Journal of Computational and Applied Mathematics 367 (2020) 112469

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A class of strongly completely monotonic functions related to gamma function

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ARTICLE INFO

Article history:
Received 15 December 2018
Received in revised form 8 July 2019

MSC:
primary 33B15
secondary 26A48

Keywords:
Strongly completely monotonic function
Completely monotonic function
Gamma function

ABSTRACT

A function $f:(0,\infty)\to\mathbb{R}$ is called strongly completely monotonic if it has derivatives of all orders and $(-1)^n x^{n+1} f^{(n)}(x)$ is nonnegative and decreasing on $(0,\infty)$ for all $n\geq 0$. In this paper, we prove the function

$$g_n(x)=\ln\Gamma\left(x+\frac{1}{2}\right)-x\ln x+x-\frac{1}{2}\ln(2\pi)+\sum_{k=1}^{n}\frac{\left(1-2^{1-2k}\right)B_{2k}}{2k(2k-1)x^{2k-1}}$$

is strongly completely monotonic on $(0,\infty)$. Using the same technique, we give an alternative proof of a known result. Moreover, two conjectures are proposed.

1. Introduction

A function f is said to be completely monotonic on an interval I, if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x)\geq 0 \text{ for all } x\in I \text{ and } n=0,1,2,\ldots. \tag{1.1}$$

If the inequality (1.1) is strict, then f is said to be strictly completely monotonic on I (see [1,2]). The classical Bernstein–Widder theorem (see [2,3]) states that a function f is completely monotonic on $(0,\infty)$ if and only if it is a Laplace transform

$$f(x)=\int_0^\infty e^{-xt}d\alpha(t),$$

where $\alpha(t)$ is non-decreasing and the above integral converges for $0<x<\infty$.

There have many results involving completely monotonic functions related to certain special functions, for example, [4–11].

In particular, by making use of Euler's summation formula [12, p.806, Eq. (23.1.30)], Alzer [13] proved the following result.

Theorem 1.1. *For given integer $n\geq 0$, let*

$$\kappa_1(x)=\ln\Gamma(x)-\left(x-\frac{1}{2}\right)\ln x+x-\frac{1}{2}\ln(2\pi), \tag{1.2}$$

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https://doi.org/10.1016/j.cam.2019.112469
0377-0427/

and the function $x \mapsto f_n(x)$ be defined on $(0, \infty)$ by

$$f_n(x) = \begin{cases} \kappa_1(x) - \sum_{k=1}^{n} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}, & \text{if } n \geq 1 \\ \kappa_1(x), & \text{if } n = 0 \end{cases}, \tag{1.3}$$

where B_n denotes the Bernoulli number. Then both of the functions $x \mapsto f_{2n}(x)$ and $x \mapsto -f_{2n+1}(x)$ are strictly completely monotonic on $(0, \infty)$.

Very recently, Chen and Paris [14] established another result as follows.

Theorem 1.2 ([14]). *For given integer $n \geq 0$, let*

$$\kappa_2(x) = \ln \frac{\Gamma(x+1)}{\Gamma(x+1/2)} - \frac{1}{2}\ln x, \tag{1.4}$$

and the function $x \mapsto h_n(x)$ be defined on $(0, \infty)$ by

$$h_n(x) = \begin{cases} \kappa_2(x) - \sum_{k=1}^{n} \frac{\left(1-2^{-2k}\right)B_{2k}}{k(2k-1)x^{2k-1}}, & \text{if } n \geq 1 \\ \kappa_2(x), & \text{if } n = 0 \end{cases}, \tag{1.5}$$

where B_n denotes the Bernoulli number. Then both of the functions $x \mapsto h_{2n}(x)$ and $x \mapsto -h_{2n+1}(x)$ are strictly completely monotonic on $(0, \infty)$.

Yang [15] has proven

Theorem 1.3. *For any integer $n \geq 0$, let*

$$\kappa_3(x) = \ln \Gamma\left(x + \frac{1}{2}\right) - x\ln x + x - \frac{1}{2}\ln(2\pi), \tag{1.6}$$

and the function $x \mapsto g_n(x)$ be defined on $(0, \infty)$ by

$$g_n(x) = \begin{cases} \kappa_3(x) + \sum_{k=1}^{n} \frac{\left(1-2^{1-2k}\right)B_{2k}}{2k(2k-1)x^{2k-1}}, & \text{if } n \geq 1 \\ \kappa_3(x), & \text{if } n = 0 \end{cases}, \tag{1.7}$$

where B_n denotes the Bernoulli number. Then (i) $g_n(x)$ can be represented in the integral form

$$g_n(x) = \int_0^\infty Q_n\left(\frac{t}{2}\right)\frac{e^{-xt}}{2t}\,dt, \tag{1.8}$$

where

$$Q_n(t) = \frac{1}{\sinh t} + \sum_{k=0}^{n} \frac{2\left(2^{2k-1}-1\right)B_{2k}}{(2k)!}t^{2k-1}; \tag{1.9}$$

(ii) both of the functions $x \mapsto g_{2n+1}$ and $x \mapsto -g_{2n}$ are strictly completely monotonic on $(0, \infty)$.

In 1989, Trimble and Wells [16] introduced the concept of the strongly completely monotonic functions as follows.

Definition 1. A function $f : (0, \infty) \to \mathbb{R}$ is called strongly completely monotonic if it has derivatives of all orders and $(-1)^n x^{n+1} f^{(n)}(x)$ is nonnegative and decreasing on $(0, \infty)$ for all $n = 0, 1, 2, \ldots$.

As remarked in [17], it is clear that being strongly completely monotonic is stronger than being completely monotonic. And the strongly completely monotonic functions are connected to the important question of superadditivity [16].

By Definition 1, we easily see that a function $f(x)$ is strongly completely monotonic on $(0, \infty)$ if and only if the function $xf(x)$ is completely monotonic on $(0, \infty)$. In [16] the authors gave another characterization of strongly completely monotonic functions.

Proposition 1.1 ([16]). *The function $f(x)$ is strongly completely monotonic if and only if*

$$f(x) = \int_0^\infty e^{-xt} p(t)\,dt$$

where $p(t)$ is nonnegative and increasing and the integral converges for all $x > 0$.

In 2009, Koumandos and Pedersen [17, Definition 1.5] introduced the notion of completely monotonic functions of order r.

