ELSEVIER

Journal of Computational and Applied Mathematics 50 (1994) 207–219

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

A shortcut to asymptotics for orthogonal polynomials

Thomas Dehn

Institut für Praktische Mathematik, D-76128 Universität Karlsruhe, Germany

Received 11 June 1992; revised 8 February 1993

Abstract

We consider the asymptotic behavior of the ratios $q_{n+1}(z)/q_n(z)$ of polynomials orthonormal with respect to some positive measure μ. Let the recurrence coefficients α_n and β_n converge to 0 and $\frac{1}{2}$, respectively. Then, $q_{n+1}(z)/q_n(z) \to \Phi(z)$, for $n \to \infty$, locally uniformly for $z \in \mathbb{C}\backslash\operatorname{supp}\mu$, where Φ maps $\mathbb{C}\backslash[-1, 1]$ conformally onto the exterior of the unit disc (Nevai (1979)). We provide a new and direct proof for this and some related results due to Nevai, and apply it to convergence acceleration of diagonal Padé approximants.

Key words: Orthogonal polynomials; Recurrence relations; Poincaré's Theorem; Ratio asymptotics; Padé approximants; Convergence acceleration; Δ^2-method

1. Introduction

Let μ be a positive measure with real support E. A polynomial q_n of degree n is called an nth *orthogonal polynomial* with respect to μ if

$$\int_E q_n(x)x^j \, \mathrm{d}\mu(x) = 0, \quad \text{for } j = 0, 1, \ldots, n-1, \tag{1}$$

and called *orthonormal* if, additionally,

$$\int_E q_n(x)^2 \, \mathrm{d}\mu(x) = 1,$$

and the leading coefficient γ_n of q_n is positive.

As it is well known, the orthonormal polynomials q_n satisfy the following property.

Property 1.1 (Recurrence relations). *There exist real α_n and positive β_n with*

$$zq_n(z) = \beta_n q_{n+1}(z) + \alpha_n q_n(z) + \beta_{n-1} q_{n-1}(z), \quad \textit{for } n \geqslant 1. \tag{2}$$

Obviously,

$$\beta_n = \frac{\gamma_n}{\gamma_{n+1}}. \tag{3}$$

0377-0427/94/\$07.00
SSDI 0377-0427(93)E0134-8

Following [13, p.10], we denote by $M(0, 1)$ the class of measures for which the recurrence coefficients α_n and β_n converge to 0 and $\frac{1}{2}$, respectively. Reference [20] gives a survey of measures contained in $M(0, 1)$.

An extensive research on recurrence relations has been carried out long ago, and various results are scattered over a multitude of papers. Therefore we have included an (incomplete) historical section.

It is instructive to write the recurrence relations (2) in matrix form:

$$\begin{pmatrix} q_{n+1}(z) \\ q_n(z) \end{pmatrix} = \begin{pmatrix} \dfrac{z-\alpha_n}{\beta_n} & -\dfrac{\beta_{n-1}}{\beta_n} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_n(z) \\ q_{n-1}(z) \end{pmatrix} =: A_n(z) \begin{pmatrix} q_n(z) \\ q_{n-1}(z) \end{pmatrix}. \tag{4}$$

If $\mu \in M(0, 1)$, then the matrices $A_n(z)$ are convergent:

$$A_n(z) \to A(z) := \begin{pmatrix} 2z & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{for } n \to \infty. \tag{5}$$

The eigenvalues of $A(z)$ are $\Phi(z)$ and $(\Phi(z))^{-1}$ with corresponding eigenvectors $(\Phi(z), 1)^{\mathrm{T}}$ and $(1, \Phi(z))^{\mathrm{T}}$. Hence Nevai's Theorem states the convergence of $q_{n+1}(z)/q_n(z)$ to the dominant eigenvalue of $A(z)$. We provide a simple method of deriving this result and related theorems in Section 4. To this end, we investigate the behavior of the zeros of the orthonormal polynomials q_n and show normal family properties for the ratios of the monic orthogonal polynomials. These preparations are made in Section 3. In Section 5 we derive new convergence results for diagonal Padé approximants.

2. Recurrence relations and Poincaré's Theorem

Each sequence of polynomials q_n which satisfies (2) with positive β_n and real α_n is a sequence of polynomials orthonormal with respect to some nonnegative measure μ with real support. This statement (usually called Favard's Theorem after a note by Favard [6]) follows from Stieltjes' [18] and Hamburger's [8] solutions to the moment problem on the real line. If the β_n are not necessarily positive, but only nonvanishing real numbers, then the q_n are orthogonal with respect to some (nonunique) signed measure with real support [17].

Poincaré [14, Chapters 1, 2, 6] was probably the first who proved ratio asymptotics for recursively defined quantities.

Theorem 2.1 (Poincaré [14]). *Let the numbers $f(n)$ satisfy*

$$f(n) + \alpha_{n,1} f(n-1) + \cdots + \alpha_{n,k} f(n-k) = 0, \quad \textit{for } n = k, k+1, \ldots, \tag{6}$$

with $\lim_{n\to\infty} \alpha_{n,j} = a_j$ for $j = 1, \ldots, k$, and let the characteristic polynomial

$$p(\lambda) = \lambda^k + a_1\lambda^{k-1} + \cdots + a_{k-1}\lambda + a_k$$

have k zeros $\zeta_1, \ldots, \zeta_k$ with different absolute values. Then either $f(n) = 0$ for all $n \geqslant n_0$, or $f(n+1)/f(n)$ converges to some zero ζ_j.

Poincaré's own proof of Theorem 2.1 was very complicated, and a shorter proof of Theorem 2.1 for the special case of three-term recurrence relations was given in [21, Section 2]. Meanwhile, Blumenthal [2, pp. 16–21] had used Poincaré's Theorem and Stieltjes' solution of the moment problem to derive ratio asymptotics for orthonormal polynomials. He formulated his results in terms of continued fractions. In modern notation, one of his results reads as follows.

