} = H_1 + A_1^H \Delta_{H_k,G_1} H_k A_1,$$

we have

$$\begin{aligned}
&S_1^{k+1} = (A_k \Delta_{G_\infty,H_k}) A_1 \Delta_{G_\infty,H_1} \\
&= A_k[I + (G_1 + A_1 G_\infty \Delta_{H_1,G_\infty} A_1^H)) H_k]^{-1} A_1 \Delta_{G_\infty,H_1} \\
&= A_k[\Delta_{G_1,H_k} - \Delta_{G_1,H_k} A_1 G_\infty (I + H_1 G_\infty + A_1^H H_k \Delta_{G_1,H_k} A_1 G_\infty)^{-1} A_1^H H_k \Delta_{G_1,H_k}] A_1 \Delta_{G_\infty,H_1} \\
&= A_{k+1} \Delta_{G_\infty,H_1} - A_{k+1} G_\infty (I + H_{k+1} G_\infty)^{-1} A_1^H H_k \Delta_{G_1,H_k} A_1 \Delta_{G_\infty,H_1} \\
&= A_{k+1} \Delta_{G_\infty,H_{k+1}} (I + G_\infty H_{k+1} - G_\infty A_1^H H_k \Delta_{G_1,H_k} A_1) \Delta_{G_\infty,H_1} \\
&= A_{k+1} \Delta_{G_\infty,H_{k+1}} = S_{k+1},
\end{aligned}$$

which concludes that S_k holds for all $k \geq 1$. Note that

$$\begin{aligned}
&H_\infty - H_{k+1} \\
&= A_1^H \left(H_\infty \Delta_{G_1,H_\infty} - H_k \Delta_{G_1,H_k}\right) A_1 \\
&= A_1^H H_\infty \Delta_{G_1,H_\infty} (I + G_1 H_k) \Delta_{G_1,H_k} A_1 - A_1^H \Delta_{H_\infty,G_1} (I + H_\infty G_1) H_k \Delta_{G_1,H_k} A_1 \\
&= A_1^H \Delta_{H_\infty,G_1} \left(H_\infty - H_k\right) \Delta_{G_1,H_k} A_1 \\
&= A_1^H \Delta_{H_\infty,G_1} (A_k^H H_\infty \Delta_{G_k,H_\infty} A_k) \Delta_{G_1,H_k} A_1 \\
&= T_1^H T_k^H H_\infty A_{k+1} \\
&= T_{k+1}^H H_\infty A_{k+1} = T_{k+1}^H (H_\infty^{-1} + G_{k+1}) T_{k+1}.
\end{aligned}$$

Also,

$$
\begin{aligned}
&G_\infty - G_{k+1} \\
&= A_1 \left(\Delta_{G_\infty,H_1} G_\infty - \Delta_{G_k,H_1} G_k\right) A_1^H \\
&= A_1 (G_\infty \Delta_{H_1,G_\infty} (I + H_1 G_k) \Delta_{H_1,G_k} A_1^H - A_1 \Delta_{G_\infty,H_1} (I + G_\infty H_1) G_k \Delta_{H_1,G_k} A_1^H \\
&= A_1 \Delta_{G_\infty,H_1} \left(G_\infty - G_k\right) \Delta_{H_1,G_k} A_1^H \\
&= A_1 \Delta_{G_\infty,H_1} (A_k \Delta_{G_\infty,H_k} G_\infty A_k^H) \Delta_{H_1,G_k} A_1^H \\
&= S_1 S_k G_\infty A_{k+1}^H \\
&= S_{k+1} G_\infty A_{k+1}^H = S_{k+1} G_\infty (I + H_{k+1} G_\infty) S_{k+1}^H .
\end{aligned}
$$

