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On the convergence of the MAOR method[1]

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Abstract

In order to solve a linear system $Ax=b$, Hadjidimos et al. (1992) defined a class of modified AOR (MAOR) method, whose special case implies the MSOR method. In this paper, some sufficient and/or necessary conditions for convergence of the MAOR and MSOR methods will be achieved, when A is a two-cyclic matrix and when A is a Hermitian positive-definite matrix, an H-, L- or M-matrix, and a strictly or irreducibly diagonally dominant matrix. The convergence results on the MSOR method are better than some known theorems. The optimum parameters and the optimum spectral radii of the MAOR and MSOR methods are obtained, which also answers the open problem given by Hadjidimos et al.

Keywords: Linear systems; MAOR method; MSOR method; Convergence; Optimum parameter

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1. Introduction

Let us consider a system of n equations

$$Ax=b, \tag{1.1}$$

where $A\in\mathbb{C}^{n\times n}$, b, $x\in\mathbb{C}^n$ with b known and x unknown.

For A satisfying Property A, in particular, for A with the form

$$A=\begin{pmatrix} D_1 & -H \\ -K & D_2 \end{pmatrix}, \tag{1.2}$$

where D_1 and D_2 are square nonsingular diagonal matrices, Young [17, Ch. 8] proposed the modified SOR (MSOR) method, where one relaxation factor ω_1 is used for the "red" equations, which

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correspond to D_1, and the other relaxation factor ω_2 is used for "black" equations, which correspond to D_2.

Clearly, the MSOR method is better than the SOR method and, therefore, the Gauss–Seidel method.

In [17, 9, 10, 2, 6, 7] some convergence theorems are proved, when A is positive definite, strictly diagonally dominant, an H-, L-, M-, or a Stieltjes matrix. In [16, 5, 4] the optimum spectral radius of the MSOR method was obtained.

In [3] a class of MAOR method was proposed whenever the matrix A is a GCO(p,q)-matrix. Some convergence conditions for two-cyclic matrix were given.

In this paper, we assume always the coefficient matrix A of (1.1) having the form (1.2), i.e., A is a two-cyclic matrix. We shall investigate the convergence of the MAOR and MSOR methods. Some sufficient and/or necessary conditions for convergence will be achieved, when A is a Hermitian positive-definite matrix, an H-, L- or M-matrix, a strictly or irreducibly diagonally dominant matrix. The convergence results on the MSOR method are better than some known theorems. The optimum parameters and the optimum spectral radii of the MAOR and MSOR methods are obtained.

For convenience we shall now briefly explain some of the terminology used in the next sections. We write $B \geqslant C$ $(B > C)$ if $b_{ij} \geqslant c_{ij}$ $(b_{ij} > c_{ij})$ holds for all entries of $B=(b_{ij})$ and $C=(c_{ij})$, calling B nonnegative if $B \geqslant 0$. The matrix $|B|=(|b_{ij}|)$ is called the absolute value of B. These definitions can be applied immediately to vectors by identifying them with $n \times 1$ matrices. We denote the spectral radius of B by $\rho(B)$.

Definition 1.1. A matrix $B=(b_{ij}) \in \mathbb{R}^{n\times n}$ is called

(a) an L-matrix if

$$b_{ii} > 0, \quad i=1,\ldots,n, \tag{1.3}$$

$$b_{ij} \leqslant 0, \quad i \neq j, \; i,j=1,\ldots,n; \tag{1.4}$$

(b) an M-matrix if (1.4) holds, B is nonsingular and $B^{-1} \geqslant 0$.

Young [17, Theorem 2-7.3] has shown that a Stieltjes matrix is also an M-matrix, and it is proved in [17, Section 2.7] that an M-matrix is an L-matrix.

Definition 1.2. A matrix $B=(b_{ij}) \in \mathbb{C}^{n\times n}$ is called

(a) an H-matrix if the comparison matrix $m(B)$ defined by

$$m_{ii} = |b_{ii}|, \quad i=1,\ldots,n,$$

$$m_{ij} = -|b_{ij}|, \quad i \neq j, \; i,j=1,\ldots,n;$$

is an M-matrix.

(b) strictly generalized diagonally dominant by rows (or columns) if there is a nonsingular positive diagonal matrix P such that AP (or PA) is strictly diagonally dominant by rows (or columns).

Clearly, an M-matrix is also an H-matrix. By [1, Theorem 6-2.3] it is easy to prove that a matrix B is an H-matrix if and only if it is strictly generalized diagonally dominant by rows or by columns. Hence, if B is strictly or irreducibly diagonally dominant (by rows or columns), then it is an H-matrix.

2. MAOR method

In [3] a class of MAOR method was defined whenever the matrix A is a GCO(p,q)-matrix. For the two-cyclic matrix A given in (1.2), let

$$A = D - C_L - C_U$$

be an usual splitting of A, where

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \qquad C_L = \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}, \qquad C_U = \begin{pmatrix} 0 & H \\ 0 & 0 \end{pmatrix}.$$

Then the Jacobi iteration matrix can be defined by

$$J = D^{-1}(C_L + C_U) = L + U \tag{2.1}$$

with

$$L = D^{-1}C_L = \begin{pmatrix} 0 & 0 \\ D_2^{-1}K & 0 \end{pmatrix}, \qquad U = D^{-1}C_U = \begin{pmatrix} 0 & D_1^{-1}H \\ 0 & 0 \end{pmatrix}.$$

The modified AOR (MAOR) method is defined as follows:

$$x^{k+1} = \mathscr{L}_{\Omega,\Gamma} x^k + \phi_{\Omega,\Gamma,b}, \quad k = 0, 1, \ldots, \tag{2.2}$$

where, for $\Omega = \operatorname{diag}(\omega_1 I_1, \omega_2 I_2)$, ω_1, $\omega_2 \neq 0$, and $\Gamma = \operatorname{diag}(\tilde{\gamma} I_1, \gamma I_2)$, the MAOR iteration matrix $\mathscr{L}_{\Omega,\Gamma}$ is defined by

$$\begin{aligned} \mathscr{L}_{\Omega,\Gamma} &= (D - \Gamma C_L)^{-1}[(I - \Omega)D + (\Omega - \Gamma)C_L + \Omega C_U] \\ &= (I - \Gamma L)^{-1}[I - \Omega + (\Omega - \Gamma)L + \Omega U] \end{aligned}$$

and

$$\phi_{\Omega,\Gamma,b} = (D - \Gamma C_L)^{-1}\Omega b = (I - \Gamma L)^{-1} D^{-1} \Omega b.$$