Theorem 2.2 (Blumenthal [2]). *Let the polynomials q_n satisfy the recurrence relations* (2) *with convergent recurrence coefficients $\alpha_n \to 0$ and $0 < \beta_n \to \frac{1}{2}$. Then the polynomials q_n are orthogonal with respect to some measure μ, whose support is the union of $[-1, 1]$ with finitely many isolated points. Pointwise, for $z \notin \operatorname{supp} \mu$, there holds*

$$\frac{q_{n+1}(z)}{q_n(z)} \to \Phi(z).$$

Blumenthal's argumentation is more heuristic than rigorous, and, as a consequence, his computation of supp μ is false. Blumenthal made the faulty assumption that $\mu \in M(0, 1)$ implied

$$\frac{q_{n+1}(1)}{q_n(1)} \to 1, \quad \text{for } n \to \infty.$$

But this is valid only if the support of μ is the union of $[-1, 1]$ with at most finitely many points. We will come back to this problem in Section 4.1.

Example 2.3. If $\alpha_n = 0$ for all n, and $\beta_n \to \frac{1}{2}(1+\epsilon)$, then $[-1-\epsilon, 1+\epsilon] \subseteq \operatorname{supp} \mu$, and hence for $n \to \infty$ the number of zeros of q_n outside $[-1, 1]$ tends to infinity. Therefore, we can choose all the $\alpha_n = 0$ and let the β_n converge so slowly to $\frac{1}{2}$ from above that the number of zeros of q_n which are greater than 1 tends to infinity, contradicting Blumenthal's computation of supp μ.

A generalization of Poincaré's Theorem has been given in [12, Theorem 2].

Theorem 2.4 (Máté and Nevai [12]). *Let $A_n \in \mathbb{C}^{k \times k}$, $n = 1, 2, \ldots$, $\boldsymbol{u}_0 \in \mathbb{C}^k$, and let the vectors $\boldsymbol{u}_n \in \mathbb{C}^k$ be defined recursively by $\boldsymbol{u}_{n+1} := A_n \boldsymbol{u}_n$. If A_n converges to $A \in \mathbb{C}^{k \times k}$ with k eigenvalues $\zeta_1, \ldots, \zeta_k$ with different absolute values, then either $\boldsymbol{u}_n = \boldsymbol{0}$ for all n large enough, or no $\boldsymbol{u}_n$ vanishes, and there exist numbers c_n such that $c_n \boldsymbol{u}_n$ converges to some eigenvector of A.*

Proof. Of course, if $\boldsymbol{u}_{n_0} = \boldsymbol{0}$, then $\boldsymbol{u}_n = \boldsymbol{0}$ for $n \geqslant n_0$. So let us presuppose $\boldsymbol{u}_n \neq \boldsymbol{0}$ for all n and $|\zeta_1| < |\zeta_2| < \cdots < |\zeta_k|$. Denote with $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ eigenvectors of A corresponding to $\zeta_1, \ldots, \zeta_k$ with $\|\boldsymbol{x}_i\|_\infty = 1$, $i = 1, \ldots, k$. These vectors form a basis of $\mathbb{C}^k$, and hence we can write

$$\boldsymbol{u}_n = \mu_{n,1}\boldsymbol{x}_1 + \cdots + \mu_{n,k}\boldsymbol{x}_k, \qquad A^m \boldsymbol{u}_n = \zeta_1^m \mu_{n,1}\boldsymbol{x}_1 + \cdots + \zeta_k^m \mu_{n,k}\boldsymbol{x}_k, \quad \text{for } m \geqslant 1.$$

Since $A_n \to A$, we obtain for all m,

$$A_{n+m-1} \cdot \ldots \cdot A_n \boldsymbol{u}_n - A^m \boldsymbol{u}_n = \boldsymbol{v}_{n,m} \quad (\text{hence } \boldsymbol{u}_{n+m} = A^m \boldsymbol{u}_n + \boldsymbol{v}_{n,m}), \tag{7}$$

with

$$\frac{\|\boldsymbol{v}_{n,m}\|_\infty}{\|\boldsymbol{u}_n\|_\infty} \leqslant \|A_{n+m-1} \cdot \ldots \cdot A_n - A^m\|_\infty \to 0, \quad \text{for } n \to \infty. \tag{8}$$

Now we take the maximal j which fulfils

$$|\mu_{n,j}| = \max_{1 \leqslant i \leqslant k} |\mu_{n,i}|, \quad \text{for infinitely many } n.$$

Let n_0 be so large that for $i > j$ and $n \geqslant n_0$ no $|\mu_{n,i}|$ is maximal.

In the next step we show the existence of n_1 with

$$|\mu_{n,j}| = \max_{1 \leqslant i \leqslant k} |\mu_{n,i}|, \quad \text{for all } n \geqslant n_1.$$

If $|\mu_{n,j}|$ is maximal, then $\|\boldsymbol{u}_n\|_\infty \leqslant k\,|\mu_{n,j}|$, and with (8) we obtain

$$\lim_{\substack{n \to \infty \\ \mu_{n,j} \text{ maximal}}} \frac{\|\boldsymbol{v}_{n,m}\|_\infty}{|\mu_{n,j}|} = 0. \tag{9}$$

Therefore we can choose $n_1 > n_0$ (with $|\mu_{n_1,j}|$ maximal) so large such that for $\boldsymbol{v}_{n,1} = \sum_{i=1}^k \nu_i \boldsymbol{x}_i$, we have

$$|\nu_j| + |\nu_i| < |\mu_{n,j}| \cdot \big| |\zeta_j| - |\zeta_i| \big|, \quad \text{for all } i \neq j \text{ and all } n \geqslant n_1,$$

for which $|\mu_{n,j}|$ is maximal. With (7) this implies

$$|\mu_{n+1,i}| = |\mu_{n,i}\zeta_i + \nu_i| < |\mu_{n,j}\zeta_j + \nu_j| = |\mu_{n+1,j}|, \quad \text{for } i < j.$$

