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Robust stability for delay Lur'e control systems with multiple nonlinearities

Yong He[a,*], Min Wu[a], Jin-Hua She[b], Guo-Ping Liu[c]

[a] *School of Information Science and Engineering, Central South University, Changsha 410083, China*
[b] *School of Bionics, Tokyo University of Technology, Tokyo, 192-0982, Japan*
[c] *School of Electronics, University of Glamorgan, Pontypridd CF37 1DL, UK*

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Abstract

This paper deals with delay Lur'e control systems with multiple nonlinearities and time-varying structured uncertainties. First, some sufficient, and necessary and sufficient conditions for the existence of a Lyapunov functional in the extended Lur'e form with a negative definite derivative that guarantees delay-independent robust absolute stability are presented. Then, some new less conservative delay-dependent absolute stability criteria are derived that employ free weighting matrices to express the relationships between the terms in the Leibniz–Newton formula. All the criteria are based on linear matrix inequalities. Finally, a numerical example is presented to illustrate the effectiveness of the method.

Keywords: Lur'e control system; Absolute stability; Lyapunov functional; Linear matrix inequality (LMI); Delay-dependent criterion

1. Introduction

The problem of the absolute stability of Lur'e control systems has been widely studied for several decades (see [8,9,14,16,21]). Since time delays are frequently encountered in such systems and are often a source of instability, a considerable number of studies have also been done on the stability of delay

* Corresponding author. Tel./fax: +86-731-883-6091.
E-mail address: heyong08@yahoo.com.cn (Y. He).

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Lur'e control systems (e.g. [1,2,7,10,17]). They have resulted in the establishment of some necessary and sufficient conditions for the existence of a Lyapunov functional with a negative definite derivative in the extended Lur'e form that guarantees the absolute stability of such systems [7]. However, it is known that the necessary and sufficient conditions in [7] cannot be extended to deal with systems with time-varying structured uncertainties. [10] employed linear matrix inequalities (LMIs) to express the necessary and sufficient conditions given in [7]. The advantage of this method is that it is easy to extend to systems with time-varying structured uncertainties.

On the other hand, a number of interesting new ideas have been proposed recently to improve the delay-dependent stability criteria for linear delay systems [4–6,12,13,15,18,20]. The most effective of these methods was first presented by [15] and then extended to a more general setting by [13]. However, their methods still require improvement in several places. One is that they treated the terms of the Leibniz–Newton formula as though they were independent of each other and did not fully consider the relationships between them. Recently, [11] presented a new method of obtaining delay-dependent stability criteria for neutral systems. The most interesting feature of this method is that it uses free weighting matrices to express those relationships.

This paper discusses the problem of the existence of a Lyapunov functional in the extended Lur'e form with a negative definite derivative that guarantees the robust absolute stability of a delay Lur'e control system with multiple nonlinearities in a bounded sector. Some necessary and sufficient conditions for its existence are derived by extending the delay-independent criteria given in [10] to a system with time-varying structured uncertainties. The method presented in [11] is also employed to derive some delay-dependent stability conditions for delay Lur'e control systems. These methods have two big advantages. One is that the free parameters in the Lyapunov functional can easily be selected by solving a group of LMIs in both delay-independent and delay-dependent criteria. The other is that the relationships between the terms in the Leibniz–Newton formula are taken into account in the delay-dependent criteria. A numerical example is presented to illustrate how much of an improvement the necessary and sufficient conditions are over the sufficient condition obtained by direct application of the S-procedure. The benefit of delay-dependent criteria is also demonstrated in the example.

2. Notation and preliminaries

Consider the following delay Lur'e control system with time-varying structured uncertainties and multiple nonlinearities:

$$\mathscr{S}_1 : \begin{cases} \dot{x}(t) = (A + \Delta A(t))x(t) + (B + \Delta B(t))x(t-\tau) \\ \qquad\quad + (D + \Delta D(t))f(\sigma(t)), \\ \sigma(t) = C^{\mathrm{T}}x(t), \end{cases} \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$ is the state vector; $\tau > 0$; $A = (a_{ij})_{n\times n}$; $B = (b_{ij})_{n\times n}$; $D = (d_{ij})_{n\times m} = (d_1, d_2, \ldots, d_m)$; $C = (c_{ij})_{n\times m} = (c_1, c_2, \ldots, c_m)$; d_j and c_j $(j = 1, 2, \ldots, m)$ are the jth column of D and C, respectively; $\sigma(t) = (\sigma_1(t), \sigma_2(t), \ldots, \sigma_m(t))^{\mathrm{T}}$; and $f(\sigma(t)) = (f_1(\sigma_1(t)), f_2(\sigma_2(t)), \ldots, f_m(\sigma_m(t)))^{\mathrm{T}}$ is a nonlinear function. The nonlinearities $f_j(\cdot)$ satisfy