It is easy to show that the MAOR iteration method is independent of $\tilde{\gamma}$ so that we can denote the iteration matrix by $\mathscr{L}_{\omega_1,\omega_2,\gamma}$, i.e., the MAOR method can be defined by

$$x^{k+1} = \mathscr{L}_{\omega_1,\omega_2,\gamma} x^k + \phi_{\omega_1,\omega_2,\gamma,b}, \quad k = 0, 1, \ldots, \tag{2.3}$$

where the iteration matrix $\mathscr{L}_{\omega_1,\omega_2,\gamma}$ is defined by

$$\begin{aligned} \mathscr{L}_{\omega_1,\omega_2,\gamma} &= (I - \gamma L)^{-1}[I - \Omega + (\omega_2 - \gamma)L + \omega_1 U] \\ &= \begin{pmatrix} (1-\omega_1)I_1 & \omega_1 D_1^{-1}H \\ (\omega_2 - \gamma\omega_1)D_2^{-1}K & (1-\omega_2)I_2 + \gamma\omega_1 D_2^{-1}KD_1^{-1}H \end{pmatrix} \end{aligned} \tag{2.4}$$

and

$$\phi_{\omega_1,\omega_2,\gamma,b} = (I - \gamma L)^{-1} D^{-1} \Omega b.$$

When the parameter γ equals ω_2 the MAOR method reduces to the MSOR method [17, Ch. 8] and the iteration matrix is denoted by $\mathscr{L}_{\omega_1,\omega_2}$, i.e.,

$$\begin{aligned}\mathscr{L}_{\omega_1,\omega_2} &= (I-\omega_2 L)^{-1}[I-\Omega+\omega_1 U]\\ &= \begin{pmatrix} (1-\omega_1)I_1 & \omega_1 D_1^{-1}H \\ \omega_2(1-\omega_1)D_2^{-1}K & (1-\omega_2)I_2+\omega_1\omega_2 D_2^{-1}KD_1^{-1}H \end{pmatrix}.\end{aligned}$$

It is easy to prove that if $\gamma\neq 0$ then the MAOR method is an extrapolated MSOR (EMSOR) method with extrapolation parameter ω_2/γ and overrelaxation factors $\omega_1\gamma/\omega_2$ and γ, i.e.,

$$\mathscr{L}_{\omega_1,\omega_2,\gamma} = \left(1-\frac{\omega_2}{\gamma}\right)I + \frac{\omega_2}{\gamma}\mathscr{L}_{(\omega_1\gamma/\omega_2),\gamma}. \tag{2.5}$$

In addition, the MAOR method is also a special case of the method given in [8] by

$$x^{k+1} = (I-\alpha\tilde{\Omega}L)^{-1}[I-\tilde{\Omega}+(1-\alpha)\tilde{\Omega}L+\tilde{\Omega}U]x^k + (I-\alpha\tilde{\Omega}L)^{-1}D^{-1}\tilde{\Omega}b, \quad k=0,1,\ldots, \tag{2.6}$$

where $\tilde{\Omega}=\operatorname{diag}(\tilde{\omega}_1,\ldots,\tilde{\omega}_n)$ and α a real parameter. In fact, if we set $\tilde{\Omega}=\Omega$ and $\alpha\omega_2=\gamma$ then (2.6) reduces to (2.3) with (2.4).

3. Convergence

In this section we discuss the convergence of the MSOR and MAOR methods.

3.1. Hermitian matrices

We assume that A is a Hermitian matrix with positive diagonal elements. In this case $K^H=H$ and, therefore, $C_L^H=C_U$. Consequently, the matrix

$$\tilde{J}=D^{-1/2}(C_L+C_U)D^{-1/2}=D^{1/2}JD^{-1/2}$$

is also a Hermitian matrix. Hence, with $\tilde{J}$ the Jacobi iteration matrix J has only real eigenvalues.

We denote the eigenvalues of J by

$$\mu_n\leqslant\cdots\leqslant\mu_1,$$

and let

$$\bar{\mu}=\rho(J).$$

Then it follows by [17, Theorem 5-4.7] that

$$\bar{\mu}=\mu_1=-\mu_n.$$

Under the assumptions above we define a set $\mathscr{J}(\omega_1,\omega_2,\gamma)$ by

$$0<\omega_1<2,\ \ 0<\omega_2<2,$$

and either γ is arbitrary whenever $\bar{\mu}=0$ or

$$\omega_2 - \frac{\omega_2}{\bar{\mu}}\left[2\min\left\{\frac{1}{\omega_1},\frac{1}{\omega_2}\right\}-1\right] < \gamma < \omega_2 + \frac{\omega_2}{\bar{\mu}}\left[2\min\left\{\frac{1}{\omega_1},\frac{1}{\omega_2}\right\}-1\right],$$

whenever $\bar{\mu}\neq 0$.

Now we describe convergence theorems.

Theorem 3.1. *Let A be a Hermitian matrix with positive diagonal elements.*

(a) *If $(\omega_1,\omega_2,\gamma)\in\mathscr{J}(\omega_1,\omega_2,\gamma)$, then the MAOR method converges if and only if A is positive definite.*

(b) *If $0<\omega_1<2$, $0<\omega_2<2$, then the MSOR method converges if and only if A is positive definite.*

Proof. We write

$$\begin{aligned} M(\gamma,\Omega) &= \begin{pmatrix} \omega_1 I_1 & 0 \\ 0 & \omega_2 I_2 \end{pmatrix}^{-1} \begin{pmatrix} D_1 & 0 \\ -\gamma K & D_2 \end{pmatrix}, \\ N(\gamma,\Omega) &= \begin{pmatrix} \omega_1 I_1 & 0 \\ 0 & \omega_2 I_2 \end{pmatrix}^{-1} \begin{pmatrix} (1-\omega_1)D_1 & \omega_1 H \\ (\omega_2-\gamma)K & (1-\omega_2)D_2 \end{pmatrix}. \end{aligned} \tag{3.1}$$

Then

$$\mathscr{L}_{\omega_1,\omega_2,\gamma} = [M(\gamma,\Omega)]^{-1}N(\gamma,\Omega)$$

and

$$A = M(\gamma,\Omega) - N(\gamma,\Omega)$$

hold. Furthermore, we have

$$\begin{aligned} [M(\gamma,\Omega)]^H + N(\gamma,\Omega) &= \begin{pmatrix} (\frac{2}{\omega_1}-1)D_1 & (1-\frac{\gamma}{\omega_2})H \\ (1-\frac{\gamma}{\omega_2})K & (\frac{2}{\omega_2}-1)D_2 \end{pmatrix} \\ &= D^{1/2}\left[\begin{pmatrix} (\frac{2}{\omega_1}-1)I_1 & 0 \\ 0 & (\frac{2}{\omega_2}-1)I_2 \end{pmatrix} + \left(1-\frac{\gamma}{\omega_2}\right) D^{1/2} J D^{-1/2}\right] D^{1/2}. \end{aligned}$$