Hence $|\mu_{n+1,j}|$ is maximal, too, and by induction $|\mu_{n,j}|$ is maximal for all $n \geqslant n_1$. Obviously,

$$\boldsymbol{u}_{n+m} = \zeta_1^m \mu_{n,1} \boldsymbol{x}_1 + \cdots + \zeta_k^m \mu_{n,k} \boldsymbol{x}_k + \boldsymbol{v}_{n,m}.$$

With (9) and $|\zeta_i| < |\zeta_j|$ for $i < j$ this yields $|\mu_{n,i}| / |\mu_{n,j}| \to 0$ for $n \to \infty$ and $i < j$. But we also obtain $|\mu_{n,i}| / |\mu_{n,j}| \to 0$ for $n \to \infty$ and $i > j$, because $|\mu_{n,j}| > |\mu_{n,i}|$ for all $n \geqslant n_1$, although $|\zeta_i| > |\zeta_j|$ for $i > j$. So finally we arrive at

$$\frac{\boldsymbol{u}_n}{\mu_{n,j}} \to \boldsymbol{x}_j, \quad \text{for } n \to \infty. \qquad \square$$

It is easy to formulate parameter-dependent versions of Theorem 2.4 which yield locally uniform convergence. Poincaré's Theorem follows easily from Theorem 2.4 when the recurrence relations (6) are written in matrix form [12]. We mention [10] for a further discussion of recurrence relations.

3. The properties of the zeros

The zeros of the polynomials q_n have many useful properties.

Property 3.1. *All zeros of q_n lie in the convex hull of the support of μ.*

To abbreviate, we set $\tilde{I}(\mu) := \mathrm{conv}(\mathrm{supp}\ \mu)$, the convex hull of the support of μ.

Property 3.2. *Each open interval F with $\mu(F) = 0$ contains at most one zero of q_n.*

Property 3.3. *The zeros $x_1, \ldots, x_n$ of q_n and the zeros $y_1, \ldots, y_{n+1}$ of q_{n+1} interlace:*

$$y_1 < x_1 < y_2 < \cdots < y_n < x_n < y_{n+1}. \tag{10}$$

Properties 3.1 and 3.2 follow directly from the orthogonality relations (1) [7, Satz 2.2, Satz 2.4], [19, Theorem 3.3.1]. Property 3.3 can be demonstrated easily by induction [19, Theorem 3.3.2]. Let $\mathrm{dist}(z, K) := \inf\{|z - x|: x \in K\}$ denote the *distance* from z to K and $\mathrm{Dist}(z, K) := \sup\{|z - x|: x \in K\}$ the *maximal distance* from z to K. For $y \in K$ and $z \in \mathbb{C}$ we have $\mathrm{dist}(z, K) \leqslant |z - y| \leqslant \mathrm{Dist}(z, K)$. Now Property 3.1 reads as follows.

Property 3.4. *If $q_n(y) = 0$ and $z \in \mathbb{C}$, then* $\mathrm{dist}(z, \tilde{I}(\mu)) \leqslant |z - y| \leqslant \mathrm{Dist}(z, \tilde{I}(\mu))$.

Our first lemma shows how Properties 3.3 and 3.4 yield bounds for the ratios

$$\frac{Q_n(z)}{Q_{n+1}(z)} \tag{11}$$

of monic polynomials. Our bounds are uniform throughout all polynomials having the interlacing property, e.g., throughout all *sequences* of monic orthogonal polynomials. The upper bound is well known and can also be obtained from the partial fraction decomposition of Q_n/Q_{n+1}, since (10) implies that all residues are positive [4]. We remark that Lemma 3.5 can be transferred to ratios of polynomials with zeros interlacing on arcs with bounded rotation.

Lemma 3.5. *Let the zeros of the polynomials $Q_n(z) = (z - x_1)(z - x_2) \cdots (z - x_n)$ and $Q_{n+1}(z) = (z - y_1)(z - y_2) \cdots (z - y_{n+1})$ satisfy the interlacing property* (10). *Then for all $z \in \mathbb{C}$ and each set K containing all x_i and y_i, there holds*

$$\frac{\mathrm{dist}(z, K)}{\mathrm{Dist}(z, K)^2} \leqslant \left| \frac{Q_n(z)}{Q_{n+1}(z)} \right| \leqslant \frac{1}{\mathrm{dist}(z, K)}. \tag{12}$$

Proof. We have to consider three cases:

(a) $\mathrm{Re}\ z \leqslant y_1$;

(b) $y_j \leqslant \mathrm{Re}\ z \leqslant y_{j+1}$ for any j, $1 \leqslant j \leqslant n$;

(c) $y_{n+1} \leqslant \mathrm{Re}\ z$.