$$f_j(\cdot) \in K_j[0, k_j] = \{f_j(\sigma_j) | f_j(0) = 0;\ 0 \leqslant \sigma_j f_j(\sigma_j) \leqslant k_j\sigma_j^2,\ \sigma_j \neq 0\}, \quad j = 1, 2, \ldots, m, \tag{2}$$

for $0 < k_j < +\infty, \ \ j = 1, 2, \ldots, m$. For simplicity, $f_j(\sigma_j(t))$ is abbreviated to $f_j(\sigma_j)$ in some places in this paper.

The uncertainties are assumed to be of the following form:

$$[\Delta A(t) \ \Delta B(t) \ \Delta D(t)] = HF(t)[E_a \ E_b \ E_d], \tag{3}$$

where H, E_a, E_b, and E_d are known real constant matrices with appropriate dimensions; E_{dj} is the jth column of E_d; and $F(t)$ is an unknown real time-varying matrix with Lebesgue measurable elements satisfying ($\| \cdot \|$ means the Euclidean norm)

$$\|F(t)\| \leqslant 1, \ \ \forall t. \tag{4}$$

The nominal system of $\mathscr{S}_1$ is given by

$$\mathscr{S}_0 : \begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau) + Df(\sigma(t)), \\ \sigma(t) = C^{\mathrm{T}}x(t). \end{cases} \tag{5}$$

Constructing a Lyapunov functional in the extended Lur'e form yields

$$V(x_t) = x^{\mathrm{T}}(t)Px(t) + \int_{t-\tau}^{t} x^{\mathrm{T}}(s)Qx(s) + 2\sum_{j=1}^{m} \lambda_j \int_0^{\sigma_j} f_j(\sigma_j)\,\mathrm{d}\sigma_j, \tag{6}$$

where $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$ and $\lambda_j \geqslant 0 \ (j = 1, 2, \ldots, m)$ need to be determined.

Definition 1. The functional $V(x_t)$ of (6) is said to be a Lyapunov functional of system $\mathscr{S}_1$ (or of the nominal system $\mathscr{S}_0$), with a negative definite derivative if

$$\dot{V}(x_t)|_{\mathscr{S}_1} < 0 \ (\text{resp. } \dot{V}(x_t)|_{\mathscr{S}_0} < 0) \tag{7}$$

for any $f_j(\cdot) \in K_j[0, k_j] \ (j = 1, 2, \ldots, m, \ (x(t), x(t-\tau)) \neq 0)$.

If condition (7) holds, $\mathscr{S}_1$ is robustly absolutely stable and the nominal system $\mathscr{S}_0$ is absolutely stable in the sector bounded by $K = \mathrm{diag}(k_1, k_2, \ldots, k_m)$.

The following lemmas are employed to derive the main results of this study.

Lemma 2 (*He and Wu [10]*). *Equation* (7) *holds for the nominal system* $\mathscr{S}_0$, *i.e.*, $\mathscr{S}_0$ *is absolutely stable in the sector bounded by* $K = \mathrm{diag}(k_1, k_2, \ldots, k_m)$, *if there exist* $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$, $T = \mathrm{diag}(t_1, t_2, \ldots, t_m) \geqslant 0$ *and* $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \geqslant 0$ *such that the LMI*

$$\Omega = \begin{bmatrix} A^{\mathrm{T}}P + PA + Q & PB & PD + A^{\mathrm{T}}C\Lambda + CKT \\ B^{\mathrm{T}}P & -Q & B^{\mathrm{T}}C\Lambda \\ D^{\mathrm{T}}P + \Lambda C^{\mathrm{T}}A + TKC^{\mathrm{T}} & \Lambda C^{\mathrm{T}}B & \Lambda C^{\mathrm{T}}D + D^{\mathrm{T}}C\Lambda - 2T \end{bmatrix} < 0 \tag{8}$$

holds. This condition is also a necessary condition when $m = 1$.

This lemma was derived by directly applying the S-procedure to the nonlinearities, and the condition is only sufficient when $m > 1$.

