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Finite analogues of Euclidean space

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Abstract

Graphs are attached to $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is the field with q elements, q odd, using an analogue of the Euclidean distance. The graphs are shown to be asymptotically Ramanujan for large q (better than Ramanujan in half the cases). Comparisons are made with finite upper half planes constructed in a similar way using an analogue of Poincaré's non-Euclidean distance. The eigenvalues of the adjacency operators of the finite Euclidean graphs are shown to be Kloosterman sums.

Keywords: Finite symmetric space; Ramanujan graph; Kloosterman sum

AMS classification: primary 11T99; 05C25; secondary 11T23; 20H30

1. Introduction

Here we study the simplest finite symmetric space, namely, the finite Euclidean space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$ with $q=p^s$ elements using a finite analogue of the usual Euclidean distance. The study of finite symmetric spaces G/K has been carried out by many authors, using the methods of group representations and association schemes (see [7, 8, 21, 37, 38]). Here we have chosen to use only the abelian group $\mathbb{F}_q^n$ rather than the non-abelian group G of isometries in order to keep the discussion elementary. This was also possible for ordinary Euclidean space over the real number field (see [39, Ch. 1]). To see the connection of q-analogues of special functions with finite symmetric spaces G/K associated to finite groups G, see Askey's preface and Stanton's article in [6].

The tools needed in the present paper are very well known. For example, we require only standard properties of the characters of the abelian group $\mathbb{F}_q^n$ to obtain the necessary formulas for the eigenvalues of our Euclidean graphs. The eigenvalues turn out to be the well-known exponential sums called Kloosterman sums (see Eq. (10)). We will also need Weil's estimate for the Kloosterman

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sums (to be found in [42, pp. 386–389]). See Schmidt [34] for a more elementary approach. And we will be interested in the results on these sums to be found in [18] and [19].

Kloosterman sums provide another connection with q-series, since they are Fourier coefficients of the modular forms known as Poincaré series (see [33, Ch. 1]). They have been used by many authors to estimate the Fourier coefficients of modular forms (see [35, pp. 506–520]) in a quest to prove the Ramanujan conjecture bounding the Fourier coefficients of holomorphic modular forms such as the discriminant function Δ. Ultimately, Deligne proved this conjecture using algebraic geometric methods (see [13]).

One of the objects of this paper is to compare our finite Euclidean spaces and graphs with the non-Euclidean analogues studied in [3, 4, 12, 40]. Recall that the *Poincaré upper half plane* is

$$H = \{z = x + iy \mid x, y \in \mathbb{R},\ y > 0\}$$

with the non-Euclidean distance

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The finite upper half plane H_q is constructed by replacing $\mathbb{R}$ with the finite field $\mathbb{F}_q$ and $y > 0$ with $y \neq 0$. The non-Euclidean distance is replaced with an analogue on H_q having values in the finite field. Then we obtain graphs by connecting points at a fixed "distance" from one another.

Part of the motivation is to find new examples of *Ramanujan graphs* (as defined in [26]). A connected k-regular graph is Ramanujan if for every eigenvalue λ of the adjacency matrix with $|\lambda| \neq k$, we have

$$|\lambda| \leqslant 2\sqrt{k-1}.$$

Ramanujan graphs are of interest, for example, in the construction of communications networks because they have good expansion properties (see [25]). The Ramanujan conjecture was used to show that the graphs in [26] were Ramanujan. Thus the name of Ramanujan was given to these graphs. The proof that the non-Euclidean graphs in [3, 4, 12, 40] are Ramanujan requires an estimate of certain exponential sums of Soto-Andrade [37] — an estimate which was made first by Katz [20] and then by Winnie Li [24], the latter using a more elementary method.

Aside from the discovery of new Ramanujan graphs, this paper is part of an attempt to find finite models for the symmetric spaces discussed in [39]. This has been of interest to physicists for some time (see e.g. [29]).

An outline of the paper follows. Let $\mathbb{F}_q$ be a finite field with $q = p^r$ elements, for p a prime, $p \neq 2$. We define a distance $d(x, y) = {}^t(x-y)\cdot(x-y)$ for column vectors $x, y \in \mathbb{F}_q^n$ with ${}^tx =$ transpose of x. Then the Euclidean graph $E_q(n, a)$ associated to $\mathbb{F}_q^n$ has as vertices the elements of $\mathbb{F}_q^n$. Two vertices $x, y \in \mathbb{F}_q^n$ are joined by an edge if $d(x, y) = a$.

For $(q, n, a) \neq (q, 2, 0)$ with -1 not a square in $\mathbb{F}_q$, the graph $E_q(n, a)$ is a connected regular graph of degree $q^{n-1} + \mathit{error}$ (see Theorem 1). Some graphs are drawn using Mathematica in Figs. 1–3. The graph in Fig. 1 is a finite analogue of a torus or doughnut.

Figs. 4 and 5 are Mathematica list density plots of the matrix of values of the distance function $d((x, y), (0, 0))$, $(x, y) \in \mathbb{F}_q^2$. The "curves" you obtain by connecting dots of the same color (or grey level) are those obtained by connecting points (x,y) so that $x^2 + y^2 =$ constant. Thus they are finite

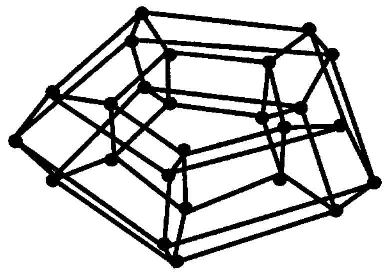

Fig. 1. The graph $E_5(2,1)$, a finite torus.