Case (a): For $y_1 \geqslant \mathrm{Re}\ z$, using the interlacing property (10), we obtain

$$|z - y_i| \leqslant |z - x_i| \leqslant |z - y_{i+1}|, \quad \text{for } 1 \leqslant i \leqslant n.$$

Together with $\mathrm{dist}(z, K) \leqslant \mathrm{Dist}(z, K)$ and Property 3.4 this implies

$$\frac{\mathrm{dist}(z, K)}{\mathrm{Dist}(z, K)^2} \leqslant \frac{1}{\mathrm{Dist}(z, K)} \leqslant \frac{1}{|z - y_{n+1}|} \leqslant \left| \frac{Q_n(z)}{Q_{n+1}(z)} \right| \leqslant \frac{1}{|z - y_1|} \leqslant \frac{1}{\mathrm{dist}(z, K)}.$$

Case (c) is similar to case (a), hence only (12) in case (b) remains to be demonstrated. Assume $x_j \leqslant \mathrm{Re}\, z$. Now the interlacing property (10) leads us to

$$|z-x_i| \leqslant |z-y_i|, \quad \text{for } 1 \leqslant i \leqslant j,$$
$$|z-y_{i+1}| \leqslant |z-x_i|, \quad \text{for } 1 \leqslant i \leqslant j-1,$$
$$|z-y_i| \leqslant |z-x_i| \leqslant |z-y_{i+1}|, \quad \text{for } j+1 \leqslant i \leqslant n.$$

From Property 3.4, we conclude

$$\frac{\mathrm{dist}(z, K)}{\mathrm{Dist}(z, K)^2} \leqslant \frac{|z-x_j|}{|z-y_1|\,|z-y_{n+1}|} \leqslant \left| \frac{Q_n(z)}{Q_{n+1}(z)} \right| \leqslant \frac{1}{|z-y_{j+1}|} \leqslant \frac{1}{\mathrm{dist}(z, K)}.$$

The case $x_j > \mathrm{Re}\, z$ is very similar. All we have to do is to substitute y_{j+1} by y_j in the inequality above. This completes the proof of Lemma 3.5. □

Lemma 3.5 gives rise to the following important definition. We emphasize that an interlacing sequence does not have to be a sequence of monic orthogonal polynomials, because no recurrence relations have to be valid.

Definition 3.6. Let $\{Q_n\}$ be a sequence of monic polynomials of degree n. We call $\{Q_n\}$ an *interlacing sequence* over the set K, if the zeros of Q_n and Q_{n+1} interlace for all n, and if all these zeros are contained in the set K. We denote the set of all interlacing sequences over K with $I(K)$ and write $I(a, b)$ for the set of all interlacing sequences over $[a, b]$. With $Q_k(a, b)$ we denote the set of all ratios

$$\frac{Q_{n+k}(z)}{z^k Q_n(z)}, \tag{13}$$

where $n, n+k \in \mathbb{N}$, $k \in \mathbb{Z}$, and $\{Q_n\}$ is an interlacing sequence over $[a, b]$. $Q_k(K)$ is defined analogously.

Lemma 3.5 has immediate consequences for the ratios (13).

Theorem 3.7. *Let $K \subset \mathbb{C}$ be compact. Then $Q_k(K)$ is uniformly bounded on closed subsets of $\overline{\mathbb{C}} \setminus (K \cup \{0\})$ and forms a normal family of meromorphic functions in $\overline{\mathbb{C}} \setminus K$. Especially, $Q_k(-1, 1)$ is a normal family in $\overline{\mathbb{C}} \setminus [-1, 1]$ for all k, $k \in \mathbb{Z}$.*

Proof. The uniform boundedness of the ratios (13) follows directly from (12) and the property

$$\lim_{z \to \infty} \frac{\mathrm{dist}(z, K)}{|z|} = \lim_{z \to \infty} \frac{\mathrm{Dist}(z, K)}{|z|} = 1. \tag{14}$$

Now Montel's Theorem [1, Theorem 5.15] implies that $Q_k(K)$ is a normal family in $\overline{\mathbb{C}} \setminus K$. □

Corollary 3.8. *Let μ have compact support $E \subseteq [a, b]$, and let $k \in \mathbb{Z}$. Then the ratios $Q_{n+k}(z)/Q_n(z)$ of the corresponding monic orthogonal polynomials are uniformly bounded on closed subsets of $\mathbb{C} \setminus [a, b]$ and form a normal family in $\overline{\mathbb{C}} \setminus [a, b]$.*

Proof. Due to Properties 3.1 and 3.3, the monic orthogonal polynomials $Q_n(z)$ form an interlacing sequence over $[a, b]$. □

Theorem 3.9. *Let $\{Q_n\} \in I(a, b)$, $k \neq 0$ be fixed, $G \subseteq \overline{\mathbb{C}} \setminus [a, b]$ a domain, φ holomorphic in $G \setminus \{\infty\}$ and $\{z_j\}$ a sequence of complex numbers with $\lim_{j\to\infty} z_j = z^* \in G \cup \{\infty\}$. Additionally, for all j let*

$$\lim_{n\to\infty} \frac{Q_{n+k}(z_j)}{Q_n(z_j)} = \varphi(z_j). \tag{15}$$

Then the following statements hold true.

(i) *φ can be continued analytically to $\overline{\mathbb{C}} \setminus [a, b]$. Furthermore,*

$$\frac{Q_{n+k}(z)}{z^k Q_n(z)} \to \frac{\varphi(z)}{z^k}, \quad \textit{for } n \to \infty, \tag{16}$$

locally uniformly on $\overline{\mathbb{C}} \setminus ([a, b] \cup \{0\})$, and

$$\frac{Q_{n+k}(z)}{Q_n(z)} \to \varphi(z), \quad \textit{for } n \to \infty, \tag{17}$$

locally uniformly on $\mathbb{C} \setminus [a, b]$.

(ii) *φ possesses the Laurent series at ∞ of the form*

$$\varphi(z) = z^k + \varphi_{k-1} z^{k-1} + \varphi_{k-2} z^{k-2} + \cdots.$$

With $Q_n(z) = z^n + q_{n-1,n} z^{n-1} + q_{n-2,n} z^{n-2} + \cdots$,

$$\lim_{n\to\infty} (q_{n+k-1,n+k} - q_{n-1,n}) = \varphi_{k-1} \tag{18}$$

is valid.