In contrast, letting $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_m)$ and

$$D_j^m = \{\alpha | \alpha_i = 0, \ \text{for } i \geqslant j; \ \ \alpha_i \in \{0, k_i\} \text{ for } i < j, \ \ (i = 1, 2, \ldots, m)\}, \ \ j = 1, 2, \ldots, m, \tag{9}$$

for 2^{j-1} elements, and assuming that

$$A(\alpha) := A + D\alpha C^{\mathrm{T}}, \ \ P(\alpha) := P + C\Lambda\alpha C^{\mathrm{T}} \tag{10}$$

yields the following lemma, which provides a necessary and sufficient condition.

Lemma 3 (*He and Wu [10]*). *Assuming $m \geqslant 1$, for the nominal system $\mathscr{S}_0$ the necessary and sufficient condition for the existence of the Lyapunov functional $V(x_t)$,* Eq. (6), *satisfying inequality* (7), *i.e., $\mathscr{S}_0$ is absolutely stable in the sector bounded by $K = \mathrm{diag}(k_1, k_2, \ldots, k_m)$, is that, for any $\alpha \in D_j^m$ $(j = 1, 2, \ldots, m)$, there exist $t_\alpha \geqslant 0$, $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$ and $\lambda_i \geqslant 0$ $(i = 1, 2, \ldots, m)$, such that the following LMIs hold.*

$$G_j(\alpha) = \begin{bmatrix} \Phi_{11}(\alpha) & P(\alpha)B & \Phi_{13,j}(\alpha) + t_\alpha k_j c_j \\ B^{\mathrm{T}}P(\alpha) & -Q & \lambda_j B^{\mathrm{T}} c_j \\ \Phi_{13,j}(\alpha) + t_\alpha k_j c_j^{\mathrm{T}} & \lambda_j c_j^{\mathrm{T}} B & 2\lambda_j c_j^{\mathrm{T}} d_j - 2t_\alpha \end{bmatrix} < 0, \quad j = 1, 2, \ldots, m, \tag{11}$$

where

$$\Phi_{11}(\alpha) = A^{\mathrm{T}}(\alpha)P(\alpha) + P(\alpha)A(\alpha) + Q,$$
$$\Phi_{13,j}(\alpha) = P(\alpha)d_j + \lambda_j A^{\mathrm{T}}(\alpha)c_j.$$

The following lemma is used to deal with the time-varying structured uncertainties in the system.

Lemma 4 (*Xie [19]*). *For given matrices $Q = Q^{\mathrm{T}}$, H, E and $R = R^{\mathrm{T}} > 0$ with appropriate dimensions,*

$$Q + HFE + E^{\mathrm{T}}F^{\mathrm{T}}H^{\mathrm{T}} < 0,$$

holds for all F satisfying $F^{\mathrm{T}}F \leqslant R$ if and only if there exists $\varepsilon > 0$ such that

$$Q + \varepsilon HH^{\mathrm{T}} + \varepsilon^{-1}E^{\mathrm{T}}RE < 0.$$

3. Delay-independent robust absolute stability

First, for the system with time-varying structured uncertainties, $\mathscr{S}_1$, the following sufficient condition is derived from Lemma 2 by applying the S-procedure directly to the nonlinearities and handling the uncertainties by means of Lemma 4.

Theorem 5. *System $\mathscr{S}_1$ is robustly absolutely stable in the sector bounded by $K = \mathrm{diag}(k_1, k_2, \ldots, k_m)$ if there exist $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \geqslant 0$, $T = \mathrm{diag}(t_1, t_2, \ldots, t_m) \geqslant 0$ and $\varepsilon \geqslant 0$ such that the LMI*

$$\begin{bmatrix} \Psi_{11} & PB + \varepsilon E_a^{\mathrm{T}} E_b & \Psi_{13} & PH \\ B^{\mathrm{T}}P + \varepsilon E_b^{\mathrm{T}} E_a & -Q + \varepsilon E_b^{\mathrm{T}} E_b & B^{\mathrm{T}} C\Lambda + \varepsilon E_b^{\mathrm{T}} E_d & 0 \\ \Psi_{13}^{\mathrm{T}} & \Lambda C^{\mathrm{T}} B + \varepsilon E_d^{\mathrm{T}} E_b & \Psi_{33} & \Lambda C^{\mathrm{T}} H \\ H^{\mathrm{T}} P & 0 & H^{\mathrm{T}} C\Lambda & -\varepsilon I \end{bmatrix} < 0 \tag{12}$$

holds, where

$$\Psi_{11} = A^{\mathrm{T}}P + PA + Q + \varepsilon E_a^{\mathrm{T}} E_a,$$
$$\Psi_{13} = PD + A^{\mathrm{T}}C\Lambda + CKT + \varepsilon E_a^{\mathrm{T}} E_d,$$
$$\Psi_{33} = \Lambda C^{\mathrm{T}} D + D^{\mathrm{T}} C\Lambda - 2T + \varepsilon E_d^{\mathrm{T}} E_d.$$