Assume that the eigenvalues of the Hermitian matrix

$$J_1 = \begin{pmatrix} (\frac{2}{\omega_1}-1)I_1 & 0 \\ 0 & (\frac{2}{\omega_2}-1)I_2 \end{pmatrix} + \left(1-\frac{\gamma}{\omega_2}\right) D^{1/2} J D^{-1/2}$$

are $\{\nu_1,\ldots,\nu_n\}$. By [15, Ch. 2, Section 44] we obtain

$$\nu_i \geqslant \min_{j=1,2}\left\{\frac{2}{\omega_j}-1\right\} + \min_{1\leqslant j\leqslant n}\left\{\left(1-\frac{\gamma}{\omega_2}\right)\mu_j\right\}, \quad i=1,\ldots,n.$$

Because $(\omega_1,\omega_2,\gamma)\in\mathscr{J}(\omega_1,\omega_2,\gamma)$ it follows that $\nu_i>0$, $i=1,\ldots,n$. This shows that the matrix J_1 and, therefore, $[M(\gamma,\Omega)]^H+N(\gamma,\Omega)$ is positive definite. Now the statement (a) follows directly by [11, Corollary 2.10] or [1, Corollary 7-5.44].

The statement (b) is a special case of (a).

The above convergence theorem on the MAOR method depends on the eigenvalues of the Jacobi iteration matrix. Now we present a sufficient condition for convergence, which only depends on the values of the parameters ω_1, ω_2 and γ.

Theorem 3.2. *Let A be a Hermitian positive-definite matrix. Then the MAOR method converges if the parameters ω_1, ω_2 and γ satisfy either*

$$0<\omega_1 \leqslant \omega_2 \leqslant \gamma \leqslant 2, \quad \omega_2<2 \tag{3.2}$$

or

$$0<\omega_2 \leqslant \omega_1 <2, \quad \omega_2 \leqslant \gamma \leqslant \frac{2\omega_2}{\omega_1}. \tag{3.3}$$

Proof. Let $M(\gamma,\Omega)$ and $N(\gamma,\Omega)$ be defined by (3.1). Then

$$\begin{aligned} M(\gamma,\Omega)^H+N(\gamma,\Omega) &= \begin{pmatrix} (\frac{2}{\omega_1}-1)D_1 & (1-\frac{\gamma}{\omega_2})H \\ (1-\frac{\gamma}{\omega_2})K & (\frac{2}{\omega_2}-1)D_2 \end{pmatrix} \\ &= \left(\frac{\gamma}{\omega_2}-1\right)A+\left(2\Omega^{-1}-\frac{\gamma}{\omega_2}I\right)D. \end{aligned}$$

We denote the eigenvalues of A and $M(\gamma,\Omega)^H+N(\gamma,\Omega)$ by $\{\eta_1,\dots,\eta_n\}$ and $\{v_1,\dots,v_n\}$, respectively. If ω_1, ω_2 and γ satisfy either (3.2) or (3.3), then

$$\begin{aligned} v_i &\geqslant \left(\frac{\gamma}{\omega_2}-1\right)\min_{1\leqslant j\leqslant n}\eta_j+\left(\min_{j=1,2}\left\{\frac{2}{\omega_j}\right\}-\frac{\gamma}{\omega_2}\right)\min_{1\leqslant j\leqslant n}a_{jj} \\ &\geqslant \left(\min_{j=1,2}\left\{\frac{2}{\omega_j}\right\}-1\right)\min_{1\leqslant j\leqslant n}\eta_j>0, \quad i=1,\dots,n, \end{aligned}$$

since $0<\min_{1\leqslant j\leqslant n}\eta_j \leqslant \min_{1\leqslant j\leqslant n}a_{jj}$.

This shows that the matrix $[M(\gamma,\Omega)]^H+N(\gamma,\Omega)$ is positive definite and the convergence of the MAOR method follows directly by [11, Corollary 2.10] or [1, Corollary 7-5.44].

3.2. *H-matrices and strictly or irreducibly diagonally dominant matrices*

First, we consider the case when $\rho(|J|)<1$ holds.

Theorem 3.3. *Let $\rho(|J|)<1$. Then the following inequalities hold:*
(a)

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) \leqslant \max_{i=1,2}\{|1-\omega_i|+\omega_i\rho(|J|)\}<1, \tag{3.4}$$

whenever $0<\omega_1<2/[1+\rho(|J|)]$, $0<\omega_2<2/[1+\rho(|J|)]$ and $0\leqslant\gamma\leqslant\omega_2$; or

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) \leqslant \left|1-\frac{\omega_2}{\gamma}\right|+\max_{i=1,2}\left\{\left|\frac{\omega_2}{\gamma}-\omega_i\right|+\omega_i\rho(|J|)\right\}<1, \tag{3.5}$$

whenever $0<\omega_2\leqslant\gamma<2/[1+\rho(|J|)]$ and $0<\omega_1<2\omega_2/\{\gamma[1+\rho(|J|)]\}$.

(b)

$$\rho(\mathscr{L}_{\omega_1,\omega_2}) \leqslant \max_{i=1,2}\{|1-\omega_i| + \omega_i\rho(|J|)\} < 1, \tag{3.6}$$

whenever $0<\omega_1<2/[1+\rho(|J|)]$, $0<\omega_2<2/[1+\rho(|J|)]$.

Proof. We first assume that

$$0<\omega_1<\frac{2}{1+\rho(|J|)}, \quad 0<\omega_2<\frac{2}{1+\rho(|J|)}, \quad 0\leqslant\gamma\leqslant\omega_2.$$

Let

$$T=(I-\gamma|L|)^{-1}[|I-\Omega|+(\omega_2-\gamma)|L|+\omega_1|U|]. \tag{3.7}$$

Then T is nonnegative and, hence, by [13, Theorem 2.7] there exists an eigenvector $x\geqslant 0$, $x\neq 0$, such that

$$Tx=\rho(T)x$$

holds, i.e.,

$$[|I-\Omega|+(\omega_2-\gamma)|L|+\omega_1|U|]x=\rho(T)(I-\gamma|L|)x.$$

Multiplying by Ω^{-1}, it follows

$$[\rho(T)\Omega^{-1}-|I-\Omega^{-1}|]x=\left(\frac{\omega_2-\gamma+\gamma\rho(T)}{\omega_2}|L|+|U|\right)x.$$

As $[(\omega_2-\gamma+\gamma\rho(T))/\omega_2]\,|L|+|U|\geqslant 0$, it follows by [12, Theorem 11] that

$$\min_{i=1,2}\{\omega_i^{-1}\rho(T)-|1-\omega_i^{-1}|\}\leqslant\rho\left(\frac{\omega_2-\gamma+\gamma\rho(T)}{\omega_2}|L|+|U|\right). \tag{3.8}$$