Proof. Formula (15) implies

$$\frac{Q_{n+k}(z_j)}{z_j^k Q_n(z_j)} \to \frac{\varphi(z_j)}{z_j^k}, \quad \text{for } n \to \infty. \tag{19}$$

By assumption, $\{Q_n\} \in I(a, b)$. Theorem 3.7 states that $\{Q_{n+k}(z)/(z^k Q_n(z))\}$ is a normal family in $\overline{\mathbb{C}} \setminus [a, b]$. Vitali's Theorem and (19) yield the convergence of

$$\frac{Q_{n+k}(z)}{z^k Q_n(z)}$$

to an analytic function ψ in $\overline{\mathbb{C}} \setminus ([a, b] \cup \{0\})$ with $\psi(z_j) = \varphi(z_j)/z_j^k$ for all j. Now the identity theorem for analytic functions completes the proof of our first assertion, because the points z_j have a limit point in $\overline{\mathbb{C}} \setminus [a, b]$. The assertions on the Laurent series of φ follow easily from (16). □

We need another simple lemma which provides bounds for $|q_n(z)|$ on E.

Lemma 3.10. *Let $\{m_n\}$ be a strictly increasing sequence of integers. Then,*

$$\int_E \inf_{n\in\mathbb{N}} \left\{ \frac{|q_{m_n}(x)|^2}{m_n} \right\} \mathrm{d}\mu(x) \leqslant \inf_{n\in\mathbb{N}} \left\{ \frac{1}{m_n} \right\} = 0 \tag{20}$$

and

$$\mu(\{x_0\}) q_{m_n}^2(x_0) \leqslant \int_E q_{m_n}^2(x)\,\mathrm{d}\mu(x) = 1, \tag{21}$$

for each x_0 with $\mu(\{x_0\}) > 0$.

Proof. Both inequalities are elementary. □

Lemma 3.11. *Let $a < \liminf_{n\to\infty}(\alpha_n - \beta_n - \beta_{n-1})$ and $b > \limsup_{n\to\infty}(\alpha_n + \beta_n + \beta_{n-1})$. Then n_0 exists so that for all $n \geqslant n_0$ the following statements hold.*

(i) *If $z \geqslant b$ and $q_n(z) \geqslant \max\{0, q_{n-1}(z)\}$, then $\{q_m(z)\}_{m\geqslant n}$ is strictly increasing.*

(ii) *If $z \leqslant a$ and $(-1)^n q_n(z) \geqslant \max\{0, (-1)^{n-1} q_{n-1}(z)\}$, then the sequence $\{(-1)^m q_m(z)\}_{m\geqslant n}$ is strictly increasing.*

(iii) *If $z \geqslant b$, then $\{q_n(z)\}$ has only finitely many sign changes. If $z \leqslant a$, then $\{(-1)^n q_n(z)\}$ has only finitely many sign changes.*

Proof. It suffices to choose n_0 so large that $a < \alpha_n - \beta_n - \beta_{n-1} < \alpha_n + \beta_n + \beta_{n-1} < b$ for all $n \geqslant n_0$. With the recurrence relations (2) our assertions (i) and (ii) easily follow by induction, and (iii) is a consequence of (i) and (ii). □

Corollary 3.12. *Let $z > \sup(\alpha_n + \beta_n + \beta_{n-1})$. Then $\{q_n(z)\}_{n\geqslant 0}$ is strictly increasing.*

4. The main results

We now have all the tools to prove our main results (cf. [13, Sections 3.3 and 4.1]). Theorem 4.1 and many related results were obtained by Nevai [13], but some of them had already been obtained by Blumenthal [2] in 1898, Van Vleck [21, p.257] in 1904, and Shohat [16, Theorem XV] in 1934.

Theorem 4.1. *Let the polynomials q_n be orthonormal with respect to $\mu \in M(0, 1)$. Then the following statements are valid.*

(i) *$\operatorname{supp}\mu$ is the union of $[-1, 1]$ and a set B consisting of at most denumerably many points x_j. B has no limit points outside $[-1, 1]$ and may be empty.*

(ii) *For $z \in \mathbb{C}\setminus\operatorname{supp}\mu$ the ratios $q_{n+1}(z)/q_n(z)$ converge locally uniformly to $\Phi(z)$:*

$$\frac{q_{n+1}(z)}{q_n(z)} \to \Phi(z), \quad \text{for } n \to \infty. \tag{22}$$

(iii) *For* $z \in \operatorname{supp} \mu \setminus [-1, 1]$ *the ratios* $q_{n+1}(z)/q_n(z)$ *converge pointwise to* $\Phi(z)^{-1}$:

$$\frac{q_{n+1}(z)}{q_n(z)} \to \frac{1}{\Phi(z)}, \quad \textit{for } n \to \infty. \tag{23}$$

(iv) *For each* $x \in \mathbb{R} \setminus \operatorname{supp} \mu$, *there exist* $\epsilon > 0$ *and* n_0 *such that the polynomials* q_n *possess no zero in* $[x-\epsilon, x+\epsilon]$ *for all* $n \geqslant n_0$. *For each* $x \in \operatorname{supp} \mu \setminus [-1, 1]$ *and sufficiently small* $\epsilon > 0$, *there exists an* n_0 *so that for all* $n \geqslant n_0$, q_n *possesses exactly one zero in* $[x-\epsilon, x+\epsilon]$.