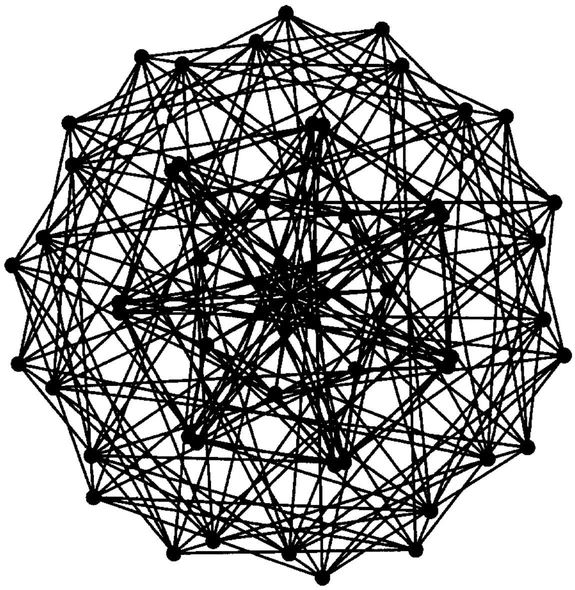

Fig. 2. The graph $E_7(2,1)$.

analogues of circles. The pictures are most beautiful in color, and should be compared with those produced for the finite analogue of the Poincaré distance in [4]. The latter graphs are much more chaotic. The Euclidean list density plots look like Fresnel diffraction patterns (see [17]). These list density plots can be considered to give finite analogues of the level curves of the eigenfunctions of the Laplacian in the real plane $\mathbb{R}^2$. See the cover of Powers [31] which shows a vibrating drum covered with dust. The dust forms circles which are nodal lines for the eigenfunctions of the Laplacian. This is another way to see that the curves of constant color in Figs. 4 and 5 are finite analogues of circles.

If λ is an eigenvalue of the adjacency operator of $E_q(n,a)$ with $\lambda \neq$ degree of the graph, then λ is a Kloosterman sum as in Eq. (10). Thus by Weil's estimate of these sums (see [42, pp. 386–389])

$$|\lambda| \leqslant 2q^{(n-1)/2}.$$

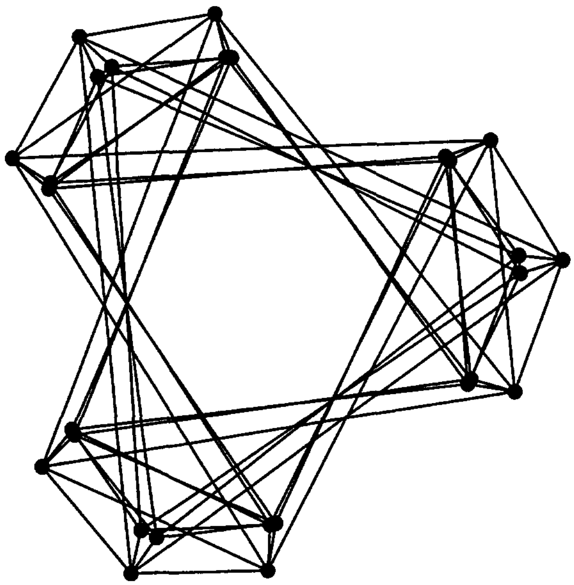

Fig. 3. The graph $E_3(3,1)$.

This estimate is asymptotic to the Ramanujan bound $2\sqrt{\text{degree}-1}$; sometimes it is better, sometimes worse. When $n=3$, the Kloosterman sums can be evaluated as either 0 or what is essentially a cosine (see Eq. (14) below, [32] and [11]). It is then easily shown that when $q=p=\text{prime} > 158$, the graphs $E_p(3,1)$ are not Ramanujan. $E_p(2,1)$ fails to be Ramanujan for $p = 17$ and 53. The distribution results in [1] and [19] also shed light on the matter.

The last two results of the paper (Proposition 4 and Theorem 5) show that there are *at most* 3 nonisomorphic graphs $E_q(n,a)$ for a given $\mathbb{F}_q^n$. If $n = 2k$ is even, there are *exactly* 2 nonisomorphic graphs $E_q(2k,0)$ and $E_q(2k,1)$. These last two results show that the finite Euclidean graphs differ somewhat from the finite upper half plane graphs in that there are fewer distinct graphs for each q.

Figs. 7 and 8 are Matlab Histograms of the Kloosterman sums, which are eigenvalues for the graphs $E_q(n,a)$ with $n = 2$ and 3 and $p = q$ near 1000. You can see from these figures that the eigenvalue distribution for $n = 3$ is very different from that for $n = 2$. The latter is proved by Katz [19] to be the semi-circle distribution (see also [1]). We have conjectured that the non-Euclidean graphs of [4] should also have spectra with the semi-circle distribution, but this is still an open question.

2. Finite Euclidean space

In this paper we seek to study a finite analogue of Ch. 1 of Terras [39]. That is, we study a finite analogue of the Euclidean space $\mathbb{R}^n$ along with an analogue of the Laplace operator and the eigenfunctions and eigenvalues of that operator.