Assume that $\rho(T)\geqslant 1$ holds. Then

$$1\leqslant\frac{\omega_2-\gamma+\gamma\rho(T)}{\omega_2}\leqslant\rho(T)$$

is true, and, therefore,

$$\min_{i=1,2}\{\omega_i^{-1}\rho(T)-|1-\omega_i^{-1}|\}\leqslant\frac{\omega_2-\gamma+\gamma\rho(T)}{\omega_2}\rho(|L|+|U|)\leqslant\rho(T)\rho(|J|).$$

This shows that there exists i, $i=1$ or 2, such that

$$\omega_i^{-1}\rho(T)-|1-\omega_i^{-1}|\leqslant\rho(T)\rho(|J|). \tag{3.9}$$

Case 1: $\omega_i\leqslant 1$. In this case it follows that

$$\omega_i^{-1}\rho(T)-\omega_i^{-1}+1\leqslant\rho(T)\rho(|J|)<\rho(T),$$

and also

$$(\omega_i^{-1}-1)[\rho(T)-1]<0,$$

which contradicts $\omega_i^{-1}-1\geqslant 0$ and $\rho(T)-1\geqslant 0$.

Case 2: $\omega_i > 1$. Now from (3.9), we have

$$\omega_i^{-1}\rho(T) + \omega_i^{-1} - 1 \leqslant \rho(T)\rho(|J|).$$

As $\omega_i < 2/[1+\rho(|J|)]$, it implies

$$\frac{1}{2}[1+\rho(|J|)][1+\rho(T)] - 1 < \rho(T)\rho(|J|),$$

and also

$$[1-\rho(|J|)][\rho(T)-1] < 0,$$

which contradicts $1-\rho(|J|)>0$ and $\rho(T)-1 \geqslant 0$.
Now we can conclude that

$$\rho(T) < 1$$

holds. Hence, we obtain

$$1 - \frac{\gamma}{\omega_2} + \frac{\gamma}{\omega_2}\rho(T) \leqslant 1$$

and (3.8) implies

$$\min_{i=1,2}\{\omega_i^{-1}\rho(T) - |1-\omega_i^{-1}|\} \leqslant \rho(|L|+|U|) = \rho(|J|).$$

Consequently,

$$\rho(T) \leqslant \max_{i=1,2}\{|1-\omega_i| + \omega_i\rho(|J|)\}. \tag{3.10}$$

On the other hand, it is easy to see that

$$|\mathscr{L}_{\omega_1,\omega_2,\gamma}| \leqslant T,$$

and [13, Theorem 2.8] ensures

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) \leqslant \rho(T). \tag{3.11}$$

Notice that if $\omega_i \leqslant 1$, we have

$$|1-\omega_i| + \omega_i\rho(|J|) = 1 - \omega_i + \omega_i\rho(|J|) < 1. \tag{3.12}$$

While if $\omega_i > 1$, we have

$$\begin{aligned} |1-\omega_i| + \omega_i\rho(|J|) &= -1+\omega_i+\omega_i\rho(|J|) \\ &= \omega_i[1+\rho(|J|)]-1 \\ &< \frac{2}{1+\rho(|J|)}[1+\rho(|J|)]-1 \\ &= 1. \end{aligned} \tag{3.13}$$

Summarizing (3.10)–(3.13) it has shown (3.4). Inequality (3.6) is a special case of (3.4). Using extrapolated principle it follows from (2.5) and (3.6) that

$$\begin{aligned} \rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) &\leqslant \left|1-\frac{\omega_2}{\gamma}\right| + \frac{\omega_2}{\gamma}\rho(\mathscr{L}_{(\omega_1\gamma/\omega_2),\gamma}) \\ &\leqslant \left|1-\frac{\omega_2}{\gamma}\right| + \frac{\omega_2}{\gamma}\max\left\{\left|1-\frac{\omega_1\gamma}{\omega_2}\right| + \frac{\omega_1\gamma}{\omega_2}\rho(|J|), |1-\gamma|+\gamma\rho(|J|)\right\} \\ &= \left|1-\frac{\omega_2}{\gamma}\right| + \max_{i=1,2}\left\{\left|\frac{\omega_2}{\gamma}-\omega_i\right| + \omega_i\rho(|J|)\right\} \end{aligned} \tag{3.14}$$

whenever

$$0<\frac{\omega_2}{\gamma}<\frac{2}{1+\rho(\mathscr{L}_{(\omega_1\gamma/\omega_2),\gamma})}, \tag{3.15}$$

and

$$0<\frac{\omega_1\gamma}{\omega_2}<\frac{2}{1+\rho(|J|)}, \quad 0<\gamma<\frac{2}{1+\rho(|J|)}, \tag{3.16}$$

Since $\rho(\mathscr{L}_{(\omega_1\gamma/\omega_2),\gamma})<1$ it implies (ω_2,γ) to satisfy (3.15) if $0<\omega_2\leqslant\gamma$. Now, from (3.14) and (3.16) we obtain

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma})<\left(1-\frac{\omega_2}{\gamma}\right)+\frac{\omega_2}{\gamma}=1,$$

and it completes the proof of (3.5).

From [14, Theorem 1] we know that if A is an H-matrix then $\rho(|J|)<1$ holds. Hence, by Theorem 3.3, we have the following convergence theorem.

Theorem 3.4. *If A is an H-matrix then the convergence results of Theorem* 3.3 *are valid.*

Since a strictly or irreducibly diagonally dominant matrix is also an H-matrix, Theorem 3.4 is valid for these kinds of matrices.

Furthermore, for these matrices, $\|J\|_\infty$ can take the place of $\rho(|J|)$ in Theorem 3.4. In order to describe convergence theorems, we denote

$$\sigma=\|J\|_\infty, \qquad \sigma_1=\|D_1^{-1}H\|_\infty, \qquad \sigma_2=\|D_2^{-1}K\|_\infty.$$

If A is strictly or irreducibly diagonally dominant by rows then

$$\rho(|J|) \leqslant \max\{\sigma_1, \sigma_2\} = \sigma < 1.$$

For irreducibly diagonally dominant matrices the parameters ω_1, ω_2 can equal $2/[1+\sigma]$.

Theorem 3.5. *Let A be irreducibly diagonally dominant by rows. Then the following inequalities hold:*

(a) $\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) < \max_{i=1,2}\{|1-\omega_i| + \omega_i\sigma\} \leqslant 1$,
whenever $0<\gamma\leqslant\omega_2\leqslant 2/[1+\sigma]$ *and* $0<\omega_1\leqslant 2/[1+\sigma]$ *or*

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) < \left|1-\frac{\omega_2}{\gamma}\right| + \max_{i=1,2}\left\{\left|\frac{\omega_2}{\gamma}-\omega_i\right| + \omega_i\sigma\right\} \leqslant 1,$$

whenever $0<\omega_2\leqslant\gamma\leqslant 2/[1+\sigma]$ *and* $0<\omega_1\leqslant 2\omega_2/[\gamma(1+\sigma)]$.