Proof. Replacing A, B and D in (8) with $A+HF(t)E_a$, $B+HF(t)E_b$ and $D+HF(t)E_d$, respectively, shows that (8) for $\mathscr{S}_1$ is equivalent to the following condition

$$\Omega + \begin{bmatrix} PH \\ 0 \\ \Lambda C^{\mathrm{T}} H \end{bmatrix} F(t)[E_a \;\; E_b \;\; E_d] + \begin{bmatrix} E_a^{\mathrm{T}} \\ E_b^{\mathrm{T}} \\ E_d^{\mathrm{T}} \end{bmatrix} F^{\mathrm{T}}(t)[H^{\mathrm{T}}P \;\; 0 \;\; H^{\mathrm{T}}C\Lambda] < 0. \tag{13}$$

By Lemma 4, a necessary and sufficient condition guaranteeing (13) is that there exists $\varepsilon > 0$ such that

$$\Omega + \varepsilon^{-1} \begin{bmatrix} PH \\ 0 \\ \Lambda C^{\mathrm{T}} H \end{bmatrix} [H^{\mathrm{T}}P \;\; 0 \;\; H^{\mathrm{T}}C\Lambda] + \varepsilon \begin{bmatrix} E_a^{\mathrm{T}} \\ E_b^{\mathrm{T}} \\ E_d^{\mathrm{T}} \end{bmatrix} [E_a \;\; E_b \;\; E_d] < 0. \tag{14}$$

Applying the Schur complement [3] shows that (14) is equivalent to (12). □

This theorem is conservative with regard to the robust absolute stability of system $\mathscr{S}_1$, which has multiple nonlinearities, since it is just a sufficient condition. The following theorem derived from Lemma 3 gives a necessary and sufficient condition.

Theorem 6. *A necessary and sufficient condition for the existence of a Lyapunov functional $V(x_t)$, Eq.* (6) *satisfying inequality* (7), *that ensures the robust absolute stability of $\mathscr{S}_1$ in the sector bounded by $K = \mathrm{diag}(k_1, k_2, \ldots, k_m)$ is that, for any $\alpha \in D_j^m$ $(j = 1, 2, \ldots, m)$, there exist $t_\alpha \geqslant 0$, $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$, $\lambda_i \geqslant 0$ $(i = 1, 2, \ldots, m)$ and $\varepsilon_\alpha \geqslant 0$, such that the following LMIs hold.*

$$\hat{G}_j(\alpha) = \begin{bmatrix} \hat{\Phi}_{11}(\alpha) & \hat{\Phi}_{12}(\alpha) & \hat{\Phi}_{13,j}(\alpha) & P(\alpha)H \\ \hat{\Phi}_{12}^{\mathrm{T}}(\alpha) & \hat{\Phi}_{22}(\alpha) & \hat{\Phi}_{23,j}(\alpha) & 0 \\ \hat{\Phi}_{13,j}^{\mathrm{T}}(\alpha) & \hat{\Phi}_{23,j}^{\mathrm{T}}(\alpha) & \hat{\Phi}_{33,j}(\alpha) & \lambda_j c_j^{\mathrm{T}} H \\ H^{\mathrm{T}}P(\alpha) & 0 & \lambda_j H^{\mathrm{T}} c_j & -\varepsilon_\alpha I \end{bmatrix} < 0, \quad j = 1, 2, \ldots, m, \tag{15}$$

where

$$\hat{\Phi}_{11}(\alpha) = \Phi_{11}(\alpha) + \varepsilon_\alpha E_a^{\mathrm{T}}(\alpha) E_a(\alpha),$$
$$\hat{\Phi}_{12}(\alpha) = P(\alpha)B + \varepsilon_\alpha E_a^{\mathrm{T}}(\alpha) E_b,$$
$$\hat{\Phi}_{13,j}(\alpha) = \Phi_{13,j}(\alpha) + t_\alpha k_j c_j + \varepsilon_\alpha E_a^{\mathrm{T}}(\alpha) E_{dj},$$
$$\hat{\Phi}_{22}(\alpha) = -Q + \varepsilon_\alpha E_b^{\mathrm{T}} E_b,$$
$$\hat{\Phi}_{23,j}(\alpha) = \lambda_j B^{\mathrm{T}} c_j + \varepsilon_\alpha E_b^{\mathrm{T}} E_{dj},$$
$$\hat{\Phi}_{33,j}(\alpha) = 2\lambda_j c_j^{\mathrm{T}} d_j - 2t_\alpha + \varepsilon_\alpha E_{dj}^{\mathrm{T}} E_{dj},$$
$$E_a(\alpha) = (E_a + E_d \alpha C^{\mathrm{T}})^{\mathrm{T}} (E_a + E_d \alpha C^{\mathrm{T}})$$