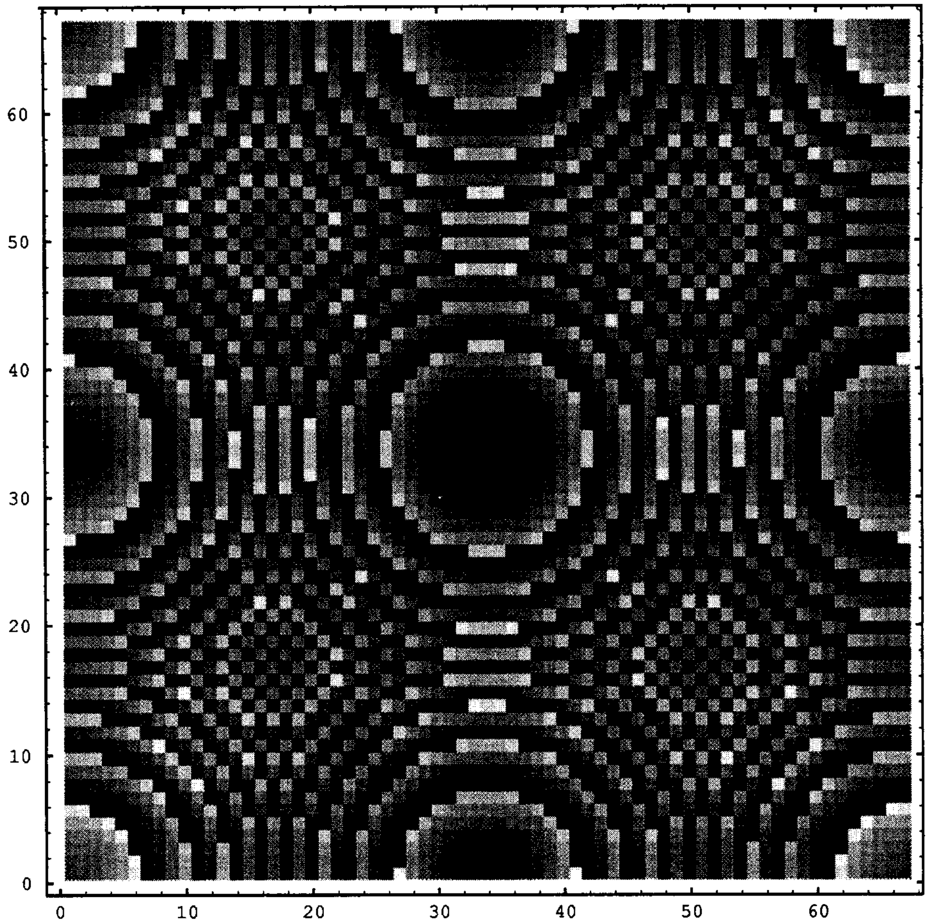

Fig. 4. List density plot in Mathematica for $p = 67$ with point (x,y) in a 67×67 grid given a color determined by $x^2 + y^2 \pmod{67}$.

Let $\mathbb{F}_q$ be a finite field with $q = p^r$ elements, where p is an odd prime. Then the finite analogue of the Euclidean n-space is defined to be

$$\mathbb{F}_q^n = \left\{ x \;\middle|\; x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},\ x_j \in \mathbb{F}_q \right\}. \tag{1}$$

We define the *distance* between two column vectors x and y in $\mathbb{F}_q^n$ by

$$d(x,y) = \sum_{j=1}^{n} (x_j - y_j)^2 = {}^{t}(x-y)(x-y). \tag{2}$$

This distance is *not* a metric in the sense of analysis since it is not real-valued, but it is a metric in the sense of algebra (see [24, p. 356]). It has values in $\mathbb{F}_q$ and it is *point-pair* invariant

$$d(x+u, y+u) = d(x,y), \quad \forall\ x, y, u \in \mathbb{F}_q^n.$$

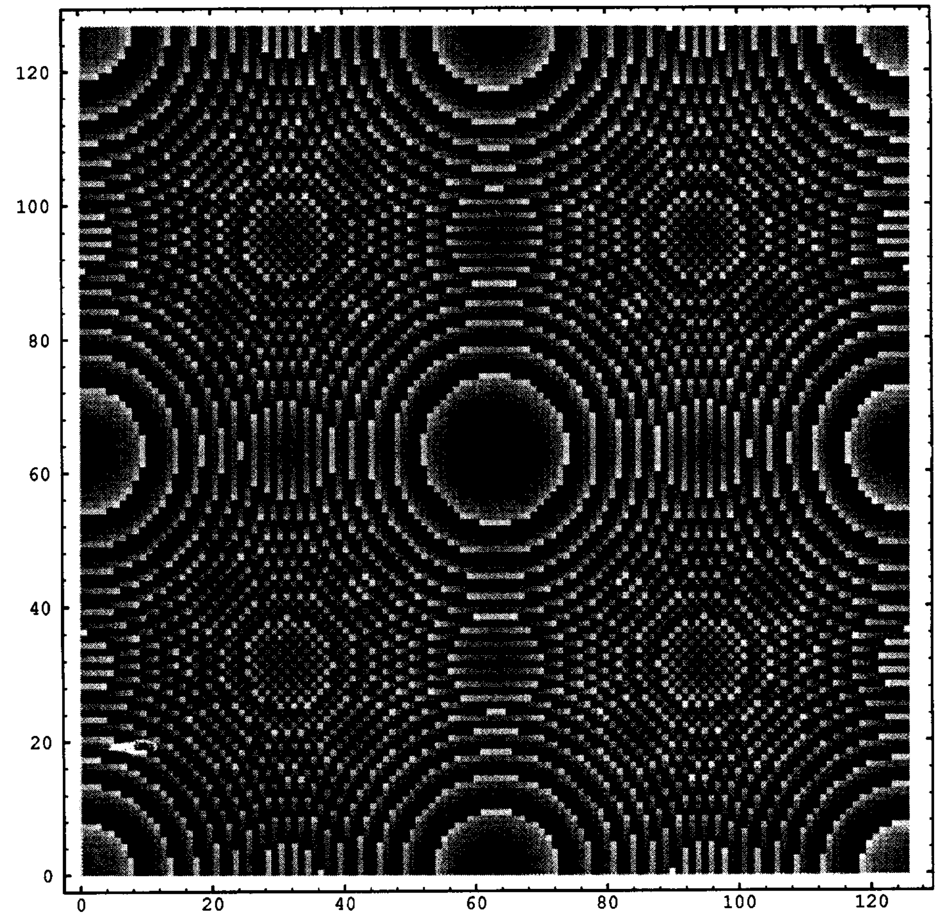

Fig. 5. List density plot in Mathematica for $p = 127$ with point (x,y) in a 127×127 grid given a color determined by $x^2 + y^2 \pmod{127}$.

It is also invariant under all the elements of the *orthogonal group*

$$\boldsymbol{O}(n,\mathbb{F}_q) = \{g \in GL(n,\mathbb{F}_q) \mid g \text{ preserves the quadratic form } x_1^2 + \cdots + x_n^2\}.$$

That is,

$$g \in \boldsymbol{O}(n,\mathbb{F}_q) \iff {}^{t}g \cdot g = I.$$

Definition. Given $a \in \mathbb{F}_q$, the *Euclidean graph* associated to $\mathbb{F}_q^n$ is $E_q(n,a)$. The vertices of the graph are the points in $\mathbb{F}_q^n$. Two vertices are adjacent iff $d(x,y) = a$. Note that this is a Cayley graph for the additive group of $\mathbb{F}_q^n$.

A *Cayley graph* for a group G and a symmetric set of generators S has as vertices the elements of G and edges between vertices x and $y = x \cdot s$, $s \in S$. The set S is *symmetric* if $s \in S$ implies $s^{-1} \in S$.

Let

$$S_q(n,a) = \{x \in \mathbb{F}_q^n \mid d(x,0) = a\}. \tag{3}$$

Note that when $a=0$ we are allowing the graph to have loops. The Euclidean graph $E_q(n,a)$ is a Cayley graph for the additive group of $\mathbb{F}_q^n$ with generating set $S_q(n,a)$ (assuming $(q,n,a) \neq (q,2,0)$ with -1 not a square in $\mathbb{F}_q$).