(b) $\rho(\mathscr{L}_{\omega_1,\omega_2}) < \max_{i=1,2}\{|1-\omega_i| + \omega_i\sigma\} \leqslant 1$,
whenever $0<\omega_1\leqslant 2/[1+\sigma]$, $0<\omega_2\leqslant 2/[1+\sigma]$.

Proof. Assume that T is defined by (3.7) and $0<\gamma\leqslant\omega_2\leqslant 2/[1+\sigma]$ and $0<\omega_1\leqslant 2/[1+\sigma]$. Since A is also an H-matrix, the proof of Theorem 3.3 is valid and, hence, the inequalities

$$\rho(T)<1 \quad \text{and} \quad 0\leqslant 1-\frac{\gamma}{\omega_2}+\frac{\gamma}{\omega_2}\rho(T)<1$$

hold. If $\rho(T)=0$, then $\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma})=0<1$, and the inequality (a) is true. Now we consider the case $\rho(T)>0$. With A the matrices J and $\{1-\gamma/\omega_2+[\gamma/\omega_2]\rho(T)\}|L|+|U|$ are irreducible. By [13, Theorem 2.1] it follows that

$$\rho\left(\left[1-\frac{\gamma}{\omega_2}+\frac{\gamma}{\omega_2}\rho(T)\right]|L|+|U|\right)<\rho(|L|+|U|)=\rho(|J|)\leqslant\sigma.$$

From (3.8) we can derive the first inequality in (a), and (b) is its special case.

Similar to the proof of Theorem 3.3, using extrapolated principle we can prove the second inequality in (a).

Remark 3.1. This result improves the one by [9, Theorem 5].

For strictly diagonally dominant matrix we have the following convergence theorem.

Theorem 3.6. *Let A be strictly diagonally dominant by rows. Then the following inequalities hold:*

(a) $\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) \leqslant \max_{i=1,2}\{|1-\omega_i| + \omega_i\sigma_i\} < 1$,
whenever $0<\omega_1<2/[1+\sigma_1]$, $0<\omega_2<2/[1+\sigma_2]$ *and* $0\leqslant\gamma\leqslant\omega_2$; *or*

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) \leqslant \left|1-\frac{\omega_2}{\gamma}\right| + \max_{i=1,2}\left\{\left|\frac{\omega_2}{\gamma}-\omega_i\right| + \omega_i\sigma_i\right\} < 1,$$

whenever $0<\omega_2\leqslant\gamma<2/[1+\sigma_2]$ *and* $0<\omega_1<2\omega_2/[\gamma(1+\sigma_1)]$.

(b) $\rho(\mathscr{L}_{\omega_1,\omega_2}) \leqslant \max_{i=1,2}\{|1-\omega_i| + \omega_i\sigma_i\} < 1$,
whenever $0<\omega_1<2/[1+\sigma_1]$, $0<\omega_2<2/[1+\sigma_2]$.

Proof. Assume that $0<\omega_1<2/[1+\sigma_1]$, $0<\omega_2<2/[1+\sigma_2]$ and $0\leqslant\gamma\leqslant\omega_2$. From (2.4) we obtain

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) \leqslant \|\mathscr{L}_{\omega_1,\omega_2,\gamma}\|_\infty$$
$$\leqslant \max\{|1-\omega_1| + \omega_1\sigma_1,\ |1-\omega_2| + |\omega_2 - \gamma\omega_1|\sigma_2 + \gamma\omega_1\sigma_1\sigma_2\}.$$

If $\omega_2 \geqslant \gamma\omega_1$ then

$$|\omega_2 - \gamma\omega_1|\sigma_2 + \gamma\omega_1\sigma_1\sigma_2 = \omega_2\sigma_2 - \gamma\omega_1\sigma_2 + \gamma\omega_1\sigma_1\sigma_2 \leqslant \omega_2\sigma_2$$

as $\sigma_1 < 1$.

If $\omega_2 < \gamma\omega_1$ then

$$\begin{aligned}|\omega_2 - \gamma\omega_1|\sigma_2 + \gamma\omega_1\sigma_1\sigma_2 &= \gamma\omega_1\sigma_2 - \omega_2\sigma_2 + \gamma\omega_1\sigma_1\sigma_2\\ &= \gamma\omega_1(1+\sigma_1)\sigma_2 - \omega_2\sigma_2\\ &\leqslant 2\gamma\sigma_2 - \omega_2\sigma_2 \leqslant \omega_2\sigma_2,\end{aligned}$$

since $\gamma \leqslant \omega_2$ and $\omega_1 < 2/[1+\sigma_1]$.

This has shown that

$$|1-\omega_2| + |\omega_2 - \gamma\omega_1|\sigma_2 + \gamma\omega_1\sigma_1\sigma_2 \leqslant |1-\omega_2| + \omega_2\sigma_2$$

and, hence,

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) \leqslant \max_{i=1,2}\{|1-\omega_i| + \omega_i\sigma_i\}.$$

In addition, clearly,

$$|1-\omega_i| + \omega_i\sigma_i < 1, \quad i = 1,2.$$

Hence, the first inequality in (a) holds, and (b) is its special case.

By (b) and using extrapolated principle we can prove the second inequality in (a).

Remark 3.2. On the MSOR method this theorem presents the same convergence region as [6, Corollary 3.1], which is better than the one in [10, Theorem 2].

3.3. M-matrices

We have known that an M-matrix is also an H-matrix, hence, the statement in Theorem 3.3 is valid for M-matrix. Here we shall prove that A being M-matrix is also a necessary condition for convergence, whenever A is an L-matrix.

We first give two lemmas.

Lemma 3.1 (Berman and Plemmons [1, Theorems 1-3.18, 1-3.35]). *Let $B \geqslant 0$ be an irreducible matrix. Then*

$$Bx \leqslant \alpha x \quad \text{for some } x \geqslant 0, \quad x \neq 0$$

implies $x > 0$ and

$$\rho(B) \leqslant \alpha.$$

Lemma 3.2. *Let A be an irreducible L-matrix. Then for $0 < \omega_1 \leqslant 1$, $0 < \omega_2 \leqslant 1$, $0 \leqslant \gamma \leqslant \omega_2$, we have*

$$\min_{i=1,2} \left\{ \frac{\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) - 1}{\omega_i} + 1 \right\} \leqslant \rho \left(\left\{ \frac{\gamma[\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) - 1]}{\omega_2} + 1 \right\} L + U \right)$$
$$\leqslant \max_{i=1,2} \left\{ \frac{\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) - 1}{\omega_i} + 1 \right\}. \tag{3.17}$$

Proof. For simplicity we denote $\rho = \rho(\mathscr{L}_{\omega_1,\omega_2,\gamma})$.