and $\Phi_{11}(\alpha)$ and $\Phi_{13,j}(\alpha)$ are defined in (11).

Proof. For the sake of simplicity, let

$$\bar{A} = A + \Delta A(t), \ \ \bar{B} = B + \Delta B(t), \ \ \bar{D} = D + \Delta D(t), \ \ \bar{A}(\alpha) = \bar{A} + \bar{D}\alpha C^{\mathrm{T}}, \tag{16}$$

and $\bar{d}_j$ be the jth column of $\bar{D}$. It is clear from Lemma 3 that the conditions (11) for $\mathscr{S}_1$ are equivalent to the statement that there exist $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$, $\lambda_i \geqslant 0$ $(i = 1, 2, \ldots, m)$ and t_α such that the following LMIs

$$\bar{G}_j(\alpha) = \begin{bmatrix} \bar{\Phi}_{11}(\alpha) & P(\alpha)\bar{B} & \bar{\Phi}_{13,j}(\alpha) + t_\alpha k_j c_j \\ \bar{B}^{\mathrm{T}} P(\alpha) & -Q & \lambda_j \bar{B}^{\mathrm{T}} c_j \\ \bar{\Phi}_{13,j}^{\mathrm{T}}(\alpha) + t_\alpha k_j c_j^{\mathrm{T}} & \lambda_j c_j^{\mathrm{T}} \bar{B} & 2\lambda_j c_j^{\mathrm{T}} \bar{d}_j - 2t_\alpha \end{bmatrix} < 0, \ \ j = 1, 2, \ldots, m \tag{17}$$

hold for any $\alpha \in D_j^m, \ \ j = 1, 2, \ldots, m$, where

$$\bar{\Phi}_{11}(\alpha) = \bar{A}^{\mathrm{T}}(\alpha) P(\alpha) + P(\alpha)\bar{A}(\alpha) + Q,$$
$$\bar{\Phi}_{13,j}(\alpha) = P(\alpha)\bar{d}_j + \lambda_j \bar{A}^{\mathrm{T}}(\alpha) c_j.$$

Replacing $\bar{A}(\alpha)$, $\bar{B}$ and $\bar{d}_j$ in (17) with $A(\alpha) + HF(t)E_a(\alpha)$, $B + HF(t)E_b$ and $d_j + HF(t)E_{dj}$, respectively, allows us to write $\bar{G}_j(\alpha)$ as

$$\begin{aligned} \bar{G}_j(\alpha) = G_j(\alpha) &+ \begin{bmatrix} P(\alpha)H \\ 0 \\ \lambda_j c_j^{\mathrm{T}} H \end{bmatrix} F(t)[E_a(\alpha) \ E_b \ E_{dj}] \\ &+ \begin{bmatrix} E_a^{\mathrm{T}}(\alpha) \\ E_b^{\mathrm{T}} \\ E_{dj}^{\mathrm{T}} \end{bmatrix} F^{\mathrm{T}}(t)[H^{\mathrm{T}} P(\alpha) \ 0 \ \lambda_j H^{\mathrm{T}} c_j], \ \ j = 1, 2, \ldots, m, \end{aligned} \tag{18}$$

where $G_j(\alpha)$ is defined in (11). By Lemma 4 and the Schur complement, $\bar{G}_j(\alpha) < 0$ if and only if LMIs (15) are true. □

4. Delay-dependent conditions

Since the criteria given in the previous section do not include any information on the delay, they are delay-independent criteria. Even though $\mathscr{S}_1$ is robustly absolutely stable for $\tau = 0$, it is not robustly absolutely stable for all $\tau > 0$. Continuity shows that $\mathscr{S}_1$ is robustly absolutely stable only for a small τ. So, the delay-independent criteria that guarantee the stability of $\mathscr{S}_1$ for any $\tau > 0$ turn out to be very conservative. Criteria that include information on the delay (i.e., delay-dependent criteria) have been widely investigated as a way of overcoming the conservatism. In particular, many papers have been devoted to the study of delay-dependent criteria for linear systems. In this study, we extended the method presented in [11] to a delay Lur'e control system with multiple nonlinearities and obtained the following theorem. It takes the relationships between the terms of the Leibniz–Newton formula into account for a delay Lur'e control system, thus enabling new delay-dependent criteria that ensure robust absolute stability to be derived.