Definition 2. Let $r \geq 0$. A function f defined on $(0, \infty)$ is said to be completely monotonic of order r if $x^r f(x)$ is completely monotonic.

From this definition we see that completely monotonic functions of order 0 are the classical completely monotonic functions, order 1 are the strongly completely monotonic functions. Moreover, Koumandos and Pedersen [17, Definition 1.5] proved a more stronger result.

Theorem 1.4. *For any $n \in \mathbb{N}$, let $f_n(x)$ be defined by (1.3). Then the function $(-1)^n x^n f_n(x)$ is completely monotonic on $(0, \infty)$. Or equivalently, both of the functions $x^{2n} f_{2n}(x)$ and $-x^{2n-1} f_{2n-1}(x)$ are completely monotonic on $(0, \infty)$.*

As a consequence of Theorem 1.4, the following corollary is immediate.

Corollary 1.1. *For any $n \in \mathbb{N}$, let $f_n(x)$ be defined by (1.3). Then both of the functions $f_{2n}(x)$ and $-f_{2n-1}(x)$ are strongly completely monotonic on $(0, \infty)$.*

Motivated by the results mentioned above, the first aim of this paper is to establish a more stronger assertion than Theorem 1.3. More precisely, we have

Theorem 1.5. *For given integer $n \geq 0$, let the function $x \mapsto g_n(x)$ be defined on $(0, \infty)$ by (1.7). Then both of the functions $g_{2n+1}(x)$ and $-g_{2n}(x)$ are strictly strongly completely monotonic on $(0, \infty)$.*

The second aim of this paper is to give an alternative proof of Corollary 1.1.

2. Lemmas

To prove our main results, we need some lemmas. The first lemma below comes from [15, Lemma 1], which will be used to prove Propositions 3.1 and 4.1. From the proof of this lemma in [15, Lemma 1], it can be seen that R_A may not be equal to R_B, and R_c is out of its own, so the lemma can be rewritten as follows.

Lemma 2.1. *Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$, $B(t) = \sum_{k=0}^{\infty} b_k t^k$ and $C(t) = \sum_{k=0}^{\infty} c_k t^k$ be real power series with radii of convergence R_A, R_B, and R_c, respectively, and $B(t) > 0$ for $t \in (0, R_B)$. Also assume that $A(t)/B(t)$ converges to $C(t)$ for $|t| < R_c \leq R = \min(R_A, R_B)$. Let n be a nonnegative integer, for $k \geq 2n+1$,*

$$\begin{aligned}
\mathcal{E}_1 &= b_k c_0 - a_k, \\
\mathcal{E}_{2,j} &= b_{k-2j+1} c_{2j-1} + b_{k-2j} c_{2j},\ 1 \leq j \leq n, \\
d_{k,2n} &= \sum_{i=0}^{2n} b_{k-i} c_i - a_k = (b_k c_0 - a_k) + \sum_{i=1}^{2n} b_{k-i} c_i \\
&= (b_k c_0 - a_k) + (b_{k-1} c_1 + b_{k-2} c_2) + (b_{k-3} c_3 + b_{k-4} c_4) + \cdots + \left(b_{k-(2n-1)} c_{2n-1} + b_{k-2n} c_{2n}\right) \\
&= (b_k c_0 - a_k) + \left(b_{k-1} c_1 + b_{k-3} c_3 + \cdots + b_{k-(2n-1)} c_{2n-1}\right) + (b_{k-2} c_2 + b_{k-4} c_4 + \cdots + b_{k-2n} c_{2n}) \\
&= (b_k c_0 - a_k) + \sum_{j=1}^{n} \left(b_{k-2j+1} c_{2j-1} + b_{k-2j} c_{2j}\right) \\
&= \mathcal{E}_1 + \sum_{j=1}^{n} \mathcal{E}_{2,j};
\end{aligned}$$

for $k \geq 2n+2$,

$$\begin{aligned}
\mathcal{O}_1 &= b_k c_0 + b_{k-1} c_1 - a_k, \\
\mathcal{O}_{2,j} &= b_{k-2j} c_{2j} + b_{k-2j-1} c_{2j+1},\ 1 \leq j \leq n, \\
d_{k,2n+1} &= \sum_{i=0}^{2n+1} b_{k-i} c_i - a_k = b_k c_0 + b_{k-1} c_1 - a_k + \sum_{i=2}^{2n+1} b_{k-i} c_i \\
&= (b_k c_0 + b_{k-1} c_1 - a_k) + \sum_{j=1}^{n} \left(b_{k-2j} c_{2j} + b_{k-2j-1} c_{2j+1}\right) \\
&= \mathcal{O}_1 + \sum_{j=1}^{n} \mathcal{O}_{2,j}.
\end{aligned}$$

Then (i) if $d_{k,2n}<0$ and $d_{k,2n+1}>0$, the double inequality

$$\sum_{k=0}^{2n} c_k t^k < \frac{A(t)}{B(t)} < \sum_{k=0}^{2n+1} c_k t^k \tag{2.1}$$

holds for all $t\in(0,R)$;

(ii) if $d_{k,2n}>0$ and $d_{k,2n+1}<0$, the double inequality

$$\sum_{k=0}^{2n+1} c_k t^k < \frac{A(t)}{B(t)} < \sum_{k=0}^{2n} c_k t^k \tag{2.2}$$

holds for all $t\in(0,R)$.

Remark 2.1. From the proof of [15, Lemma 1], we clearly see that $\sum_{i=0}^{k} b_{k-i}c_i = a_k$.

The following lemma is due to Qi [18], which will be used in proofs of Propositions 3.1 and 4.1.

Lemma 2.2 (*[18]*). *For $k\in\mathbb{N}$, Bernoulli numbers B_{2k} satisfy*

$$\frac{2^{2k-1}-1}{2^{2k+1}-1}\frac{(2k+1)(2k+2)}{\pi^2} < \frac{|B_{2k+2}|}{|B_{2k}|} < \frac{\left(2^{2k}-1\right)}{\left(2^{2k+2}-1\right)}\frac{(2k+1)(2k+2)}{\pi^2}. \tag{2.3}$$

In particular, letting $k=2j-1, 2j$, we have

$$\frac{\left|B_{4j}\right|}{\left|B_{4j-2}\right|} > \frac{2^{4j-3}-1}{2^{4j-1}-1}\frac{4j(4j-1)}{\pi^2}, \tag{2.4}$$

$$\frac{(4j+1)(4j+2)}{\pi^2}\frac{2^{4j}-1}{2^{4j+2}-1} > \frac{\left|B_{4j+2}\right|}{\left|B_{4j}\right|} > \frac{2^{4j-1}-1}{2^{4j+1}-1}\frac{(4j+2)(4j+1)}{\pi^2}. \tag{2.5}$$

Lemma 2.3 is a powerful tool to deal with the monotonicity of the ratio between two power series. An improvement of Lemma 2.3 has been presented in [19, Theorem 2.1.].