We define χ to be the *quadratic character*

$$\chi(a) = \begin{cases} 1 & \text{for } a \neq 0, \quad a = u^2, \quad u \in \mathbb{F}_q, \\ -1 & \text{for } a \neq 0, \quad a \neq u^2, \quad u \in \mathbb{F}_q, \\ 0 & \text{for } a = 0. \end{cases}$$

Theorem 1. *The Euclidean graph $E_q(n,a)$, where q is odd, is a regular graph with q^n vertices of degree given by $|S_q(n,a)|$. Here $S_q(n,a)$ is defined by* (3).

If $a \neq 0$,

$$|S_q(n,a)| = \begin{cases} q^{n-1} + \chi((-1)^{(n-1)/2}a)q^{(n-1)/2} & \text{for } n \text{ odd}, \\ q^{n-1} - \chi((-1)^{n/2}))q^{(n-2)/2} & \text{for } n \text{ even}. \end{cases}$$

If $a = 0$,

$$|S_q(n,0)| = \begin{cases} q^{n-1} & \text{for } n \text{ odd}, \\ q^{n-1} + \chi((-1)^{n/2}))(q-1)q^{(n-2)/2} & \text{for } n \text{ even}. \end{cases}$$

Note that $|S_q(n,a)| > 1$ if $n \geqslant 3$. When $n = 2$, $|S_q(2,a)| > 1$ if $a \neq 0$ or if $a = 0$ and $\chi(-1) = 1$. The graphs are connected unless $(q,n,a) = (q,2,0)$ with $\chi(-1) = -1$. In the latter case, the graph is just a set of loops on each point in $\mathbb{F}_q^n$.

Proof. One way to show that the graph is connected is to show that the degree occurs as an eigenvalue of the adjacency matrix with multiplicity 1. We will prove this later. The rest of the proof can be found in [11, 15, or 36, pp. 86–91, 145–146]. □

Figs. 1–3 show some of these graphs. They were drawn using Mathematica. The first figure is a finite analogue of a torus. Mathematica has a command to draw a torus in the Graphics Package Shapes. Torus[r,s,a,b] draws a torus with radii r and s, using an a by b mesh. We have also used part of the Mathematica DiscreteMath "Combinatorica" package to draw the graphs from a given adjacency matrix. The SpringEmbedding command was used in both Figs. 2 and 3. This models the graph as a system of vibrating masses and attempts to minimize energy.

Figs. 4 and 5 are analogous to Figs. 1–5 in Angel et al. [4]. They are list density plots made by Mathematica of a matrix of values of the function $d((x,y),(0,0)) = x^2 + y^2$, for $1 \leqslant x, y \leqslant p$. The point (x, y) is colored according to the value of x^2+y^2. The graphs here are reminiscent of Fresnel diffraction patterns (see [17] for similar pictures). Note that Figs. 1–5 in [4] look much more chaotic than the corresponding Euclidean figures.

Define the *finite Euclidean group G* to be the $2n \times 2n$ matrices with block form

$$g = \begin{bmatrix} k & u \\ 0 & 1 \end{bmatrix},$$

where $k \in \boldsymbol{O}(n, \mathbb{F}_q)$, u is an $n \times 1$ matrix, 0 is a $1 \times n$ matrix of 0's. Then g acts on $x \in \mathbb{F}_q^n$ by

$$x \longmapsto kx + u \text{ or by multiplying } \begin{bmatrix} k & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

Note that this is a group action: $g(g'x) = (gg')x$, $Ix = x$, where I is the identity matrix. This action preserves the distance defined by formula (2).

Now define K to be the subgroup of G consisting of matrices of the form

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \quad k \in \boldsymbol{O}(n, \mathbb{F}_q).$$

Then,

$$G/K \cong \mathbb{F}_q^n.$$

The space $G/K \cong \mathbb{F}_q^n$ is a symmetric space since we have a commutative algebra $L^2(\mathbb{F}_q^n)$ of functions $f : \mathbb{F}_q^n \to \mathbb{C}$ with multiplication defined by *convolution*

$$(f * g)(x) = \sum_{y \in \mathbb{F}_q^n} f(y) g(x - y). \tag{4}$$

Rudvalis notes that if v and w are nonzero elements of $S_q(n, a)$, the set defined by formula (3), then there is an element $k \in K$ so that $kv = w$. Here we use Witt's theorem (see [22]). Thus the K-orbits of points in $\mathbb{F}_q^n - \{0\}$ are the sets $S_q(n, a)$ with 0 removed, if necessary. We must remove 0 since if $a = 0$, we have $K \cdot 0 = \{0\}$, but $|S_q(n, 0)| = q^{n-1}$, for n odd, and $q^{n-1} + \mathit{error}$, for n even, by Theorem 1. Only when $(q, n, a) = (q, 2, 0)$ with $\chi(-1) = -1$ does $S_q(2, 0) = \{0\}$.

2.1. Remarks on other graphs one can view as finite analogues of Euclidean space

(a) Grids and connections with Riemann surface theory. (Motivated by finite difference approximations to the Laplace operator.)

Consider a $p \times p$ square with lower left vertex at the origin. Identify the vertical sides and the horizontal sides to get a torus $\mathbb{R}^2/(p\mathbb{Z})^2$. This can be identified with a union of translates of the unit square with sides identified

$$\mathbb{R}^2/(p\mathbb{Z})^2 = \bigcup_{a \in (\mathbb{Z}/p\mathbb{Z})^2} (\mathbb{R}^2/\mathbb{Z}^2 + a).$$

Obtain a graph from this grid by considering the vertices to be the unit squares in the grid (which can be thought of as the $a \in (\mathbb{Z}/p\mathbb{Z})^2$). Call 2 vertices *adjacent* if the unit squares have a side in common. In Fig. 6 we show the case $p = 11$, with the adjacent squares to the black square shaded grey.

This construction also gives a Cayley graph for $\mathbb{F}_p^2$. The set of generators is $\{(\pm 1, 0), (0, \pm 1)\}$. Fig. 1 shows the graph $E_5(2, 1)$. In this case, the graph from the grid is the same as that from our Euclidean "distance". The same thing happens for $E_3(2, 1)$.