Since A is an L-matrix, $L \geqslant 0$, $U \geqslant 0$ and $\mathscr{L}_{\omega_1,\omega_2,\gamma} \geqslant 0$. Hence, there is $x \geqslant 0$, $x \neq 0$, such that

$$\mathscr{L}_{\omega_1,\omega_2,\gamma} x = \rho x,$$

i.e.,

$$[I - \Omega + (\omega_2 - \gamma)L + \omega_1 U]x = (I - \gamma L)\rho x.$$

Multiplying by Ω^{-1}, it derives

$$[(\rho - 1)\Omega^{-1} + I]x = \left(\frac{\omega_2 - \gamma + \gamma\rho}{\omega_2} L + U \right) x.$$

Hence,

$$\left[\min_{i=1,2} \left(\frac{\rho - 1}{\omega_i} + 1 \right) \right] x \leqslant \left(\frac{\omega_2 - \gamma + \gamma\rho}{\omega_2} L + U \right) x \leqslant \left[\max_{i=1,2} \left(\frac{\rho - 1}{\omega_i} + 1 \right) \right] x. \tag{3.18}$$

Notice that $\omega_2 - \gamma + \gamma\rho \geqslant 0$, it implies

$$\max_{i=1,2} \left(\frac{\rho - 1}{\omega_i} + 1 \right) \geqslant 0$$

and

$$\frac{\omega_2 - \gamma + \gamma\rho}{\omega_2} L + U \geqslant 0.$$

By (3.18) and [1, Theorem 2-1.11] we derive directly the first inequality in (3.17). If $\omega_2 - \gamma + \gamma\rho = 0$, then $\omega_2 = \gamma$ and $\rho = 0$ so that the second inequality in (3.17) is trivial. If $\omega_2 - \gamma + \gamma\rho \neq 0$, then with J the matrix $[(\omega_2 - \gamma + \gamma\rho)/\omega_2]L + U$ is irreducible. It follows from the second inequality in (3.18) and Lemma 3.1 we obtain the second inequality in (3.17).

Using Lemma 3.2 we give the connection between the convergence of J and $\mathscr{L}_{\omega_1,\omega_2,\gamma}$.

Theorem 3.7. *Let A be an L-matrix. Then, for $0<\omega_1\leqslant 1$, $0<\omega_2\leqslant 1$, $0<\gamma\leqslant\omega_2$, the following statements are true*:

(a) $0\leqslant\rho(J)<1$ *if and only if* $1-\max_{i=1,2}\omega_i\leqslant\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma})<1$, *and in this case we have*

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma})\leqslant\max_{i=1,2}\{1-\omega_i+\omega_i\rho(J)\}<1.$$

(b) $\rho(J)\geqslant 1$ *if and only if* $\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma})\geqslant 1$, *and in this case we have*

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma})\geqslant\min_{i=1,2}\{1-\omega_i+\omega_i\rho(J)\}\geqslant 1.$$

Proof. First we assume A is irreducible. If $\rho(J)<1$, by [17, Theorem 2-7.2], A is an M-matrix. Theorem 3.3 implies

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma})\leqslant\max_{i=1,2}\{1-\omega_i+\omega_i\rho(J)\}<1.$$

Further, as $0\leqslant\gamma(\rho-1)/\omega_2+1<1$, by Lemma 3.2 it follows that

$$0\leqslant\left[\frac{\gamma(\rho-1)}{\omega_2}+1\right]\rho(J)\leqslant\frac{\rho-1}{\max_{i=1,2}\omega_i}+1,$$

and, hence,

$$\rho\geqslant 1-\max_{i=1,2}\omega_i.$$

Conversely, assume that $\rho<1$ holds. If $\rho(J)=0$, then the proof is completed. Now we consider the case when $\rho(J)>0$ holds. In this case, as $\gamma(\rho-1)/\omega_2+1<1$, Lemma 3.2 derives the following inequality:

$$\left[\frac{\gamma(\rho-1)}{\omega_2}+1\right]\rho(J)<\frac{\rho-1}{\max_{i=1,2}\omega_i}+1. \tag{3.19}$$

If $\gamma(\rho-1)/\omega_2+1=0$, then $\rho=0$ and, consequently, $(\rho-1)/\max_{i=1,2}\omega_i+1\leqslant 0$, which contradicts (3.19). Hence $\gamma(\rho-1)/\omega_2+1\neq 0$, and from (3.19) it derives

$$\rho(J)<\left(\frac{\rho-1}{\max_{i=1,2}\omega_i}+1\right)\Big/\left(\frac{\gamma(\rho-1)}{\omega_2}+1\right)\leqslant 1.$$

This shows (a).

From (a) it follows immediately that $\rho(J)\geqslant 1$ if and only if $\rho\geqslant 1$. In this case $\gamma(\rho-1)/\omega_2+1\geqslant 1$, and, hence, Lemma 3.2 derives that

$$\rho(J)\leqslant\frac{\rho-1}{\min_{i=1,2}\omega_i}+1,$$

and also

$$\rho\geqslant 1+[\rho(J)-1]\left[\min_{i=1,2}\omega_i\right]=\min_{i=1,2}\{1-\omega_i+\omega_i\rho(J)\}\geqslant 1.$$

Up to now, we have proved the statement under the condition that A is irreducible. Now, we assume that A is reducible. We construct an irreducible L-matrix A_ε by replacing all zero elements of $-K$ and $-H$ by a small negative number $-\varepsilon$. Let

$$J_\varepsilon = D - A_\varepsilon,$$

and let $\mathscr{L}_{\omega_1,\omega_2,\gamma}(\varepsilon)$ be the MAOR iteration matrix with respect to A_ε.

Obviously,

$$\rho(J_\varepsilon) \to \rho(J), \quad \rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}(\varepsilon)) \to \rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) \quad \text{as } \varepsilon \to 0.$$

Since A_ε is an irreducible L-matrix, by the proof above, $\rho(J_\varepsilon)$ and $\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}(\varepsilon))$ satisfy (a) and (b). Putting $\varepsilon \to 0$ we obtain (a) and (b) hold for $\rho(J)$ and $\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma})$. This shows the statement to be true when A is reducible.

Now, using [17, Theorem 2-7.2], Theorem 3.3, Theorem 3.7 we prove an equivalence theorem.

Theorem 3.8. *Let A be an L-matrix. Then the following statements are equivalent:*

(a) *A is an M-matrix.*

(b) $\rho(J) < 1$.

(c) *The* MAOR *method is convergent, whenever* $0 < \omega_1 < 2/[1 + \rho(J)]$, $0 < \omega_2 < 2/[1 + \rho(J)]$, $0 < \gamma \leqslant \omega_2$.