Theorem 7. *For a given scalar $\tau > 0$, system $\mathscr{S}_0$ is absolutely stable if there exist $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$, $Z = Z^{\mathrm{T}} \geqslant 0$, $X = X^{\mathrm{T}} = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^{\mathrm{T}} & X_{22} & X_{23} \\ X_{13}^{\mathrm{T}} & X_{23}^{\mathrm{T}} & X_{33} \end{bmatrix} \geqslant 0$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \geqslant 0$, $T = \mathrm{diag}(t_1, t_2, \ldots, t_m) \geqslant 0$ and any matrices N_i $(i = 1, 2, 3)$ such that the following LMIs* (19) *and* (20) *hold.*

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} + CKT & \tau A^{\mathrm{T}} Z \\ \Gamma_{12}^{\mathrm{T}} & \Gamma_{22} & \Gamma_{23} & \tau B^{\mathrm{T}} Z \\ \Gamma_{13}^{\mathrm{T}} + TKC^{\mathrm{T}} & \Gamma_{23}^{\mathrm{T}} & \Gamma_{33} - 2T & \tau D^{\mathrm{T}} Z \\ \tau Z A & \tau Z B & \tau Z D & -\tau Z \end{bmatrix} < 0, \tag{19}$$

$$\Pi = \begin{bmatrix} X_{11} & X_{12} & X_{13} & N_1 \\ X_{12}^{\mathrm{T}} & X_{22} & X_{23} & N_2 \\ X_{13}^{\mathrm{T}} & X_{23}^{\mathrm{T}} & X_{33} & N_3 \\ N_1^{\mathrm{T}} & N_2^{\mathrm{T}} & N_3^{\mathrm{T}} & Z \end{bmatrix} \geqslant 0, \tag{20}$$

where

$$\begin{aligned}
\Gamma_{11} &= A^{\mathrm{T}} P + PA + Q + N_1 + N_1^{\mathrm{T}} + \tau X_{11}, \\
\Gamma_{12} &= PB + N_2^{\mathrm{T}} - N_1 + \tau X_{12}, \\
\Gamma_{13} &= PD + A^{\mathrm{T}} C \Lambda + N_3^{\mathrm{T}} + \tau X_{13}, \\
\Gamma_{22} &= -Q - N_2 - N_2^{\mathrm{T}} + \tau X_{22}, \\
\Gamma_{23} &= B^{\mathrm{T}} C \Lambda - N_3^{\mathrm{T}} + \tau X_{23}, \\
\Gamma_{33} &= \Lambda C^{\mathrm{T}} D + D^{\mathrm{T}} C \Lambda + \tau X_{33}.
\end{aligned}$$

Proof. Choose a Lyapunov functional candidate to be

$$V_d(x_t) = V(x_t) + \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s) Z \dot{x}(s)\, \mathrm{d}s\, \mathrm{d}\theta, \tag{21}$$

where $V(x_t)$ is defined in (6) and $Z = Z^{\mathrm{T}} \geqslant 0$ needs to be determined.

Using the Leibniz–Newton formula yields

$$x(t) - x(t-\tau) - \int_{t-\tau}^{t} \dot{x}(s)\, \mathrm{d}s = 0. \tag{22}$$

Then, for any constant matrices N_i $(i = 1, 2, 3)$ with appropriate dimensions, the following is true.

$$2[x^{\mathrm{T}}(t) N_1 + x^{\mathrm{T}}(t-\tau) N_2 + f^{\mathrm{T}}(\sigma(t)) N_3] \left[x(t) - x(t-\tau) - \int_{t-\tau}^{t} \dot{x}(s)\, \mathrm{d}s \right] = 0. \tag{23}$$

On the other hand, for any constant matrix X with appropriate dimensions, the following is also true.

$$\begin{bmatrix} x(t) \\ x(t-\tau) \\ f(\sigma(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \tau(X_{11} - X_{11}) & \tau(X_{12} - X_{12}) & \tau(X_{13} - X_{13}) \\ \tau(X_{12} - X_{12})^{\mathrm{T}} & \tau(X_{22} - X_{22}) & \tau(X_{23} - X_{23}) \\ \tau(X_{13} - X_{13})^{\mathrm{T}} & \tau(X_{23} - X_{23})^{\mathrm{T}} & \tau(X_{33} - X_{33}) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \\ f(\sigma(t)) \end{bmatrix} = 0. \tag{24}$$