One can do a similar Cayley graph for $\mathbb{F}_p^n$ with *generating set*

$$S_q^G = \{\pm e_j \mid j = 1, \ldots, n\}.$$

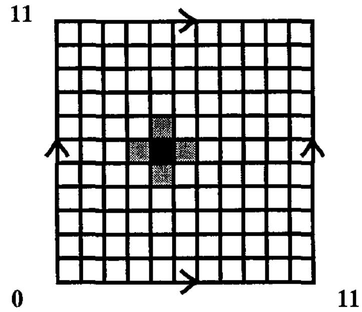

Fig. 6. A grid from which a graph can be created with vertices the 121 squares and edges between adjoining squares; for example, the black square is adjacent to the grey ones adjoining it.

Here e_j is the vector with 0 everywhere except in the jth place, where there is a 1. Thus,

$$|S_q^G| = 2n.$$

(b) Hamming distance in coding theory.

From *coding theory*, we have the *Hamming distance*: For $a, b \in \mathbb{F}_q^n$, set

$$d^H(a,b) = \#\{i \mid a_i \neq b_i\}.$$

Set

$$S_q^H = \{a \mid d(a,0) = 1\} \supset S_q^G,$$

$$|S_q^H| = n(q-1).$$

Again we can construct a Cayley graph using S_q^H as a set of generators. Call this graph a *Hamming graph*. Only in the case $q = 3$ do we have $S_q^G = S_q^H$.

The Hamming graphs have been much studied thanks to their importance for coding theory. The associated spherical functions are Krawtchouk polynomials (see [8, 14, 38], as well as Stanton's article in [6], and [41]).

(c) Finite Euclidean graphs.

Our generating set $S_q(n,1)$ is usually a much larger set of generators than S_q^G or S_q^H—*except when* $n = 2$. Then we find

$$|S_q(2,1)| = q - \chi(-1).$$

This will be smaller than $|S_q^H| = 2(q-1)$ for $q > 3$. If $q = 3$, we get $3 - \chi(-1) = 3 + 1 = 4$ and $S_3(2,1) = S^G = S^H$. But, if $n \geqslant 3$,

$$|S_q(n,1)| = \begin{cases} q^{n-1} + \chi((-1)^{(n-1)/2})\, q^{(n-1)/2} & \text{for } n \text{ odd,} \\ q^{n-1} + \chi((-1)^{n/2})\, q^{(n-2)/2} & \text{for } n \text{ even,} \end{cases}$$

and

$$q^{n-1} \geqslant n(q-1) + q^{(n-1)/2} \iff q^{(n-1)/2} \geqslant \frac{n(q-1)}{q^{(n-1)/2}} + 1,$$

for all $q \geqslant 3$, $n \geqslant 3$. Note that $S_q(n,1) \supset S_q^G$, since $1^2 = 1$. Thus, $S_q(n,1)$ generates $\mathbb{F}_q^n$.

3. Spectrum of adjacency operators

We want to study the spectrum of the *adjacency operator* A acting on $f : \mathbb{F}_q^n \to \mathbb{C}$ via

$$A = A_a f(x) = \sum_{d(x,y)=a} f(y). \tag{5}$$

The *combinatorial Laplacian* is $\Delta_a = A_a - kI$ if k is the degree of the graph and I is the identity operator; i.e. $Ig = g$. So Δ_a has the same eigenfunctions as A_a, for all $a \in \mathbb{F}_q$.

Since our group $\mathbb{F}_q^n$ is abelian, it is easy to find *simultaneous eigenfunctions* of A_a for all $a \in \mathbb{F}_q$. We will use the notation

$$e(u) = \exp\{2\pi \mathrm{i} \mathrm{Tr}(u)/p\}.$$

Here, $\mathrm{Tr}(u) = \mathrm{Trace}_{\mathbb{F}_q/\mathbb{F}_p}(u) = u + u^p + \cdots + u^{p^{s-1}}$ if $q = p^s$ and $u \in \mathbb{F}_q$. For each $b \in \mathbb{F}_q^n$, define

$$e_b(x) = e({}^t b \cdot x) \quad \text{for } x \in \mathbb{F}_q^n. \tag{6}$$

The following result is very old. Any book on applications of group representations contains some version of it as well as many number theory books (e.g., [10, p. 421]). Many papers on Ramanujan graphs have used it as well (see e.g., [23]).

Proposition 2. *For $b \in \mathbb{F}_q^n$, e_b is an eigenfunction of A_a corresponding to the eigenvalue*

$$\lambda_b = \sum_{d(s,0)=a} e_b(s).$$

Moreover, as b runs through $\mathbb{F}_q^n$ we obtain a complete orthonormal set of eigenfunctions of A. Thus, every eigenvalue of A has the form λ_b for some $b \in \mathbb{F}_q^n$. The inner product on $f, g \in L^2(\mathbb{F}_q^n)$ is

$$(f, g) = \sum_{x \in \mathbb{F}_q^n} f(x) \overline{g(x)}.$$

Proof. Note that

$$\begin{aligned} Ae_b(x) &= \sum_{d(x,y)=a} e_b(y) = \sum_{y=s+x,\, d(s,0)=a} e_b(s+x) \\ &= \Big(\sum_{d(s,0)=a} e_b(s) \Big) e_b(x). \end{aligned}$$

The rest comes from standard facts about Fourier analysis on $\mathbb{F}_q^n$ (see [14, 41]). □

Our next problem is to estimate the eigenvalues λ_b, $b \in \mathbb{F}_q^n$. Note that by Theorem 1, the bound given in Theorem 3 is asymptotic to the Ramanujan bound $2\sqrt{|S_q(n,a)| - 1}$ as $q \longrightarrow \infty$. Sometimes this bound is as good as or *better* than the Ramanujan bound (when the error in Theorem 1 is positive), and sometimes it is *worse*. It is possible for the graphs to be non-Ramanujan (e.g., $E_P(3,1)$

for p sufficiently large, as we will see later). After writing this paper, we found part of the proof of the following Theorem in [11]. We will thus give only a sketch of that part of the proof.

Theorem 3. *Let λ_b denote the eigenvalue of the adjacency operator A_a of the graph $E_q(n,a)$ corresponding to the eigenfunction $e_b(x)$ defined in* (6). *Then*

$$|\lambda_b| \leqslant 2q^{(n-1)/2},$$

for $b \neq 0$ in $\mathbb{F}_q^n$. Moreover, the eigenvalues λ_b, for $b \neq 0$, are expressed as generalized Kloosterman sums in Eq. (11).