By the similar discussion of Theorem 2.1 and Theorem 2.4 for Eq. (3), we can show with no difficulty that $\rho(\Delta_{H_1,G_\infty} A_1^H) < 1$ with respect to Eq. (27), that is,

$$\rho(S_1) = \rho((\Delta_{H_1,G_\infty} A_1^H)^H) < 1.$$

On the other hand, let $\mathcal{M}$ and $\mathcal{L}$ be two matrices defined by

$$\mathcal{M} = \begin{bmatrix} A_1 & 0 \\ -H_1 & I_n \end{bmatrix}, \quad \mathcal{L} := \begin{bmatrix} I_n & G_1 \\ 0 & A_1^H \end{bmatrix}.$$

Let $\mathcal{J}$ be a skew-symmetric matrix defined by

$$\mathcal{J} \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

It can be seen that $\mathcal{M}\mathcal{J}\mathcal{M}^H = \mathcal{L}\mathcal{J}\mathcal{L}^H$, since $G_1 = G_1^H$ and $H_1 = H_1^H$. Let $\lambda \in \sigma(\mathcal{M} - \lambda\mathcal{L})$ and $\mathbf{x}$ be a nonzero eigenvector satisfying $\mathcal{M}^H \mathbf{x} = \overline{\lambda}\mathcal{L}^H \mathbf{x}$. It follows that

$$\mathcal{L}\mathcal{J}\mathcal{L}^H \mathbf{x} = \mathcal{M}\mathcal{J}\mathcal{M}^H \mathbf{x} = \overline{\lambda}\mathcal{M}\mathcal{J}\mathcal{L}^H x,$$

First, if $\lambda \neq 0$, we have $\mathcal{M}(\mathcal{J}\mathcal{L}^H \mathbf{x}) = (1/\overline{\lambda})\mathcal{L}(\mathcal{J}\mathcal{L}^H \mathbf{x})$. Once $\mathcal{J}\mathcal{L}^H \mathbf{x} \neq 0$, this implies that $1/\overline{\lambda} \in \sigma(\mathcal{M} - \lambda\mathcal{L})$; otherwise, $\mathcal{M}^H \mathbf{x} = 0$ if $\mathcal{J}\mathcal{L}^H \mathbf{x} = 0$, which contradicts that $\mathbf{x}$ is nonzero. Second, if $\lambda = 0$, there exists a nonzero vector $\mathbf{x}$ such that $\mathcal{M}^H \mathbf{x} = 0$. Since $\operatorname{rank}(\mathcal{M}) = \operatorname{rank}(\mathcal{L})$, it follows that there exists a nonzero vector $\mathbf{y}$ such that $\mathcal{L}\mathbf{y} = 0$ and, hence, $\infty := 1/0 \in \sigma(\mathcal{M} - \lambda\mathcal{L})$. Thus, the eigenvalues of $\mathcal{M} - \lambda\mathcal{L}$ come in pairs, i.e., $1/\overline{\lambda} \in \sigma(\mathcal{M} - \lambda\mathcal{L})$ if $\lambda \in \sigma(\mathcal{M} - \lambda\mathcal{L})$.

Let $U = \begin{bmatrix} I_n \\ H_\infty \end{bmatrix}$ and $V = \begin{bmatrix} -G_\infty \\ I_n \end{bmatrix}$. It is true that

$$\mathcal{M}U = \mathcal{L}UT_1, \quad \mathcal{M}V{S_1}^H = \mathcal{L}V.$$

This implies that $\sigma(T_1) \subset \sigma(\mathcal{M} - \lambda\mathcal{L})$ and $\sigma(S_1) \subset \sigma(\mathcal{L} - \lambda\mathcal{M})$. Furthermore, there are exactly n eigenvalues of $\mathcal{M} - \lambda\mathcal{L}$ inside the unit circle and the other outside the unit circle, since $\rho(T_1) < 1$.

If $\lambda \in \sigma(T_1)$ (i.e., $1/\overline{\lambda} \in \sigma(\mathcal{M} - \lambda\mathcal{L})$), then there exists a $x \neq 0$ such that $\overline{\lambda}\mathcal{M}x = \mathcal{L}x$ and, hence, $\overline{\lambda} \in \sigma(S_1)$. The converse is also true and concludes that $\sigma(T_1) = \sigma(S_1^H)$. $\square$

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