Calculating the derivative of $V_d(x_t)$ along the solutions of $\mathscr{S}_0$ and adding the terms on the left sides of (23) and (24) to it yields

$$\dot{V}_d(x_t)|_{\mathscr{S}_0} = \xi^{\mathrm{T}}(t)\Gamma\xi(t) - \int_{t-\tau}^{t} \zeta^{\mathrm{T}}(t,s)\Pi\zeta(t,s)\,\mathrm{d}s, \tag{25}$$

where

$$\xi(t) = [x^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t-\tau)\ f^{\mathrm{T}}(\sigma)]^{\mathrm{T}}, \zeta(t,s) = [\xi^{\mathrm{T}}(t)\ \dot{x}^{\mathrm{T}}(s)]^{\mathrm{T}},$$
$$\Gamma = \begin{bmatrix} \Gamma_{11} + \tau A^{\mathrm{T}}ZA & \Gamma_{12} + \tau A^{\mathrm{T}}ZB & \Gamma_{13} + \tau A^{\mathrm{T}}ZD \\ \Gamma_{12}^{\mathrm{T}} + \tau B^{\mathrm{T}}ZA & \Gamma_{22} + \tau B^{\mathrm{T}}ZB & \Gamma_{23} + \tau B^{\mathrm{T}}ZD \\ \Gamma_{13}^{\mathrm{T}} + \tau D^{\mathrm{T}}ZA & \Gamma_{23}^{\mathrm{T}} + \tau D^{\mathrm{T}}ZB & \Gamma_{33} + \tau D^{\mathrm{T}}ZD \end{bmatrix},$$

Γ_{ij} $(i, j = 1, 2, 3; i \leqslant j \leqslant 3)$ are defined in (19) and Π is defined in (20).

In addition, the conditions (2) are equivalent to

$$f_j(\sigma_j(t))(f_j(\sigma_j(t)) - k_j c_j^{\mathrm{T}} x(t)) \leqslant 0, \quad j = 1, 2, \ldots, m \tag{26}$$

and it is easy to show that

$$\{\xi(t) | (x(t), x(t-\tau))) \neq 0 \quad \text{and (2)}\} = \{\xi(t) | \xi(t) \neq 0 \quad \text{and (2)}\}. \tag{27}$$

Now, using (26) and (27) and applying the S-procedure shows that, if there exists $T = \mathrm{diag}(t_1, t_2, \ldots, t_m) \geqslant 0$ such that

$$\begin{aligned} &\xi^{\mathrm{T}}(t)\Gamma\xi(t) - \int_{t-\tau}^{t} \zeta^{\mathrm{T}}(t,s)\Pi\zeta(t,s)\,\mathrm{d}s \\ &- 2\sum_{j=1}^{m} t_j f_j(\sigma_j(t))(f_j(\sigma_j(t)) - k_j c_j^{\mathrm{T}} x(t)) < 0, \end{aligned} \tag{28}$$

then $\dot{V}_d(x_t)|_{\mathscr{S}_0} < 0$ for $\xi(t) \neq 0$ and $(x(t), x(t-\tau)) \neq 0$ under the condition (2). Thus, $\mathscr{S}_0$ is absolutely stable. Equation (28) gives (19) and (20). □

From Theorem 7, a stability criterion for a system with time-varying structured uncertainties is easily obtained by using Lemma 4.

Theorem 8. *For a given scalar* $\tau > 0$, $\mathscr{S}_1$ *is robustly absolutely stable if there exist* $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$, $Z = Z^{\mathrm{T}} \geqslant 0$, $X = X^{\mathrm{T}} = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^{\mathrm{T}} & X_{22} & X_{23} \\ X_{13}^{\mathrm{T}} & X_{23}^{\mathrm{T}} & X_{33} \end{bmatrix} \geqslant 0$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \geqslant 0$, $T = \mathrm{diag}(t_1, t_2, \ldots, t_m) \geqslant 0$, *any matrices* N_i $(i = 1, 2, 3)$ *and a scalar* $\varepsilon > 0$ *such that the following LMI and* (20) *hold.*