Lemma 2.3 (*[20, Theorem 2.1.]*). *Let $A(t)=\sum_{k=0}^{\infty} a_k t^k$ and $B(t)=\sum_{k=0}^{\infty} b_k t^k$ be two real power series converging on $(-r,r)$ $(r>0)$ with $b_k>0$ for all k. If the sequence $\{a_k/b_k\}$ is increasing (decreasing) for all k, then the function $t\mapsto A(t)/B(t)$ is also increasing (decreasing) on $(0,r)$.*

The following lemma is called the monotonicity rules for the ratio of two Laplace transforms, which was established in [21, Lemma 4], [22, Theorem 4] (see also [23]).

Lemma 2.4. *Let the functions A, B be defined on $(0,\infty)$ such that their Laplace transforms exist with $B(t)\neq 0$ for all $t>0$ Then the function*

$$x\mapsto U(x)=\frac{\int_0^\infty A(t)e^{-xt}dt}{\int_0^\infty B(t)e^{-xt}dt}$$

is decreasing (increasing) on $(0,\infty)$ if A/B is increasing (decreasing) on $(0,\infty)$ with

$$U(0)=\lim_{x\to 0} U(x)=\lim_{t\to\infty}\frac{A(t)}{B(t)} \quad \text{and} \quad U(\infty)=\lim_{x\to\infty} U(x)=\lim_{t\to 0}\frac{A(t)}{B(t)}$$

provide the indicated limits exist.

Lemma 2.5 (*[24]*). *The two given sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ satisfy the conditions*

$$b_n>0;\ \sum_{n=0}^{\infty} b_n t^n \text{ converges for all values of } t;\ \lim_{n\to\infty}\frac{a_n}{b_n}=s.$$

Then $\sum_{n=0}^{\infty} a_n t^n$ converges too for all values of t and in addition

$$\lim_{t\to\infty}\frac{\sum_{n=0}^{\infty} a_n t^n}{\sum_{n=0}^{\infty} b_n t^n}=s.$$

3. Proof of Theorem 1.5

We first prove the following proposition.

Proposition 3.1. *Let integer $n, m \geq 0$, and*

$$R(t,m) = \sum_{k=0}^{m} \frac{2(2k-2)\left(2^{2k-1}-1\right)B_{2k}}{(2k)!} t^{2k-1} = \sum_{k=0}^{m} (-1)^{k-1} \frac{2(2k-2)\left(2^{2k-1}-1\right)|B_{2k}|}{(2k)!} t^{2k-1}.$$

Then the double inequality

$$R(t,2n) < \frac{\sinh t + t\cosh t}{\sinh^2 t} < R(t,2n+1) \tag{3.1}$$

holds for all $t > 0$.

Proof. Let

$$Z(t,m) = \frac{t}{2}R(t,m) = \sum_{k=0}^{m} (-1)^{k-1} \frac{(2k-2)\left(2^{2k-1}-1\right)|B_{2k}|}{(2k)!} t^{2k} = \sum_{k=0}^{\infty} \frac{(2k-2)\left(2^{2k-1}-1\right)B_{2k}}{(2k)!} t^{2k}.$$

Then inequalities (3.1) can be written as

$$Z(t,2n) < \frac{(\sinh t + t\cosh t)/t}{(\cosh 2t - 1)/t^2} < Z(t,2n+1).$$

Let

$$A(t) = \frac{\sinh t}{t} + \cosh t = \sum_{k=0}^{\infty} \frac{2k+2}{(2k+1)!} t^{2k} := \sum_{k=0}^{\infty} a_k t^{2k}, \quad |t| < \infty$$

$$B(t) = \frac{\cosh 2t - 1}{t^2} = \sum_{k=0}^{\infty} \frac{2^{2k+2}}{(2k+2)!} t^{2k} := \sum_{k=0}^{\infty} b_k t^{2k}, \quad |t| < \infty$$

$$C(t) = \frac{A(t)}{B(t)} = \sum_{k=0}^{\infty} \frac{(2k-2)\left(2^{2k-1}-1\right)B_{2k}}{(2k)!} t^{2k} := \sum_{k=0}^{\infty} c_k t^{2k}, \quad |t| < \pi.$$

(1) We prove

$$d_{k,2n} = \sum_{i=1}^{2n} b_{k-i}c_i = (b_k c_0 - a_k) + \sum_{j=1}^{n} \left(b_{k-2j+1}c_{2j-1} + b_{k-2j}c_{2j}\right) < 0$$

for $k \geq 2n+1$ by distinguishing two cases.

Case 1.1: $k = 2n+1$. Since $\sum_{i=0}^{k} b_{k-i}c_i = a_k$, we see that

$$\begin{aligned} d_{k,2n} &= \sum_{i=0}^{2n} b_{k-i}c_i - a_k = \sum_{i=0}^{2n} b_{2n+1-i}c_i - a_{2n+1} \\ &= \sum_{i=0}^{2n} b_{2n+1-i}c_i - \sum_{i=0}^{2n+1} b_{2n+1-i}c_i \\ &= \sum_{i=0}^{2n} b_{2n+1-i}c_i - \left(\sum_{i=0}^{2n} b_{2n+1-i}c_i + b_0 c_{2n+1}\right) \\ &= -b_0 c_{2n+1} = -2\frac{4n\left(2^{4n+1}-1\right)B_{4n+2}}{(4n+2)!} < 0. \end{aligned}$$