Proof. It suffices to estimate λ_{2b} (as q is odd). Define for $r \in \mathbb{F}_q$

$$B_r(b) = \sum_{x \in \mathbb{F}_q^n} e\left(2\,{}^{t}b \cdot x + r\,{}^{t}x \cdot x\right). \tag{7}$$

Then for $b \neq 0$

$$q\lambda_{2b} = \sum_{r \in \mathbb{F}_q^*} B_r(b)\, e(-ar). \tag{8}$$

For $r \neq 0$, we can rewrite (7) as follows by completing the square:

$$B_r(b) = (\chi(r)\, G_1)^n\, e\left(-\frac{{}^{t}bb}{r}\right). \tag{9}$$

Here G_1 is a *Gauss sum* defined for $r \neq 0$ as follows:

$$\begin{aligned} G_r &= \sum_{y \in \mathbb{F}_q} e(ry^2) = \sum_{x \in \mathbb{F}_q} (1 + \chi(x))\, e(rx) \\ &= \chi(r)\, G_1, \end{aligned}$$

where χ is the quadratic character of Theorem 1. The Gauss sum can be viewed as a finite analogue of a *gamma function*.

It follows from (8) and (9) that for $b \neq 0$

$$q\lambda_{2b} = G_1^n \sum_{r \in \mathbb{F}_q^*} \chi^n(r)\, e\left(-ar + \frac{{}^{t}bb}{r}\right).$$

The sum over r is a generalized *Kloosterman sum*: defined for $a, a' \in \mathbb{F}_q$ and κ a multiplicative character of $\mathbb{F}_q$ by

$$K(\kappa \mid a, a') = \sum_{r \in \mathbb{F}_q^*} \kappa(r)\, e\left(-ar + \frac{a'}{r}\right). \tag{10}$$

In our case, $a' = \sum_{j=1}^{n} b_j^2 = d(b,0)$, $\kappa = \chi^n$. This may be viewed as a finite analogue of a *Bessel function*.

So we have proved the useful formula saying that for $b \neq 0$, *the eigenvalue* λ_{2b} *is essentially a generalized Kloosterman sum*:

$$\lambda_{2b} = \frac{1}{q} G_1^n K(\chi^n \mid a, d(b,0)). \tag{11}$$

To evaluate G_1 over $\mathbb{F}_q$ with $q = p^s$, we use the Davenport–Hasse theorem, as given in [9, formula (10.2), p. 122] (and the evaluation of G_1 over $\mathbb{F}_p$ by Gauss).

$$G_1 = \begin{cases} (-1)^{s-1}\sqrt{q} & \text{if } p \equiv 1 \pmod 4, \\ -(-i)^s\sqrt{q} & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{12}$$

The Kloosterman sum is bounded as in [42, pp. 386–389] (or see [34], or [23, Theorem 16]):

$$|K(\chi^n \mid a, d(b,0))| \leqslant 2\sqrt{q}. \tag{13}$$

Put together (11)–(13) and we have Theorem 3. □

Note that if you run through the proof of Theorem 3 with $b = 0$, you obtain a proof of Theorem 1, provided you know that $\chi(-1) = (-1)^{s(p-1)/2}$, if $q = p^s$.

When n is odd, the sums in formula (10) are called Salié sums (see [11, 32]). It turns out that the Salié sums are either 0 or essentially cosines. More precisely, we have *for odd dimensions n, when* $a \cdot d(b,0) \neq 0$:

$$\lambda_{2b} = \begin{cases} 2G_1^{n-1}\chi(d(b,0))\cos(4\pi \mathrm{Tr}(c)/p) & \text{if } a \cdot d(b,0) = c^2, \\ 0 & \text{if } a \cdot d(b,0) \text{ is not a square.} \end{cases} \tag{14}$$

For *odd dimensions n,* if $a \cdot d(b,0) = 0$, with $b \neq 0$,

$$\lambda_{2b} = \begin{cases} q\,\chi(-a) & \text{if } d(b,0) = 0, \\ q\,\chi(-d(b,0)) & \text{if } a = 0,\ d(b,0) \neq 0. \end{cases} \tag{15}$$

From this, it is not hard to see that if $p \equiv 3 \pmod 4$ for $p > 158$, the graphs $E_p(3,1)$ are not Ramanujan. Of course the graphs $E_p(3,1)$ for $p \equiv 1 \pmod 4$ are Ramanujan.

Remarks. (a) Comparison with finite upper half planes. Our non-Euclidean finite upper half plane graphs in [3, 4, 12, 40] were all Ramanujan (see [20, 24]), but this is only asymptotically true for the Euclidean graphs $E_p(3,1)$, when $p \equiv 3 \pmod 4$, as seen above. For $E_p(2,1)$, with $p \equiv 1 \pmod 4$, we find when $p = 17$ and 53 the graphs are not Ramanujan.

Some of the eigenfunctions for the non-Euclidean finite upper half planes were Kloosterman sums (see [12, Proposition 2]). All of the eigenfunctions for the finite upper half planes could be viewed as spherical functions; i.e., K-invariant eigenfunctions of all the G-invariant (adjacency) operators on G/K (see [4]). For the finite upper half plane graphs, the eigenvalues λ, such that $\lambda \neq q+1$, have multiplicity $\geqslant q-1$ (see [3, 4, 12]). This is the case for the Euclidean graphs as well. However, we do not pursue the matter here, as it involves the representation theory of the non-Abelian group G of all Euclidean motions.

Table 1
Eigenvalues of finite Euclidean graphs $E_q(2,a)$ (all graphs tabulated are Ramanujan when connected)

(n, p, a)	Eigenvalue	Multiplicity
(2,3,0) (not connected)	1	9
(2,3,a), $a \neq 0$	−2	4
	1	4
	4	1
(2,5,0)	−1	16
	4	8
	9	1
(2,5,a), $a \neq 0$	−3.2361	4
	−1	8
	0.3820	4
	1.2361	4
	2.6180	4
	4	1
(2,7,0) (not connected)	1	49
(2,7,a), $a \neq 0$	−4.4940	8
	−2.0489	8
	−1.1099	8
	1.6039	8
	2.3569	8
	2.6920	8
	8	1

For the finite upper half plane graphs, eigenvalues are also eigenfunctions (see [4, Theorem 1, Part 3]). This is also the case for our finite Euclidean graphs. By Proposition 2, λ_b is a sum over s of $e_b(s) = e_s(b)$ and $e_s(b)$ is an eigenfunction of the adjacency operator.