Proof. (a) First, we prove (22) for all $z \notin \tilde{I}(\mu)$. Because of (3), Theorem 3.9 and $\beta_n \to \frac{1}{2}$, it is sufficient to demonstrate (22) pointwise for infinitely many $z > b > \sup\{x: x \in \tilde{I}(\mu)\}$. Corollary 3.12 and $\Phi(z) > 1$ for $z > 1$ yield

$$\begin{pmatrix} q_{n+1}(z) \\ q_n(z) \end{pmatrix} = \lambda_{1,n} \begin{pmatrix} \Phi(z) \\ 1 \end{pmatrix} + \lambda_{2,n} \begin{pmatrix} 1 \\ \Phi(z) \end{pmatrix},$$

with $\lambda_{1,n} > \lambda_{2,n} > 0$ for all $z > \sup(\alpha_n + \beta_n + \beta_{n-1})$. With $A_n(z) \to A(z)$, (22) follows for $z > \sup(\alpha_n + \beta_n + \beta_{n-1})$ (cf. Theorem 2.4). This proves (22) for $z \in \mathbb{C} \setminus \tilde{I}(\mu)$.

(b) Secondly, we show $[-1, 1] \subseteq \operatorname{supp} \mu$. Let us suppose that there exists an open interval $F \subseteq [-1, 1]$ with μ-measure 0. Then Property 3.3 entails that no q_n has more than one zero in F. Hence we can choose a subinterval $F_1 \subseteq F$ which contains no zeros of q_{n_m} and q_{n_m+1} for some subsequence $\{q_{n_m}\}$ of $\{q_n\}$. By Theorem 3.7 and (a),

$$\frac{q_{n_m+1}(z)}{q_{n_m}(z)} \to \Phi(z),$$

locally uniformly for $z \in F \cup (\mathbb{C} \setminus \tilde{I}(\mu))$. But $\Phi(z)$ cannot even be continued continuously to F, a contradiction.

(c) By assumption, $\alpha_n \to 0$ and $\beta_n \to \frac{1}{2}$. Lemma 3.11(iii) yields that for real $z \notin [-1, 1]$, $\{q_n(z)\}$ has only finitely many sign changes. Now Property 3.4 shows that for each $\epsilon > 0$, there exists some $N = N(\epsilon)$ such that each q_n possesses no more than N zeros outside of $[-1-\epsilon, 1+\epsilon]$.

Let $x \in \operatorname{supp} \mu \setminus [-1-\epsilon, 1+\epsilon]$. By $|\Phi(x)| > 1$ and (21), $q_{n+1}(x)/q_n(x) \to \Phi(x)$ is impossible. As in (b), by Theorem 3.7 we obtain that for each $\delta > 0$ there exists an n_0 with $q_n(x_n) = 0$ for some $x_n \in [x-\delta, x+\delta]$ for all $n \geqslant n_0$. Hence $\operatorname{supp} \mu \setminus [-1-\epsilon, 1+\epsilon]$ contains N points at most, and to each of these (isolated) points one zero of q_n converges. Now we conclude with Property 3.2 that there are no other zeros of q_n outside of $[-1-\epsilon, 1+\epsilon]$. This completes the demonstration of (iv).

(d) (iii) is true, because otherwise (22) would be valid in an environment of x, contradicting (c) and (21). Now (ii) follows from (a) and Theorem 3.7. □

Our method allows us to prove a generalization of another theorem of Nevai [13, Theorem 4.1.12]. We remark that the condition of μ having compact support can be dropped if $z^* \neq \infty$.

Theorem 4.2. *Let* supp μ *be compact and* $z^* \in \overline{\mathbb{C}} \setminus \tilde{I}(\mu)$. *If there exists any sequence* $\{z_j\}$ *with* $\lim_{j\to\infty} z_j = z^*$ *and the property*

$$\frac{q_{n+1}(z_j)}{q_n(z_j)} \to \Phi(z_j), \quad \textit{for all } j, \tag{24}$$

then $\mu \in M(0, 1)$.

Proof. We choose an arbitrary convergent subsequence $\{\beta_{n_m}\}$ of $\{\beta_n\}$ with limit β, $0 \leqslant \beta \leqslant \infty$. Then, by (3) and (24), for each z_j the ratios of the monic orthogonal polynomials Q_{n_m} are convergent, too:

$$\frac{Q_{n_m+1}(z_j)}{z_j Q_{n_m}(z_j)} \to \frac{\Phi(z_j)}{\beta z_j}, \quad \text{for } m \to \infty.$$

But these ratios form a normal family in $\overline{\mathbb{C}} \setminus \tilde{I}(\mu)$. Hence $\beta > 0$, and the Vitali Theorem implies

$$\frac{Q_{n_m+1}(z)}{z Q_{n_m}(z)} \to \frac{\Phi(z)}{\beta z}, \quad \text{for } m \to \infty,$$

locally uniformly in $\overline{\mathbb{C}} \setminus (\tilde{I}(\mu) \cup \{0\})$. From Lemma 3.5, $\Phi(z) = 2z + \mathrm{O}(z^{-1})$ for $z \to \infty$ and (14), we easily conclude $\beta = \frac{1}{2}$. Thus $\beta_n \to \frac{1}{2}$ and, by (18), we obtain $\alpha_n = q_{n-1,n} - q_{n,n+1} \to 0$. $\square$

4.1. The endpoints −1 and 1

Theorem 4.3. *Let* $\mu \in M(0, 1)$ *and* supp $\mu = [-1, 1] \cup B$. *If B contains only finitely many points* > 1, *then*

$$\frac{q_{n+1}(1)}{q_n(1)} \to 1.$$

If B contains only finitely many points < -1, *then*

$$\frac{q_{n+1}(-1)}{q_n(-1)} \to -1.$$

Proof. If B contains exactly N points > 1, then by Property 3.2 at most N zeros of q_n are greater than 1. Hence, by the interlacing property, the sign of $q_n(1)$ is constant for all $n \geqslant n_0$, say positive. Let a be any limit point of $\{q_{n+1}(1)/q_n(1)\}$. We already know $a \geqslant 0$. We can prove by induction

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}^m \begin{pmatrix} a & 1 \end{pmatrix} = \begin{pmatrix} (m+1)a - m \\ ma - (m-1) \end{pmatrix}, \quad \text{for all } m \geqslant 0.$$