Case 1.2: $k \geq 2n+2$. We first check that $d_{k,2n} < 0$ if $n = 1$. A simple computation yields

$$d_{k,2} = (b_k c_0 - a_k) + (b_{k-1}c_1 + b_{k-2}c_2)$$

$$
\begin{aligned}
&= \frac{2^{2k+2}}{(2k+2)!} - \frac{2k+2}{(2k+1)!} - \frac{7}{360}\frac{2^{2k-2}}{(2k-2)!} \\
&= \frac{2^{2k-2}}{(2k+2)!}\left(16 - \frac{7}{360}(2k+2)(2k+1)(2k)(2k-1)\right) - \frac{(2k+2)}{(2k+1)!} \\
&< 0.
\end{aligned}
$$

We next show that $d_{k,2n} < 0$ for $n \geq 2$. To this end, we write $d_{k,2n}$ as

$$
\begin{aligned}
d_{k,2n} &= (b_k c_0 - a_k) + (b_{k-1}c_1 + b_{k-2}c_2) + \sum_{j=2}^{n}\left(b_{k-2j+1}c_{2j-1} + b_{k-2j}c_{2j}\right) \\
&:= d_{k,2} + \sum_{j=2}^{n}\mathcal{E}_{2,j}.
\end{aligned}
$$

It has been shown that $d_{k,2} < 0$ for $k \geq 4$, and it suffices to show that $\mathcal{E}_{2,j} = b_{k-2j+1}c_{2j-1} + b_{k-2j}c_{2j} < 0$ for $2 \leq j \leq n$. Using inequality (2.4) we have

$$
\begin{aligned}
\frac{\mathcal{E}_{2,j}}{\left|B_{4j-2}\right|} &= \frac{b_{k-2j+1}c_{2j-1} + b_{k-2j}c_{2j}}{\left|B_{4j-2}\right|} \\
&= \frac{(4j-4)\,2^{2k-4j+4}\left(2^{4j-3}-1\right)}{(2k-4j+4)!\,(4j-2)!} - \frac{(4j-2)\,2^{2k-4j+2}\left(2^{4j-1}-1\right)}{(2k-4j+2)!\,(4j)!}\frac{\left|B_{4j}\right|}{\left|B_{4j-2}\right|} \\
&< \frac{(4j-4)\,2^{2k-4j+4}\left(2^{4j-3}-1\right)}{(2k-4j+4)!\,(4j-2)!} \\
&\quad - \frac{(4j-2)\,2^{2k-4j+2}\left(2^{4j-1}-1\right)}{(2k-4j+2)!\,(4j)!}\frac{2^{4j-3}-1}{2^{4j-1}-1}\frac{4j\,(4j-1)}{\pi^2} \\
&= \frac{\left(2^{4j-3}-1\right)2^{2k-4j+4}}{(2k-4j+4)!\,(4j-2)!}F_1(k,j),
\end{aligned}
$$

where

$$
F_1(k,j) = 4j - 4 - \frac{(4j-2)(2k-4j+4)(2k-4j+3)}{4\pi^2}.
$$

Due to $k \geq 2n+2$ and $2 \leq j \leq n$, we see that $2k - 4j \geq 4$, which yields

$$
F_1(k,j) \leq 4j - 4 - \frac{(4j-2)(4+4)(4+3)}{4\pi^2} = -\frac{14-\pi^2}{\pi^2}4j - \frac{4\left(\pi^2-7\right)}{\pi^2} < 0.
$$

This leads to $d_{k,2n} < 0$ for $k \geq 2n+2$ and $n \geq 2$, which proves $d_{k,2n} < 0$ for $k \geq 2n+2$. Cases 1.1 and 1.2 result in $d_{k,2n} < 0$ for $k \geq 2n+1$.

(2) We now prove

$$
d_{k,2n+1} = (b_k c_0 + b_{k-1}c_1 - a_k) + \sum_{j=1}^{n}\left(b_{k-2j}c_{2j} + b_{k-2j-1}c_{2j+1}\right) > 0
$$

for $k \geq 2n+2$. Similarly, we distinguish two cases to prove it.

Case 2.1. $k = 2n+2$. Since $\sum_{i=0}^{k} b_{k-i}c_i = a_k$, we see that

$$
d_{k,2n+1} = \sum_{i=0}^{2n+1} b_{k-i}c_i - a_k = -b_0 c_{2n+2} = -2\frac{(4n+2)\left(2^{4n+3}-1\right)}{(4n+4)!}B_{4n+4} > 0.
$$

Case 2.2. $k \geq 2n+3$. We have

$$
\begin{aligned}
\mathcal{O}_1 &= b_k c_0 + b_{k-1}c_1 - a_k = \frac{2^{2k+2}}{(2k+2)!} - \frac{2k+2}{(2k+1)!} \\
&= \frac{2^{2k+2} - (2k+2)^2}{(2k+2)!} > 0
\end{aligned}
$$

for $k \geq 2n+3$. Using inequality (2.5) we have

$$
\frac{\mathcal{O}_{2,j}}{\left|B_{4j}\right|} = \frac{b_{k-2j}c_{2j} + b_{k-2j-1}c_{2j+1}}{\left|B_{4j}\right|}
$$

$$
\begin{aligned}
&= -\frac{(4j-2)\,2^{2k-4j+2}\left(2^{4j-1}-1\right)}{(2k-4j+2)!\,(4j)!} + \frac{4j2^{2k-4j}\left(2^{4j+1}-1\right)}{(2k-4j)!\,(4j+2)!}\frac{\left|B_{4j+2}\right|}{\left|B_{4j}\right|} \\
&> -\frac{(4j-2)\,2^{2k-4j+2}\left(2^{4j-1}-1\right)}{(2k-4j+2)!\,(4j)!} \\
&\quad + \frac{4j2^{2k-4j}\left(2^{4j+1}-1\right)}{(2k-4j)!\,(4j+2)!}\frac{2^{4j-1}-1}{2^{4j+1}-1}\frac{(4j+2)\,(4j+1)}{\pi^2} \\
&= \frac{\left(2^{4j-1}-1\right)2^{2k-4j+2}}{(4j)!\,(2k-4j+2)!}F_2\,(k,j)\,,
\end{aligned}
$$

where

$$F_2\,(k,j) = \frac{(2k-4j+2)\,(2k-4j+1)\,j}{\pi^2} - (4j-2)\,.$$

In view of $k \geq 2n+3$ and $1 \leq j \leq n$, we see that $2k-4j \geq 6$, which indicates that

$$F_2\,(k,j) > \frac{(6+2)\,(6+1)\,j}{\pi^2} - (4j-2) = \left(\frac{56}{\pi^2}-4\right)j+2 > 0.$$

This proves $\mathcal{O}_{2,j} > 0$ for $k \geq 2n+3$ and $1 \leq j \leq n$, and so $d_{k,2n+1} > 0$ for $k \geq 2n+2$.