Formula (11) says that when $b \neq 0$, λ_{2b} is a function of $d(b,0)$ only. Thus, λ_b, $b \neq 0$, is a "radial" eigenfunction of the adjacency operator A_a, if we view $d(b,0)$ as a radial coordinate. This is what happened for the finite upper half plane graphs also.

It follows that the eigenvalues λ_b, $b \neq 0$, of A_a, can have at most q distinct values, each with multiplicity given by $|S_q(n,d(n,0))| = q^{n-1} + error$. See Theorem 1 for the formula. Thus, the spectrum of a graph $E_q(n,a)$ has at most $q+1$ points, each eigenvalue λ_b, $b \neq 0$, having high multiplicity. See Tables 1 and 2 for eigenvalues and their multiplicities for some small graphs.

Figs. 7 and 8 (created using Matlab) show the distribution of the Kloosterman sums giving the eigenvalues λ_b, $b \neq 0$, for the graphs $E_{1021}(2,1)$ and $E_{1019}(3,1)$. The two distributions look very different. For $n = 2$, Katz [20] shows that, as p goes to infinity, the distribution of Kloosterman sums approaches the Wigner semi-circle distribution (alias the Sato–Tate distribution) (see also [1]). This is the limiting distribution of the spectrum of a large random graph according to McKay [27]. And we conjecture that the spectrum of the finite non-Euclidean upper half plane graphs approach the semi-circle distribution in [4]. But this question is still open.

(b) Comparison with Euclidean spaces over $\mathbb{R}$ and Kloosterman sums as finite Bessel functions

Table 2
Eigenvalues of finite Euclidean graphs $E_q(4,a)$ (all graphs tabulated are Ramanujan)

(n,p,a)	Eigenvalue	Multiplicity
(4,3,0)	−3	48
	6	32
	33	1
(4,3,a), $a \neq 0$	−3	56
	6	24
	24	1
(4,5,0)	−5	480
	20	144
	145	1
(4,5,a), $a \neq 0$	−16.1803	120
	−5	144
	1.9098	120
	6.1803	120
	13.0902	120
	120	1

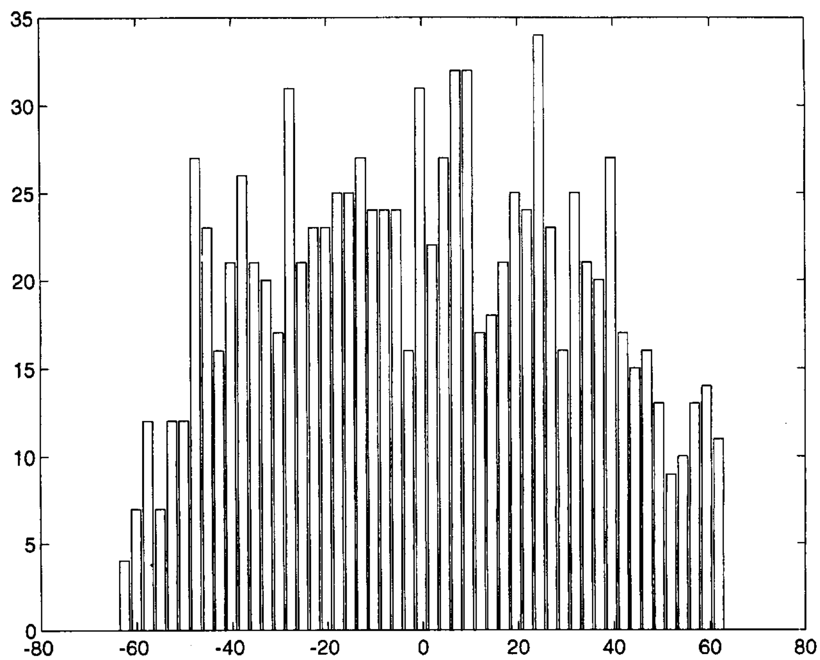

Fig. 7. Histogram of values of Kloosterman sums for $E_{1021}(2,1)$.

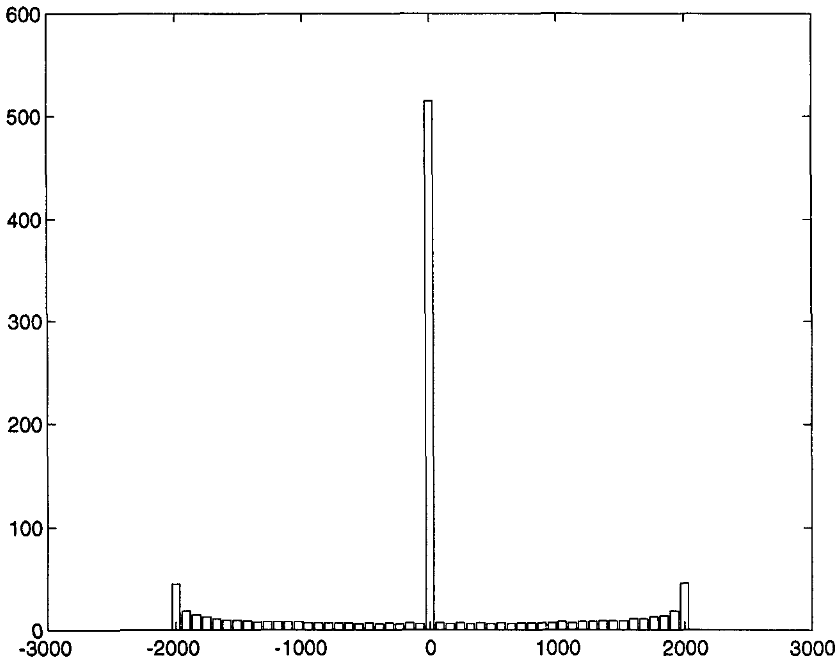

Fig. 8. Histogram of values of Kloosterman (Salié) sums for $E_{1019}(3,1)$.