(Obviously $\lim_{m\to\infty}((m+1)a - m)/(ma - (m-1)) = 1$ for all a.) Since the matrix–vector product is continuous and $q_n(1) > 0$ for $n \geqslant n_0$ (that is, $q_{n+m}(x) \approx ((m+1)a - m)q_{n-1}(x)$), this implies $a \geqslant 1$. But then $a > 1$ is impossible, too, because $((m+1)a - m)/(ma - (m-1)) \to 1$. So we obtain $\lim_{n\to\infty} q_{n+1}(1)/q_n(1) = 1$. The proof of our second claim is similar. $\square$

We remark that $\{q_n(1)\}$ has infinitely many sign changes, if supp μ contains infinitely many points greater than 1. Therefore $q_{n+1}(1)/q_n(1)$ cannot converge in this case.

5. Application to Padé approximants

We define the *Hamburger function* f_μ by

$$f_\mu(z) := \int_E \frac{d\mu(z)}{1-xz}. \tag{25}$$

f_μ is analytic in the domain $\overline{\mathbb{C}}\setminus\{1/(\text{supp }\mu)\}$. The $(n-1, n)$-Padé approximant $[n-1/n]_{f_\mu}$ to f_μ is the unique rational function $u_{n-1,n}(z)/v_{n-1,n}(z)$ with

$$v_{n-1,n}(z)f_\mu(z) - u_{n-1,n}(z) = O(z^{2n}), \quad \text{for } z \to 0. \tag{26}$$

It is well known that the polynomials $q_{n-1,n}(z) := z^n v_{n-1,n}(1/z)$ are orthogonal with respect to μ and that the polynomials $p_{n-1,n}(z) := z^{n-1}u_{n-1,n}(1/z)$ are the corresponding associated polynomials. We orthonormalize $q_{n-1,n}$. From Theorem 4.1 we obtain

$$\lim_{n\to\infty} \frac{v_{n,n+1}(z)}{v_{n-1,n}(z)} = \lim_{n\to\infty} \frac{u_{n,n+1}(z)}{u_{n-1,n}(z)} = z \lim_{n\to\infty} \frac{q_{n,n+1}(1/z)}{q_{n-1,n}(1/z)} = z\Phi\left(\frac{1}{z}\right), \tag{27}$$

locally uniformly for $1/z \notin \text{supp }\mu$. We also consider the residuals

$$R_{n-1,n}(z) := v_{n-1,n}(z)f_\mu(z) - u_{n-1,n}(z).$$

The polynomials $v_{n-1,n}$ and $u_{n-1,n}$ and the residuals satisfy the same recurrence relations

$$v_{n-1,n}(z) = z\alpha_n v_{n-1,n}(z) + \beta_n v_{n,n+1}(z) + z^2\beta_{n-1}v_{n-2,n-1}(z).$$

For $\mu \in M(0, 1)$, the characteristic polynomial of this difference equation is

$$p_z(\lambda) = \tfrac{1}{2}\lambda^2 - \lambda + \tfrac{1}{2}z^2,$$

with zeros $z\Phi(1/z)$ and $z/\Phi(1/z)$. By Theorem 2.2 we conclude

$$\frac{R_{n,n+1}(z)}{R_{n-1,n}(z)} \to \frac{z}{\Phi(1/z)}, \tag{28}$$

pointwise for $z \notin \text{supp }\mu$. The limit $z\Phi(1/z)$ is impossible, because by Markov's Theorem the diagonal Padé approximants converge to $f_\mu(z)$ locally uniformly for $1/z \notin \text{supp }\mu$, and we have

$$\frac{f_\mu(z) - [n/n+1]_{f_\mu}(z)}{f_\mu(z) - [n-1/n]_{f_\mu}(z)} = \frac{R_{n,n+1}(z)}{R_{n-1,n}(z)}\frac{v_{n-1,n}(z)}{v_{n,n+1}(z)}. \tag{29}$$

The ratios (29) form a normal family, and therefore the convergence (28) is locally uniform. We mention that Markov's Theorem follows immediately from the interlacing property of the zeros of $p_{n-1,n}$ and $q_{n-1,n}$ and the interpolation property (26) (cf. [9, p.47]).

Sequences with ratio asymptotics can be accelerated by the Δ^2-method [3]. We define the *Δ^2-accelerated* diagonal Padé approximants by

$$\begin{aligned}[n+k/n]^{\Delta}_{f_\mu}(z) &:= [n+k-2/n-2]_{f_\mu}(z) \\ &\quad - \frac{\left([n+k-2/n-2]_{f_\mu}(z)-[n+k-1/n-1]_{f_\mu}(z)\right)^2}{[n+k-2/n-2]_{f_\mu}(z)-2[n+k-1/n-1]_{f_\mu}(z)+[n+k/n]_{f_\mu}(z)}.\end{aligned}$$

We summarize the results in the following theorem.

Theorem 5.1. *Let $\mu \in M(0,1)$. Then*

$$\frac{f_\mu(z)-[n/n+1]_{f_\mu}(z)}{f_\mu(z)-[n-1/n]_{f_\mu}(z)} \to \frac{1}{\Phi(1/z)^2}, \tag{30}$$

locally uniformly for $z \notin \operatorname{supp}\mu$. Hence the convergence of the $(n-1, n)$-Padé approximants to f_μ can be accelerated by the Δ^2-method:

$$\frac{f_\mu(z)-[n/n+1]^{\Delta}_{f_\mu}(z)}{f_\mu(z)-[n/n+1]_{f_\mu}(z)} \to 0. \tag{31}$$

Here we have given the proof for the uniform convergence. Parts of the proof of the pointwise convergence are due to [2]. The acceleration by the Δ^2-method is locally uniform, too [5]. A similar theorem for row sequences was proved in [15, Section 4].