By (i) of Lemma 2.1 we obtain that the double inequality (3.1) holds for all $t > 0$.
We are now in a position to prove Theorem 1.5.

Proof of Theorem 1.5. By (1.8) $g_n\,(x)$ can be expressed as

$$g_n\,(x) = \frac{1}{4}\int_0^\infty q_n\left(\frac{t}{2}\right)e^{-xt}dt,$$

where

$$q_n\,(t) = \frac{1}{t}Q_n\,(t) = \frac{1}{t\sinh t} + \sum_{k=0}^{n}\frac{2\left(2^{2k-1}-1\right)B_{2k}}{(2k)!}t^{2k-2}. \tag{3.2}$$

It suffices to prove $q_{2n+1}\,(t)$ and $-q_{2n}\,(t)$ are nonnegative and increasing on $(0,\infty)$. Differentiation yields

$$t^2q_n'\,(t) = -\frac{\sinh t + t\cosh t}{\sinh^2 t} + \sum_{k=0}^{n}\frac{2\,(2k-2)\left(2^{2k-1}-1\right)B_{2k}}{(2k)!}t^{2k-1}.$$

Proposition 3.1 indicates that $q_{2n+1}'\,(t) > 0$ for $t \in (0,\infty)$, which gives $q_{2n+1}\,(t) > \lim_{t\to 0} q_{2n+1}\,(t) = 0$. Analogously, we have $-q_{2n}'\,(t) > 0$ for $t \in (0,\infty)$, so $-q_{2n}\,(t) > -\lim_{t\to 0} q_{2n}\,(t) = 0$. By Proposition 1.1 the desired results follow, which completes the proof.

4. An alternative proof of Corollary 1.1

We begin with the following proposition.

Proposition 4.1. *Let integer $n, m \geq 0$, and*

$$S(t,m) = -\sum_{k=0}^{m}\frac{(k-1)\,2^{2k+2}B_{2k}}{(2k)!}t^{2k-1} = \sum_{k=0}^{m}(-1)^k\,\frac{(k-1)\,2^{2k+2}\,|B_{2k}|}{(2k)!}t^{2k-1}.$$

Then the double inequality

$$S(t,2n+1) < \frac{2t+\sinh(2t)}{\sinh^2 t} < S(t,2n) \tag{4.1}$$

holds for all $t > 0$.

Proof. Let

$$
\begin{aligned}
V(t,m) &= \frac{t}{2}S(t,m) = -\frac{t}{2}\sum_{k=0}^{m}\frac{(k-1)\,2^{2k+2}B_{2k}}{(2k)!}t^{2k-1} \\
&= -\sum_{k=0}^{\infty}\frac{(2k-2)\,2^{2k}B_{2k}}{(2k)!}t^{2k}.
\end{aligned}
$$

Then inequalities (4.1) can be written as

$$V(t,2n+1)<\frac{1+(\sinh 2t)/(2t)}{(\cosh 2t-1)/\left(2t^2\right)}<V(t,2n).$$

Let

$$\begin{aligned}\overline{A}(t)&=1+\frac{\sinh 2t}{2t}=2+\sum_{k=1}^{\infty}\frac{2^{2k}}{(2k+1)!}t^{2k},\\ \overline{B}(t)&=\frac{\cosh 2t-1}{2t^2}=\sum_{k=0}^{\infty}\frac{2^{2k+1}}{(2k+2)!}t^{2k},\\ \overline{C}(t)&=-\sum_{k=0}^{\infty}\frac{(2k-2)\,2^{2k}B_{2k}}{(2k)!}t^{2k}.\end{aligned}$$

(1) We prove

$$d_{k,2n}=\sum_{i=1}^{2n}b_{k-i}c_i-a_k=(b_kc_0-a_k)+\sum_{j=1}^{n}\left(b_{k-2j+1}c_{2j-1}+b_{k-2j}c_{2j}\right)>0$$

for $k\geq 2n+1$ by distinguishing two cases.

Case 1.1: $k=2n+1$. Since $\sum_{i=0}^{k}b_{k-i}c_i=a_k$, we see that

$$d_{k,2n}=\sum_{i=0}^{2n}b_{k-i}c_i-a_k=-b_0c_{2n+1}=\frac{4n2^{4n+2}}{(4n+2)!}B_{4n+2}>0.$$

Case 1.2: $k\geq 2n+2$. We first check that $d_{k,2n}<0$ if $n=1$. A simple computation yields then

$$\begin{aligned}d_{k,2}&=(b_kc_0-a_k)+(b_{k-1}c_1+b_{k-2}c_2)\\ &=\frac{2^{2k+2}}{(2k+2)!}-\frac{2^{2k}}{(2k+1)!}+\frac{2}{45}\frac{2^{2k-3}}{(2k-2)!}\\ &=\frac{2^{2k+2}}{(2k+2)!}+\frac{(2k+1)\,k\,(2k-1)-90}{90}\frac{2^{2k}}{(2k+1)!}>0\end{aligned}$$

for $k\geq 2n+2=4$. We next show that $d_{k,2n}<0$ for $n\geq 2$. To this end, we write $d_{k,2n}$ as

$$\begin{aligned}d_{k,2n}&=(b_kc_0-a_k)+(b_{k-1}c_1+b_{k-2}c_2)+\sum_{j=2}^{n}\left(b_{k-2j+1}c_{2j-1}+b_{k-2j}c_{2j}\right)\\ &:=d_{k,2}+\sum_{j=2}^{n}\mathcal{E}_{2,j}.\end{aligned}$$