Radial eigenfunctions of $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ on $\mathbb{R}^n$ are J-Bessel functions; see [39, Ch. 2, p. 109, Exercise 5], for the case $n=3$.

The Kloosterman sums are usually viewed as finite analogues of Bessel functions. There are finite counterparts for many special functions e.g. the Gauss sum is an analogue of the gamma function (see [16]).

Kloosterman sums have been almost as important to number theorists as Bessel functions have been to engineers. One reason is that they occur as Fourier coefficients (or q-expansion coefficients) of modular forms (see [33, 35, pp. 506–520]).

(c) A natural question is: *How many nonisomorphic graphs $E_q(n,a)$ are there for fixed q and n?* Proposition 4 says that there are at most 3 nonisomorphic $E_q(n,a)$ for each $\mathbb{F}_q^n$. Theorem 5 says that for even n, there are exactly 2 nonisomorphic graphs $E_q(n,a)$ for each fixed $\mathbb{F}_q^n$. For finite upper half planes H_q we found that there appear to be q distinct graphs $X_q(\delta,a)$ for each $\mathbb{F}_q$. It remains to be proved that they are really nonisomorphic in general, however (see [3, 12]). So there appear to be more finite upper half plane graphs for fixed q. The Euclidean case differs from the non-Euclidean here.

Proposition 4 (Some graph isomorphisms). *All graphs $E_q(n,a)$, for square $a \neq 0$ are isomorphic; i.e., $a=b^2$, for some $b \neq 0$. Also, all graphs for nonsquare a are isomorphic. So, there are at most 2 isomorphism classes of graphs for $a \neq 0$.*

Proof. If $c \in \mathbb{F}_q - 0$, let $y^* = cy$, for $y \in \mathbb{F}_q^n$. Then,

$$d(y^*,0) = c^2 d(y,0).$$

Thus, the mapping $y \mapsto y^* = cy$ provides a graph isomorphism between the graphs $E_q(n,a)$ and $E_q(n,c^2a)$. $\square$

Theorem 5 (More graph isomorphisms in even dimensions). *For even n the graphs $E_q(n,a)$ are isomorphic for all nonzero a and fixed q. Thus, for each $\mathbb{F}_q^n$ we have exactly 2 nonisomorphic graphs when n is even: $E_q(n,0)$ and $E_q(n,1)$.*

Proof.

The case $n=2$.

For $c \in \mathbb{F}_q - \{0\}$ take $x = {}^t(x_1,x_2)$ solving $x_1^2 + x_2^2 = c$. Set

$$M_x = \begin{bmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{bmatrix}. \tag{16}$$

Note that $\det(M_x) = x_1^2 + x_2^2 = d(x,0)$.

Given a solution $y = {}^t(y_1,y_2)$ of $y_1^2 + y_2^2 = a$, $a \neq 0$, set

$$y^* = M_x y = \begin{bmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1y_1 - x_2y_2 \\ x_2y_1 + x_1y_2 \end{bmatrix}. \tag{17}$$

Then $My^* = M_xM_y$ and $\det(M_{y^*}) = \det(M_x)\det(M_y) \Longrightarrow d(y^*,0) = d(x,0)d(y,0) = c \cdot a$.

Thus, the map $y \mapsto M_x y = y^*$ gives a graph isomorphism of $E_q(2,a)$ onto $E_q(2,c\cdot a)$. Note that if y,t are adjacent vertices of $E_q(2,a)$, we have

$$\begin{aligned} d(y^*,t^*) &= d(M_xy, M_xt) = d(M_xy - M_xt, 0) \\ &= d(M_x(y-t),0) = c\cdot a. \end{aligned}$$

So, y^*, t^* are adjacent vertices in $E_q(2,c\cdot a)$.

General even $n = 2k$.

We know that for $c \in \mathbb{F}_q - \{0\}$, there is $x = {}^t(x_1,x_2)$ with $x_1^2 + x_2^2 = c$. Set

$$M_x = \begin{bmatrix} x_1 & -x_2 & & & & & \\ x_2 & x_1 & & & & \mathbf{0} & \\ & & x_1 & -x_2 & & & \\ & & x_2 & x_1 & & & \\ & & & & \ddots & & \\ & \mathbf{0} & & & & x_1 & -x_2 \\ & & & & & x_2 & x_1 \end{bmatrix} \quad (k\ 2\times 2 \text{ blocks down the diagonal}).$$

Given $y \in \mathbb{F}_q^{2k}$, let $y^* = M_x y$. Then

$$\begin{aligned} d(y^*,0) &= \left(y_1^{*^2} + y_2^{*^2}\right) + \left(y_3^{*^2} + y_4^{*^2}\right) + \cdots + \left(y_{n-1}^{*^2} + y_n^{*^2}\right) \\ &= c\,(y_1^2 + y_2^2) + c\,(y_3^2 + y_4^2) + \cdots + c\,(y_{n-1}^2 + y_n^2) \;= c \cdot d(y,0). \end{aligned}$$

Thus, the map $y \mapsto M_x y = y^*$ gives a graph isomorphism of $E_q(2,a)$ with $E_q(2,c\cdot a)$.

To see that the graphs $E_q(2k,0)$ and $E_q(2k,1)$ are not isomorphic, use Theorem 1 to show that they have different degrees. $\square$

Last Remarks. There is another interesting question to be asked. Is 0 an eigenvalue for $E_q(n,a)$? We have seen that the answer to this question is "yes" for n odd. What if n is even? Then the answer is "no" using a result of Katz [18, p. 13].

Rudvalis notes that we should look at more general distances than that given in Eq. (2); e.g., for $c \in \mathbb{F}_q^n$ consider

$$d_c(x,y) = \sum_{j=1}^{n} c_j(x_j - y_j)^2.$$

With given q and n, one can produce more Ramanujan graphs by varying c. The figures analogous to Figs. 4 and 5 turn out to be even more interesting (see [28]).

One can also ask what happens if $\mathbb{F}_q$ is replaced by $\mathbb{Z}/q\mathbb{Z}$. We answered the analogous question for the non-Euclidean finite upper half planes in [5]. The graphs over rings were shown to be mostly non-Ramanujan, unlike those over fields.

Finally, one wonders if one can create analogous graphs corresponding to the symmetric space which is the sphere. In particular, what is the finite analogue of the Riemannian metric on the sphere?

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