(d) *The* MSOR *method is convergent, whenever* $0 < \omega_1 < 2/[1 + \rho(J)]$, $0 < \omega_2 < 2/[1 + \rho(J)]$.

Proof. The equivalence between (a) and (b) is proved by [17, Theorem 2-7.2]. If A is an M-matrix then it is also an H-matrix. By Theorem 3.3 we derive (c).

Conversely, we assume (c) to be true. If ω_1 and ω_2 satisfy $0 < \omega_1 \leqslant 1$, $0 < \omega_2 \leqslant 1$ then, by Theorem 3.7, we obtain (b). If either ω_1 or ω_2 is larger than 1 then

$$\frac{2}{1 + \rho(J)} > 1,$$

and, therefore, $\rho(J) < 1$. Hence, if (c) holds then (b) is true.

We have proved (c) is equivalent to (a) and (b). (d) is a special case of (c).

Remark 3.3. The result on the MSOR method here is better than the ones by [9, Theorem 3], where the parameters ω_1 and ω_2 only satisfy $0 < \omega_1 \leqslant \omega_2 \leqslant 1$.

Remark 3.4. In this section we assume that A has the form (1.2), where D_1 and D_2 are square nonsingular diagonal matrices. For general cases that the matrices D_1 and D_2 are only nonsingular we can prove that all the convergence theorems above are valid, but some additional conditions for Theorems 3.1, 3.7 and 3.8 are necessary. For Theorem 3.1 we shall assume that D is positive definite. For Theorems 3.7 and 3.8 the additional condition is D being an M-matrix.

4. Optimum choice of parameter factors

In this section we study the optimum factors and the optimum spectral radii of the MSOR and MAOR methods.

First the following lemma was obtained in [3].

Lemma 4.1. *Let the eigenvalues of $\mathscr{L}_{\omega_1,\omega_2,\gamma}$ and J be, respectively, $\{\lambda\}$ and $\{\mu\}$. Then it holds*

$$(\lambda+\omega_1-1)(\lambda+\omega_2-1)=\omega_1(\omega_2-\gamma+\gamma\lambda)\mu^2,$$

i.e.,

$$\lambda^2-(2-\omega_1-\omega_2+\gamma\omega_1\mu^2)\lambda+(\omega_1-1)(\omega_2-1)+\omega_1(\gamma-\omega_2)\mu^2=0.$$

We denote the eigenvalues of the Jacobi iteration matrix by $\{\mu_i,\ i=1,\ldots,n\}$, and let

$$\bar{\mu}=\max_{1\leqslant i\leqslant n}|\mu_i|,\qquad \underline{\mu}=\min_{1\leqslant i\leqslant n}|\mu_i|,$$

$$\omega_{1b}=1+\left(\frac{\bar{\mu}-\underline{\mu}}{\sqrt{1-\underline{\mu}^2}+\sqrt{1-\bar{\mu}^2}}\right)^2,\qquad \omega_{2b}=1+\left(\frac{\bar{\mu}+\underline{\mu}}{\sqrt{1-\underline{\mu}^2}+\sqrt{1-\bar{\mu}^2}}\right)^2,$$

$$\beta=\frac{\sqrt{1-\underline{\mu}^2}-\sqrt{1-\bar{\mu}^2}}{\sqrt{1-\underline{\mu}^2}+\sqrt{1-\bar{\mu}^2}}.$$

In [16] the optimum virtual spectral radius of the MSOR method and the corresponding optimum parameter factors were derived. With the completely similar proof we can prove the following theorem.

Theorem 4.1. *If the eigenvalues of J are real and $\bar{\mu}<1$, then*

$$\rho(\mathscr{L}_{\omega_{1b},\omega_{2b}})=\rho(\mathscr{L}_{\omega_{2b},\omega_{1b}})=\beta$$

and unless $(\omega_1,\omega_2)=(\omega_{1b},\omega_{2b})$ or $(\omega_1,\omega_2)=(\omega_{2b},\omega_{1b})$ we have

$$\rho(\mathscr{L}_{\omega_1,\omega_2})>\beta.$$

Now we consider the MAOR method.

Theorem 4.2. *If the eigenvalues of J are real and $\bar{\mu}<1$, then*

$$\min_{\omega_1,\omega_2,\gamma}\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma})=\beta,$$

and the optimum parameter factors $\omega_{1o},\omega_{2o},\gamma_o$ are defined as follows:

(a) *For $\bar{\mu}=\underline{\mu}=0$,*

$$\omega_{1o}=\omega_{2o}=1,\quad \gamma_o \text{ arbitrary}.$$

(b) *For* $\bar{\mu} = \underline{\mu} > 0$,

$$\omega_{1o} = \frac{1}{\omega_{2o}(1-\bar{\mu}^2)}, \qquad \gamma_o = \omega_{2o}(2-\omega_{2o}) + \frac{1}{\bar{\mu}^2}(1-\omega_{2o})^2, \qquad \omega_{2o} \neq 0 \quad \textit{arbitrary}.$$

(c) *For* $\bar{\mu} > \underline{\mu} \geqslant 0$,

$$\gamma_o = \omega_{2o},$$

and either

$$(\omega_{1o}, \omega_{2o}) = (\omega_{1b}, \omega_{2b})$$

or

$$(\omega_{1o}, \omega_{2o}) = (\omega_{2b}, \omega_{1b}).$$

Proof. Similar to the proof of [17, Theorem 8-3.3], by [17, Theorem 8-2.1, Lemma 6-2.9] we can obtain

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) = \max_{1\leqslant i\leqslant n} \rho(\omega_1, \omega_2, \gamma; \mu_i^2),$$

where $\rho(\omega_1, \omega_2, \gamma; \mu^2)$ is the root radius of

$$\lambda^2 - b(\mu^2)\lambda + c(\mu^2) = 0$$

with

$$b(\mu^2) = 2 - \omega_1 - \omega_2 + \gamma\omega_1\mu^2 = 1 + c(\mu^2) - \omega_1\omega_2(1-\mu^2)$$

and

$$c(\mu^2) = (\omega_1 - 1)(\omega_2 - 1) + \omega_1(\gamma - \omega_2)\mu^2.$$

Obviously, one and only one of the three cases

$$\bar{\mu} = \underline{\mu} = 0, \quad \bar{\mu} = \underline{\mu} > 0, \quad \bar{\mu} > \underline{\mu} \geqslant 0$$

appears.