It has been shown that $d_{k,2}>0$ for $k\geq 4$, and it suffices to show that $\mathcal{E}_{2,j}=b_{k-2j+1}c_{2j-1}+b_{k-2j}c_{2j}>0$ for $2\leq j\leq n$. Using inequality (2.4) we have

$$\begin{aligned}\frac{\mathcal{E}_{2,j}}{\left|B_{4j-2}\right|}&=\frac{b_{k-2j+1}c_{2j-1}+b_{k-2j}c_{2j}}{\left|B_{4j-2}\right|}\\ &=\frac{2^{2k+1}(4j-2)}{(2k-4j+2)!\,(4j)!}\frac{\left|B_{4j}\right|}{\left|B_{4j-2}\right|}-\frac{2^{2k+1}(4j-4)}{(2k-4j+4)!\,(4j-2)!}\\ &>\frac{2^{2k+1}(4j-2)}{(2k-4j+2)!\,(4j)!}\frac{2^{4j-3}-1}{2^{4j-1}-1}\frac{4j(4j-1)}{\pi^2}\\ &\quad-\frac{2^{2k+1}(4j-4)}{(2k-4j+4)!\,(4j-2)!}\\ &=\frac{2^{2k+1}}{(2k-4j+4)!\,(4j-2)!\left(2^{4j-1}-1\right)}F_3(k,j),\end{aligned}$$

where

$$F_3(k,j)=(2k-4j+4)(2k-4j+3)(4j-2)\left(2^{4j-3}-1\right)\frac{1}{\pi^2}-\left(2^{4j-1}-1\right)(4j-4).$$

In view of $k \geq 2n+2$ and $2 \leq j \leq n$, we see that $2k-4j \geq 4$, which together with $\pi^2 < 79/8$ yields

$$\begin{aligned} F_3(k,j) &\geq 8 \times 7(4j-2)\left(2^{4j-3}-1\right)\frac{1}{79/8}-\left(2^{4j-1}-1\right)(4j-4) \\ &= \frac{2}{79}\left[\left((23+33j)\,2^{4j}-738j+290\right)\right] > 0. \end{aligned}$$

This leads to $d_{k,2n} < 0$ for $k \geq 2n+2$ and $n \geq 2$, which proves $d_{k,2n} < 0$ for $k \geq 2n+2$. Cases 1.1 and 1.2 result in $d_{k,2n} < 0$ for $k \geq 2n+1$.

(2) We now prove

$$d_{k,2n+1} = (b_k c_0 + b_{k-1}c_1 - a_k) + \sum_{j=1}^{n}\left(b_{k-2j}c_{2j} + b_{k-2j-1}c_{2j+1}\right) < 0$$

for $k \geq 2n+2$. Similarly, we distinguish two cases to prove it.

Case 2.1. $k = 2n+2$. Since $\sum_{i=0}^{k} b_{k-i}c_i = a_k$, we see that

$$d_{k,2n+1} = \sum_{i=0}^{2n+1} b_{k-i}c_i - a_k = -b_0 c_{2n+2} = \frac{(4n+2)\,2^{4n+4}B_{4n+4}}{(4n+4)!} < 0.$$

Case 2.2. $k \geq 2n+3$. We have

$$\begin{aligned} \mathcal{O}_1 &= b_k c_0 + b_{k-1}c_1 - a_k = \frac{2^{2k+2}}{(2k+2)!} - \frac{2^{2k}}{(2k+1)!} \\ &= -(k-1)\frac{2^{2k+1}}{(2k+2)!} < 0 \end{aligned}$$

for $k \geq 2n+3$. Using inequality (2.5) we have

$$\begin{aligned} \frac{\mathcal{O}_{2,j}}{\left|B_{4j}\right|} &= \frac{b_{k-2j}c_{2j} + b_{k-2j-1}c_{2j+1}}{\left|B_{4j}\right|} \\ &= \frac{2^{2k+1}(4j-2)}{(2k-4j+2)!\,(4j)!} - \frac{2^{2k+1}4j}{(2k-4j)!\,(4j+2)!}\frac{\left|B_{4j+2}\right|}{\left|B_{4j}\right|} \\ &< \frac{2^{2k+1}(4j-2)}{(2k-4j+2)!\,(4j)!} \\ &\quad - \frac{2^{2k+1}4j}{(2k-4j)!\,(4j+2)!}\frac{2^{4j-1}-1}{2^{4j+1}-1}\frac{(4j+2)(4j+1)}{\pi^2} \\ &= \frac{2^{2k+1}}{\left(2^{4j+1}-1\right)(4j)!\,(2k-4j+2)!}F_4(k,j), \end{aligned}$$

where

$$F_4(k,j) = (4j-2)\left(2^{4j+1}-1\right) - (2k-4j+2)(2k-4j+1)\frac{4j\left(2^{4j-1}-1\right)}{\pi^2}.$$

Since $k \geq 2n+3$ and $1 \leq j \leq n$, we see that $2k-4j \geq 6$, which together with $\pi^2 < 79/8$ indicates that

$$\begin{aligned} F_4(k,j) &< (4j-2)\left(2^{4j+1}-1\right) - 8 \times 7\frac{4j\left(2^{4j-1}-1\right)}{79/8} \\ &= -\frac{2}{79}\left[(158+132j)\,2^{4j} - 738j - 79\right] < 0. \end{aligned}$$

This proves $\mathcal{O}_{2,j} > 0$ for $k \geq 2n+3$ and $1 \leq j \leq n$, and so $d_{k,2n+1} > 0$ for $k \geq 2n+2$.

By (ii) of Lemma 2.1 we can obtain that the double inequality (4.1) holds for all $t > 0$.

By means of Proposition 4.1, we easily give an alternative proof of Corollary 1.1.

Proof of Corollary 1.1. As shown in [15, Eq. (3.10)], $f_n(x)$ can be written as

$$f_n(x) = \int_0^\infty P_n\left(\frac{t}{2}\right)\frac{e^{-xt}}{2t}dt = \frac{1}{4}\int_0^\infty p_n\left(\frac{t}{2}\right)e^{-xt}dt, \tag{4.2}$$

where

$$p_n(t) = \frac{P_n(t)}{t} = \frac{\coth t}{t} - \sum_{k=0}^{n}\frac{2^{2k}B_{2k}}{(2k)!}t^{2k-2}. \tag{4.3}$$

Then it suffices to prove $p_{2n}(t)$ and $-p_{2n+1}(t)$ are nonnegative and increasing on $(0,\infty)$. Differentiation leads us to

$$2t^2 p_n'(t) = -\frac{2t+\sinh(2t)}{\sinh^2 t} - \sum_{k=0}^{n} \frac{(k-1)\,2^{2k+2}B_{2k}}{(2k)!} t^{2k-1}.$$

By Proposition 4.1 we find that $p_{2n}'(t) > 0$ for $t \in (0,\infty)$, which gives $p_{2n}(t) > \lim_{t\to 0} p_{2n}(t) = 0$. Likewise, we have $-p_{2n+1}(t) > 0$ for $t \in (0,\infty)$, so $-p_{2n+1}(t) > -\lim_{t\to 0} p_{2n+1}(t) = 0$. From Proposition 1.1 the desired assertion follow, which ends the proof.