$$\begin{bmatrix} \Gamma_{11} + \varepsilon E_a^{\mathrm{T}} E_a & \Gamma_{12} + \varepsilon E_a^{\mathrm{T}} E_b & \tilde{\Gamma}_{13} & \tau A^{\mathrm{T}} Z & PH \\ \Gamma_{12}^{\mathrm{T}} + \varepsilon E_b^{\mathrm{T}} E_a & \Gamma_{22} + \varepsilon E_b^{\mathrm{T}} E_b & \tilde{\Gamma}_{23} & \tau B^{\mathrm{T}} Z & 0 \\ \tilde{\Gamma}_{13}^{\mathrm{T}} & \tilde{\Gamma}_{23}^{\mathrm{T}} & \tilde{\Gamma}_{33} & \tau D^{\mathrm{T}} Z & \Lambda C^{\mathrm{T}} H \\ \tau ZA & \tau ZB & \tau ZD & -\tau Z & \tau ZH \\ H^{\mathrm{T}} P & 0 & H^{\mathrm{T}} C \Lambda & \tau H^{\mathrm{T}} Z & -\varepsilon I \end{bmatrix} < 0, \tag{29}$$

where

$$\tilde{\Gamma}_{13} = \Gamma_{13} + CKT + \varepsilon E_a^{\mathrm{T}} E_d,$$
$$\tilde{\Gamma}_{23} = \Gamma_{23} + \varepsilon E_b^{\mathrm{T}} E_d,$$
$$\tilde{\Gamma}_{33} = \Gamma_{33} - 2T + \varepsilon E_d^{\mathrm{T}} E_d,$$

and Γ_{ij} $(i, j = 1, 2, 3; i \leqslant j \leqslant 3)$ *are defined in* (19).

5. A numerical example

Example 9. Consider system $\mathscr{S}_1$ with

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -0.5 & -0.1 \\ 0.1 & -0.5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad C = I$$

and $\Delta A(t)$, $\Delta B(t)$ and $\Delta D(t)$ being

$$\|\Delta A(t)\| \leqslant 0.2, \quad \|\Delta B(t)\| \leqslant 0.05, \quad \|\Delta D(t)\| \leqslant 0.05.$$

This can be transformed into (3) and (4) with

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_a = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad E_b = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad E_d = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}.$$

Since $m = 2$, assuming $k_1 = 1$ and $k_2 = 2.23$ gives

$$D_1^2 = \{\mathrm{diag}(0, 0)\}, \quad D_2^2 = \{\mathrm{diag}(0, 0), \mathrm{diag}(k_1, 0)\}.$$

Solving LMI (19) yields

$$P = \begin{bmatrix} 16.4678 & -9.3711 \\ -9.3711 & 29.0463 \end{bmatrix}, \quad Q = \begin{bmatrix} 9.2568 & -6.1537 \\ -6.1537 & 27.4173 \end{bmatrix},$$
$$\lambda_1 = 0.3889, \quad \lambda_2 = 28.5416.$$

Thus, $\mathscr{S}_1$ is robustly absolutely stable.

However, LMI (12) in Theorem 5 is not true when $k_1 = 1$ and $k_2 = 2.09$. This means that it is conservative to directly apply the S-procedure to check the stability of an uncertain system with multiple nonlinearities. In contrast, the LMIs (15) in Theorem 6 do hold. While Theorem 5 only provides a sufficient condition for the stability of $\mathscr{S}_1$, Theorem 6 provides a necessary and sufficient condition. This example clearly illustrates that Theorem 6 is a big improvement over Theorem 5.

For $k_1 = 1$ and $k_2 = 3$, LMIs (15) in Theorem 6 are not true. That means that no Lyapunov functional in the extended Lur'e form that guarantees the delay-independent robust absolute stability of $\mathscr{S}_1$ can be found. In contrast, Theorem 8 shows that $\mathscr{S}_1$ is robustly absolutely stable for $\tau \leqslant 1.5789$. This demonstrates that the delay-dependent criterion, Theorem 8, is less conservative than the delay-independent criterion, Theorem 6.

6. Conclusion

This paper has presented some sufficient, and necessary and sufficient conditions for the existence of a Lyapunov functional in the extended Lur'e form with a negative definite derivative that guarantees the delay-independent absolute, or robust absolute, stability of delay Lur'e control systems with multiple nonlinearities. The existence problem has been converted to the simple problem of solving a set of LMIs. In order to overcome the conservatism, some delay-dependent criteria have been derived for absolute, or robust absolute, stability. A numerical example demonstrated that the delay-dependent criteria thus obtained are less conservative than the delay-independent criteria.

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