Case I: $\bar{\mu} = \underline{\mu} = 0$. In this case we have $\beta = 0$ and

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) \geqslant 0.$$

Clearly, $\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) = 0$ iff

$$b(0) = 2 - \omega_1 - \omega_2 = 0, \qquad c(0) = (\omega_1 - 1)(\omega_2 - 1) = 0,$$

i.e.,

$$\omega_1 = \omega_2 = 1.$$

Case II: $\bar{\mu} = \underline{\mu} > 0$. Now, we have $\beta = 0$. Furthermore, we have that $\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) = 0$ holds iff

$$2 - \omega_1 - \omega_2 + \gamma\omega_1\bar{\mu}^2 = 0, \qquad (\omega_1 - 1)(\omega_2 - 1) + \omega_1(\gamma - \omega_2)\bar{\mu}^2 = 0.$$

Solving these equations we obtain

$$\omega_2 \neq 0, \qquad \omega_1 = \frac{1}{\omega_2(1-\bar{\mu}^2)}, \qquad \gamma = \omega_2(2-\omega_2) + \frac{(1-\omega_2)^2}{\bar{\mu}^2}.$$

Case III: $\bar{\mu} > \underline{\mu} \geqslant 0$. In this case, we have $\beta > 0$.
If either $|c(\underline{\mu}^2)| > \beta^2$ or $|c(\bar{\mu}^2)| > \beta^2$ is true, then

$$\rho(\omega_1, \omega_2, \gamma; \underline{\mu}^2) \geqslant \sqrt{|c(\underline{\mu}^2)|} > \beta$$

or

$$\rho(\omega_1, \omega_2, \gamma; \bar{\mu}^2) \geqslant \sqrt{|c(\bar{\mu}^2)|} > \beta$$

and, hence,

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) > \beta. \tag{4.1}$$

Now we consider the case when $|c(\underline{\mu}^2)| \leqslant \beta^2$ and $|c(\bar{\mu}^2)| \leqslant \beta^2$ are true.
For

$$\omega_1\omega_2 \geqslant \frac{4}{\left(\sqrt{1-\underline{\mu}^2} + \sqrt{1-\bar{\mu}^2}\right)^2}$$

we get

$$b(\underline{\mu}^2) = 1 + c(\underline{\mu}^2) - \omega_1\omega_2(1-\underline{\mu}^2) \leqslant 1 + \beta^2 - \frac{4(1-\underline{\mu}^2)}{\left(\sqrt{1-\underline{\mu}^2} + \sqrt{1-\bar{\mu}^2}\right)^2} = -2\beta,$$

which implies

$$\rho(\omega_1, \omega_2, \gamma; \underline{\mu}^2) \geqslant \tfrac{1}{2}|b(\underline{\mu}^2)| \geqslant \beta. \tag{4.2}$$

If

$$\omega_1\omega_2 \leqslant \frac{4}{\left(\sqrt{1-\underline{\mu}^2} + \sqrt{1-\bar{\mu}^2}\right)^2}$$

we can derive

$$\omega_1\omega_2 \leqslant \frac{4}{\left(\sqrt{1-\underline{\mu}^2} + \sqrt{1-\bar{\mu}^2}\right)^2} = \frac{(1-\beta)^2}{1-\bar{\mu}^2}$$
$$= \frac{(1-\beta)(\beta-\beta^2)}{\beta(1-\bar{\mu}^2)} \leqslant \frac{(1-\beta)[\beta - c(\bar{\mu}^2)]}{\beta(1-\bar{\mu}^2)}$$

and, therefore,

$$1 + c(\bar{\mu}^2) - \omega_1\omega_2(1-\bar{\mu}^2) \geqslant \beta + \frac{1}{\beta}c(\bar{\mu}^2),$$

i.e.,

$$b(\bar{\mu}^2) \geqslant \beta + \frac{1}{\beta} c(\bar{\mu}^2) \geqslant 0.$$

It follows that

$$\begin{aligned}
\rho(\omega_1, \omega_2, \gamma; \bar{\mu}^2) &= \frac{1}{2}\left[b(\bar{\mu}^2) + \sqrt{[b(\bar{\mu}^2)]^2 - 4c(\bar{\mu}^2)}\right] \\
&\geqslant \frac{1}{2}\left[\beta + \frac{1}{\beta}c(\bar{\mu}^2) + \sqrt{\left[\beta + \frac{1}{\beta}c(\bar{\mu}^2)\right]^2 - 4c(\bar{\mu}^2)}\right] \\
&= \frac{1}{2}\left[\beta + \frac{1}{\beta}c(\bar{\mu}^2) + \left(\beta - \frac{1}{\beta}c(\bar{\mu}^2)\right)\right] \\
&= \beta.
\end{aligned} \tag{4.3}$$

From (4.1)–(4.3) we have shown that

$$\max\{\rho(\omega_1, \omega_2, \gamma; \bar{\mu}^2),\ \rho(\omega_1, \omega_2, \gamma; \underline{\mu}^2)\} \geqslant \beta.$$

Furthermore, by the proof above, it is easy to prove that the equality

$$\max\{\rho(\omega_1, \omega_2, \gamma; \bar{\mu}^2),\ \rho(\omega_1, \omega_2, \gamma; \underline{\mu}^2)\} = \beta \tag{4.4}$$

holds if and only if

$$c(\underline{\mu}^2) = (\omega_1 - 1)(\omega_2 - 1) + \omega_1(\gamma - \omega_2)\underline{\mu}^2 = \beta^2,$$

$$c(\bar{\mu}^2) = (\omega_1 - 1)(\omega_2 - 1) + \omega_1(\gamma - \omega_2)\bar{\mu}^2 = \beta^2,$$

$$\omega_1\omega_2 = \frac{4}{\left(\sqrt{1 - \underline{\mu}^2} + \sqrt{1 - \bar{\mu}^2}\right)^2}.$$

Solving these equations we obtain

$$\gamma = \omega_2 \tag{4.5}$$

and either

$$\omega_1 = \omega_{1b}, \qquad \omega_2 = \omega_{2b} \tag{4.6}$$

or

$$\omega_1 = \omega_{2b}, \qquad \omega_2 = \omega_{1b}. \tag{4.7}$$

Now, by the proof above, we obtain

$$\rho(\mathscr{L}_{\omega_1, \omega_2, \gamma}) \geqslant \max\{\rho(\omega_1, \omega_2, \gamma; \bar{\mu}^2),\ \rho(\omega_1, \omega_2, \gamma; \underline{\mu}^2)\} \geqslant \beta.$$

If

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) = \beta,$$

then (4.4) holds. Hence (4.5) and either (4.6) or (4.7) are true.

Conversely, when (4.5) and either (4.6) or (4.7) hold, the MAOR method reduces the MSOR method and by Theorem 4.1 we get

$$\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) = \beta.$$

Remark 4.1. We have shown that the optimum spectral radius of the MAOR method is equal β. When $\bar{\mu} > \underline{\mu} \geqslant 0$ the optimum MAOR method is just the optimum MSOR method. The results here also answer partly the open problem in [3].

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