5. Remarks and conjectures

Remark 5.1. As shown in [15, Eq. (3.12)], $h_n(x)$ has the following integral representation:

$$h_n(x) = f_n(x) - g_n(x) = \int_0^\infty W_n\left(\frac{t}{4}\right)\frac{e^{-xt}}{2t}dt = \frac{1}{8}\int_0^\infty w_n\left(\frac{t}{4}\right)e^{-xt}dt, \tag{5.1}$$

where

$$w_n(t) = \frac{W_n(t)}{t} = \frac{\tanh t}{t} - \sum_{k=0}^{n} \frac{2^k\left(2^{2k}-1\right)B_{2k}}{(2k)!} t^{2k-2}. \tag{5.2}$$

It is easy to check that

$$\begin{aligned} p_n(t) - q_n(t) &= \frac{\coth t}{t} - \sum_{k=0}^{n} \frac{2^{2k}B_{2k}}{(2k)!} t^{2k-2} - \frac{1}{t\sinh t} - \sum_{k=0}^{n} \frac{2\left(2^{2k-1}-1\right)B_{2k}}{(2k)!} t^{2k-2} \\ &= \frac{1}{2}\left(\frac{\tanh(t/2)}{t/2} - \sum_{k=0}^{n} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{(2k)!}\left(\frac{t}{2}\right)^{2k-2}\right) = \frac{1}{2} w_n\left(\frac{t}{2}\right). \end{aligned}$$

It has been shown in the proofs of Theorem 1.5 and Corollary 1.1 that all $p_{2n}'(t), p_{2n}(t) - q_{2n}'(t), -q_{2n}'(t)$ are positive for $t \in (0,\infty)$, which yields

$$\begin{aligned} \frac{1}{2} w_{2n}\left(\frac{t}{2}\right) &= p_{2n}(t) + \left[-q_{2n}(t)\right] > 0, \\ \frac{1}{2} w_{2n}'\left(\frac{t}{2}\right) &= p_{2n}'(t) + \left[-q_{2n}'(t)\right] > 0 \end{aligned}$$

for $t > 0$. Similarly, we easily obtain $-w_{2n+1}(t/2), -w_{2n+1}'(t/2) > 0$ for $t > 0$. It then follows from Proposition 1.1 that $h_{2n}(x)$ and $-h_{2n+1}(x)$ are strictly strongly completely monotonic on $(0,\infty)$. This can be stated as a theorem.

Theorem 5.1. *For given integer $n \geq 0$, let the function $x \mapsto h_n(x)$ be defined on $(0,\infty)$ by (1.5). Then both the functions $x \mapsto h_{2n}(x)$ and $x \mapsto -h_{2n+1}(x)$ are strictly strongly completely monotonic on $(0,\infty)$.*

In 2012, Guo and Qi [25] further introduced the concept of completely monotonic degrees of nonnegative functions on $(0,\infty)$, which was slightly modified in [26] as follows.

Definition 3. Let $f(x)$ be a completely monotonic function on $(0,\infty)$ and denote $f(\infty) = \lim_{x\to\infty} f(x)$. If for some $r \in \mathbb{R}$ the function $x^r[f(x) - f(\infty)]$ is completely monotonic on $(0,\infty)$ but $x^{r+\varepsilon}[f(x) - f(\infty)]$ is not for any positive number $\varepsilon > 0$, then we say that the number r is the completely monotonic degree of $f(x)$ with respect to $x \in (0,\infty)$, and denote r by $\deg_{cm}^x[f(x)]$; if for all $r \in \mathbb{R}$ each and every $x^r[f(x)-f(\infty)]$ is completely monotonic on $(0,\infty)$, then we say that the completely monotonic degree of $f(x)$ with respect to $x \in (0,\infty)$ is ∞, and denote by $\deg_{cm}^x[f(x)] = \infty$.

From this definition and Theorem 1.4 we pose the following open problem.

Conjecture 5.1. *For given integer $n \geq 0$, let the function $x \mapsto f_n(x)$ be defined on $(0,\infty)$ by (1.3). Then we have*

$$r = \deg_{cm}^x\left[(-1)^n f_n(x)\right] = \begin{cases} 0, & \text{if } n = 0 \\ 2n-1, & \text{if } n \geq 1 \end{cases}.$$

Remark 5.2. We claim that Conjecture 5.1 holds for $n = 1$. In fact, Theorem 1.4 implies that

$$r = \deg_{cm}^x\left[-f_1(x)\right] \geq 1.$$

On the other hand, we have

$$\left[-x_1^r f(x)\right]' = -x^{r-1} f_1(x)\left(r - \frac{-xf_1'(x)}{f_1(x)}\right) \le 0,$$

which indicates that

$$r \le \inf_{x>0}\left(\frac{-xf_1'(x)}{f_1(x)}\right).$$

If we prove the function $x \mapsto -xf_1'(x)/f_1(x)$ is increasing from $(0,\infty)$ onto $(1,3)$, then

$$r = \deg_{cm}^x[-f_1(x)] \le 1,$$

and then $r = \deg_{cm}^x[-f_1(x)] = 1$. Now, using the integral representation (4.2) we obtain

$$f_1(x) = \frac{1}{4}\int_0^\infty p_1\left(\frac{t}{2}\right)e^{-xt}dt,$$

where

$$p_1(t) = \frac{\coth t}{t} - \frac{1}{t^2} - \frac{1}{3}.$$

Then integration by parts yields

$$xf_1'(x) = -\frac{1}{4}x\int_0^\infty tp_1\left(\frac{t}{2}\right)e^{-xt}dt = \frac{1}{4}\left[tp_1\left(\frac{t}{2}\right)e^{-xt}\right]_{t=0}^\infty$$
$$-\frac{1}{4}\int_0^\infty\left[tp_1\left(\frac{t}{2}\right)\right]'e^{-xt}dt = -\frac{1}{4}\int_0^\infty\left[tp_1\left(\frac{t}{2}\right)\right]'e^{-xt}dt,$$