Example 5.2. Let $f_\mu(z)=1/\sqrt{(1+z)}$. Then the polynomials $q_{n-1,n}$ are scaled Chebyshev polynomials of the second kind, and it is easy to compute

$$\frac{f_\mu(z)-[n/n+1]^{\Delta}_{f_\mu}(z)}{f_\mu(z)-[n/n+1]_{f_\mu}(z)} \to \frac{1}{\Phi(2/z+1)^2}, \quad \text{for } z \notin (-\infty, -1].$$

Hence in this case the Δ^2-method doubles the speed of convergence.

Example 5.3. Another, more typical example is $f_\mu(z)=\log(1+z)/z$. Here the polynomials $q_{n-1,n}$ are scaled Legendre polynomials, and our numerical results strongly indicate

$$\left|\frac{f_\mu(z)-[n-1/n]^{\Delta}_{f_\mu}(z)}{f_\mu(z)-[n-2/n-1]^{\Delta}_{f_\mu}(z)}\,\frac{f_\mu(z)-[n-2/n-1]_{f_\mu}(z)}{f_\mu(z)-[n-1/n]_{f_\mu}(z)}\right| - 1 + \frac{6}{2n-1} \to 0,$$

locally uniformly for $z \notin (-\infty, -1]$ (independently of z!). We can neither prove this relation nor conjecture the acceleration for arbitrary $\mu \in M(0,1)$.

In [5, Section 1.4] we derive asymptotics for polynomials with asymptotically periodic recurrence coefficients

$$\alpha_{nm+k} \to a_k \quad \text{and} \quad \beta_{nm+k} \to b_k, \qquad m \text{ fixed}, \ n \to \infty,$$

of period m. It is possible to generalize Theorem 5.1 to this class of measures. Theorem 5.1 also holds for the Markov-type functions with complex weights considered in [11].

Acknowledgements

I am very grateful to Michael Eiermann and Martin Hanke for their helpful advices, and to Walter Van Assche for answering many of my questions and for providing me with important references.

References

[1] L.V. Ahlfors, *Complex Analysis* (McGraw-Hill, New York, 3rd ed., 1979).
[2] O. Blumenthal, Ueber die Entwicklung einer willkürlichen Funktion nach den Nennern des Kettenbruchs für $\int_{-\infty}^{0}\Phi(\xi)\,d\xi/(z-\xi)$, Inaugural Dissertation, Georg-August-Univ. Göttingen, 1898.
[3] C. Brezinski, *Algorithmes d'Accélération de la Convergence, Étude Numérique,* Collect. Lang. Algorithmes Inform. (Technip, Paris, 1978).
[4] T.S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York, 1978).
[5] T. Dehn, Nennerpolynome von Padé und Padé-Typ-Approximationen an Hamburger-Funktionen, Ph.D. Dissertation, Univ. Karlsruhe, 1992.
[6] J. Favard, Sur les polynomes de Tchebicheff, *C.R. Acad. Sci. Paris* **200** (1935) 2052–2053.
[7] G. Freud, *Orthogonale Polynome,* Math. Reihe **33** (Birkhäuser, Basel, 1969).
[8] H. Hamburger, Über eine Erweiterung des Stieltjesschen Momentenproblems I, II, III, *Math. Ann.* **81** (1920) 235–319; **82** (1921) 121–164; 169–187.
[9] A. Iserles and E.B. Saff, Bi-orthogonality in rational approximation, *J. Comput. Appl. Math.* **19** (1) (1987) 47–54.
[10] R.J. Kooman and R. Tijdeman, Convergence properties of linear recurrence sequences, *Nieuw Arch. Wisk. (4)* **8** (1990) 13–25.
[11] A.P. Magnus, Toeplitz matrix techniques and convergence of complex weight Padé approximants, *J. Comput. Appl. Math.* **19** (1) (1987) 23–38.
[12] A. Máté and P.G. Nevai, A generalization of Poincaré's theorem for recurrence equations, *J. Approx. Theory* **63** (1) (1990) 92–97.
[13] P.G. Nevai, *Orthogonal Polynomials,* Mem. Amer. Math. Soc. **213** (Amer. Mathematical Soc., Providence, RI, 1979).
[14] H. Poincaré, Sur les equations linéaires aux differentielles ordinaires et aux différences finies, *Amer. J. Math.* **VII** (1885) 203–258.
[15] M. Prevost, Calculation of poles of meromorphic functions with q-d, r-s and ϵ-algorithms. Acceleration of these processes, *J. Comput. Appl. Math.* **19** (1) (1987) 89–98.
[16] J.A. Shohat, *Théorie Générale des Polynomes Orthogonaux de Tchebichef,* Mém. Sci. Math. **LXVI** (L'Académie des Sciences, Paris, 1934).
[17] J.A. Shohat, Sur les polynomes orthogonaux généralisés, *C.R. Acad. Sci. Paris* **207** (1938) 556–558.
[18] T.-J. Stieltjes, Recherches sur les fractions continues, *Ann. Fac. Sci. Toulouse Math.* **8** (1894) J1–122; **9** (1895) A5–47; also: in: *Oeuvres Complètes, Vol. 2* (Groningen, 1918) 398–566.
[19] G. Szegő, *Orthogonal Polynomials,* Amer. Math. Soc. Colloq. Publ. **23** (Amer. Mathematical Soc., Providence, RI, 4th ed., 1975).
[20] V. Totik, Orthogonal polynomials with ratio asymptotics, Technical Report ICM #90-009, Inst. Constr. Math., Tampa, FL, 1990.
[21] E.B. Van Vleck, On the convergence of algebraic continued fractions whose coefficients have limiting values, *Trans. Amer. Math. Soc.* **17** (1904) 253–262.