and then

$$\frac{-xf_1'(x)}{f_1(x)} = \frac{\int_0^\infty \frac{d}{ds}[sp_1(s)]e^{-xt}dt}{\int_0^\infty p_1(s)e^{-xt}dt},$$

where $s = t/2$. Direct computations give

$$\frac{\frac{d}{ds}[sp_1(s)]}{p_1(s)} = \frac{\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)'}{\frac{\coth s}{s} - \frac{1}{s^2} - \frac{1}{3}} = \frac{3s^2\cosh^2 s - 2s^2\sinh^2 s - 3\sinh^2 s}{(\sinh s)\left(3\sinh s - 3s\cosh s + s^2\sinh s\right)},$$
$$= \frac{s^2\cosh 2s - 3\cosh 2s + 5s^2 + 3}{s^2\cosh 2s + 3\cosh 2s - 3s\sinh 2s - s^2 - 3} = \frac{\sum_{n=3}^\infty \frac{(2n+3)(n-2)}{(2n)!}(2s)^{2n}}{\sum_{n=3}^\infty \frac{(2n-3)(n-2)}{(2n)!}(2s)^{2n}}$$
$$:= \frac{\sum_{n=3}^\infty a_n(2s)^{2n}}{\sum_{n=3}^\infty b_n(2s)^{2n}}.$$

Clearly, the sequence $\{a_n/b_n\}_{n\ge3}$ is strictly decreasing in view of

$$\frac{a_n}{b_n} = \frac{(2n+3)(n-2)}{(2n)!}\Big/\frac{(2n-3)(n-2)}{(2n)!} = \frac{2n+3}{2n-3},$$

so is the ratio $\frac{d}{ds}[sp_1(s)]/p_1(s)$ on $(0,\infty)$ by Lemma 2.3. Using the monotonicity rule for ratio of Laplace transforms given in Lemma 2.4, we deduce that $x \mapsto -xf_1'(x)/f_1(x)$ is strictly increasing on $(0,\infty)$ with

$$\lim_{x\to0}\frac{-xf_1'(x)}{f_1(x)} = \lim_{s\to\infty}\frac{[sp_1(s)]'}{p_1(s)} = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n+3}{2n-3} = 1,$$
$$\lim_{x\to\infty}\frac{-xf_1'(x)}{f_1(x)} = \lim_{s\to0}\frac{[sp_1(s)]'}{p_1(s)} = \frac{a_3}{b_3} = \left[\frac{2n+3}{2n-3}\right]_{n=3} = 3,$$

where the first limit holds due to Lemma 2.5. Consequently, $r = \deg_{cm}^x[-f_1(x)] = 1$.

Remark 5.3. By the same method and technique, we can prove that the function $x \mapsto -xg_1'(x)/g_1(x)$ is increasing from $(0,\infty)$ onto $(1,3)$, which implies that

$$r = \deg_{cm}^x[g_1(x)] \le 1.$$

This together with Theorem 1.5 yields

$$r = \deg_{cm}^x[g_1(x)] = 1.$$

The assertion allows us to pose the second conjecture.

Conjecture 5.2. *For given integer $n \geq 0$, let the function $x \mapsto g_n(x)$ be defined on $(0,\infty)$ by (1.7). Then we have*

$$r = \deg_{cm}^{x}\left[(-1)^{n-1} g_n(x)\right] = \begin{cases} 0, & \text{if } n = 0 \\ 2n-1, & \text{if } n \geq 1 \end{cases}.$$

Moreover, the increasing property of $x \mapsto -xf_1'(x)/f_1(x)$ on $(0,\infty)$ implies the following assertion.

Proposition 5.1. *Both of the functions*

$$x \mapsto xf_1'(x) + f_1(x) = \ln\Gamma(x) + x\psi(x) + x + \frac{1}{2} - \left(2x - \frac{1}{2}\right)\ln x - \frac{1}{2}\ln(2\pi)$$

$$\begin{aligned} x \mapsto\ & -3f_1(x) - xf_1'(x) \\ =\ & -3\ln\Gamma(x) - x\psi(x) - 3x + \frac{1}{6x} - \frac{1}{2} + \left(4x - \frac{3}{2}\right)\ln x + \frac{3}{2}\ln(2\pi) \end{aligned}$$

are strictly completely monotonic on $(0,\infty)$. In particular, the double inequality

$$\sqrt{\frac{2\pi}{x}}x^{2x}\exp\left[-x\psi(x) - x + \frac{1}{2}\right] < \Gamma(x) < \sqrt{\frac{2\pi}{x}}x^{4x/3}\exp\left(-\frac{1}{3}x\psi(x) - x + \frac{1}{18x} - \frac{1}{6}\right)$$

holds for $x > 0$.

Analogously, the increasing property of $x \mapsto -xg_1'(x)/g_1(x)$ on $(0,\infty)$ indicates the following assertion.

Proposition 5.2. *Both of the functions*

$$\begin{aligned} x \mapsto\ & -xg_1'(x) - g_1(x) \\ =\ & -\ln\Gamma\left(x + \frac{1}{2}\right) - x\psi\left(x + \frac{1}{2}\right) - x + 2x\ln x + \frac{1}{2}\ln(2\pi) \\ x \mapsto\ & 3g_1(x) + xg_1'(x) \\ =\ & 3\ln\Gamma\left(x + \frac{1}{2}\right) + x\psi\left(x + \frac{1}{2}\right) + 3x + \frac{1}{12x} - 4x\ln x - \frac{3}{2}\ln(2\pi) \end{aligned}$$

are strictly completely monotonic on $(0,\infty)$. In particular, the double inequality

$$\begin{aligned} & \sqrt{2\pi}x^{4x/3}\exp\left[-\frac{x}{3}\psi\left(x + \frac{1}{2}\right) - x - \frac{1}{36x}\right] \\ <\ & \Gamma\left(x + \frac{1}{2}\right) \\ <\ & \sqrt{2\pi}x^{2x}\exp\left[-x\psi\left(x + \frac{1}{2}\right) - x\right] \end{aligned}$$

holds for $x > 0$.

Acknowledgments

The author is thankful to reviewers for their careful corrections and valuable comments on the original version of this paper.

This paper is supported by the Natural Science Foundation of China grants No. 61772025.

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