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On extreme zeros of classical orthogonal polynomials

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Abstract

Let x_1 and x_k be the least and the largest zeros of the Laguerre or Jacobi polynomial of degree k. We shall establish sharp inequalities of the form $x_1 < A$, $x_k > B$, which are uniform in all the parameters involved. Together with inequalities in the opposite direction, recently obtained by the author, this locates the extreme zeros of classical orthogonal polynomials with a high precision.

MSC: 33C45

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1. Introduction

Study of extreme zeros of the Hermite, Laguerre and Jacobi polynomials has a long history and most of the classical results are collected in [16]. But only recently attention has been shifted to the case when the parameters may vary with the degree k of a polynomial [2–4,7,10,13,15]. Most of these results are of the asymptotic nature (with [7] and [13] being a remarkable exception) and hold under certain restrictions on the parameters. Recently the author obtained the following explicit uniform bounds [11] (similar inequalities for the Laguerre case were given earlier in [10]).

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Theorem 1. *Let x_1 and x_k be the least and the largest zero of the Laguerre polynomial $L_k^{(\alpha)}(x)$, respectively, $\alpha > -1$. Then*

$$x_1 > V^2 + 3V^{4/3}(U^2 - V^2)^{-1/3}, \tag{1}$$

$$x_k < U^2 - 3U^{4/3}(U^2 - V^2)^{-1/3} + 2, \tag{2}$$

where $V = \sqrt{k+\alpha+1} - \sqrt{k}$, $U = \sqrt{k+\alpha+1} + \sqrt{k}$.

Theorem 2. *Let x_1 and x_k be the least and the largest zero of the Jacobi polynomial $P_k^{(\alpha,\beta)}(x)$, respectively, $\alpha \geqslant \beta > -1$. Then*

$$x_1 > A + 3(1-A^2)^{2/3}(2R)^{-1/3}, \tag{3}$$

$$x_k < B - 3(1-B^2)^{2/3}(2R)^{-1/3} + \frac{4q(s+1)}{(r^2+2s+1)^{3/2}}, \tag{4}$$

where

$$s = \alpha + \beta + 1, \quad q = \alpha - \beta, \quad r = 2k + \alpha + \beta + 1, \quad R = \sqrt{(r^2 - q^2 + 2s + 1)(r^2 - s^2)},$$

and

$$A = -\frac{R + q(s+1)}{r^2 + 2s + 1}, \quad B = \frac{R - q(s+1)}{r^2 + 2s + 1}.$$

As the zeros of the Hermite polynomials can be easily expressed through the zeros of the corresponding Laguerre polynomials we will not consider them in this paper.

Previously known results give, roughly speaking, $V^2 < x_1 < x_k < U^2$, for Laguerre polynomials [5,7,16], and $A < x_1 < x_k < B$, for the Jacobi case [7,13]. It is also known that these bounds are asymptotically correct under certain assumptions on the parameters. On the other hand one can expect that much sharper results similar to these of Theorems 1 and 2 hold in a more general situation. In particular, inequalities analogous to (1)–(4) are known for the zeros of Charlier [9] and binary Krawtchouk polynomials [8].

The aim of this paper is to show that the bounds given by Theorems 1 and 2 are essentially sharp, thus locating the extreme zeros of the classical orthogonal polynomials with a high precision. Namely we shall establish (in a rather elementary way) two following theorems giving similar inequalities in the opposite direction. Our method is based on so-called Bethe ansatz equations, having some important applications to orthogonal polynomials [6,12]. It is also worth noticing that the above simple bounds $V^2 < x_1 < x_k < U^2$, and $A < x_1 < x_k < B$, for the Laguerre and Jacobi polynomials, respectively, are an immediate corollary of the Bethe ansatz equation we use here (see Lemma 1 below).

Theorem 3. *Let $\delta = 1/k + 1/(\alpha+1) < 1/50$, then in the notation of Theorem 1,*

$$x_1 < V^2 + \frac{9V^{4/3}}{(U^2 - V^2)^{1/3}(2 - 27\delta^{2/3})}. \tag{5}$$

Let $k \geqslant 30$, then

$$x_k > U^2 - \frac{9U^{4/3}}{2(U^2 - V^2)^{1/3}} \tag{6}$$

provided $\alpha \leqslant 2(3+2\sqrt{3})k-1$, *and*

$$x_k > U^2 - \frac{9U^{4/3}}{(U^2-V^2)^{1/3}(2-3k^{-2/3})}, \tag{7}$$

otherwise.

Theorem 4. *Let* $\alpha \geqslant \beta > -1$, *then in the notation of Theorem* 2, *for* $k \geqslant 5$,

$$x_1 < A + 9(1-A^2)^{2/3}(2R)^{-1/3}, \tag{8}$$

and for $k \geqslant 56$,

$$x_k > B - 9(1-B^2)^{2/3}(2R)^{-1/3}. \tag{9}$$

It seems that the bounds in this direction received much less attention. Yet there are some rather weak classical inequalities which will be used here ([16, Sections 6.2, 6.31]).

Theorems 1–4 yield the asymptotics for the extreme zeros given in the next theorem (in the Jacobi case x_k and B may vanish what leads to more complicated expressions). The meaning of O-terms here is that for sufficiently large k, say $k > 100$, one can replace them by absolute constants.

Theorem 5. (i) *In the notation of Theorem* 1, *for sufficiently large k and* $\alpha > 50$, *the extreme zeros of the Laguerre polynomial* $L_k^{(\alpha)}(x)$ satisfy

$$\frac{x_1}{V^2} = 1 + O\left((\alpha+1)^{-1/2}\left(\frac{1}{\alpha+1}+\frac{1}{k}\right)^{1/6}\right), \tag{10}$$

$$\frac{x_k}{U^2} = 1 - O(k^{-1/6}(k+\alpha)^{-1/2}). \tag{11}$$

(ii) *In the notation of Theorem* 2, *for sufficiently large k and* $\alpha \geqslant \beta > -1$, *the extreme zeros of the Jacobi polynomial* $P_k^{(\alpha,\beta)}(x)$ *satisfy*

$$\frac{x_1}{A} = 1 + O\left(\left(\frac{(\beta+1)^2}{k(k+\alpha)(k+\beta)}\right)^{2/3}\right); \quad r^2 \geqslant q^2 + r^2, \tag{12}$$

$$\frac{x_1}{A} = 1 + O\left(\frac{(\beta+1)^{4/3}}{k^{2/3}(k+\beta)^{5/6}\sqrt{k+\alpha}}\right); \quad r^2 < q^2 + s^2, \tag{13}$$

Let $r^2 = q^2 + s^2 + \gamma(s+1)^{2/3}(r^2-s^2)^{1/3}$, *then*

$$\frac{x_k}{B} = 1 - O(\gamma^{-1} + \gamma^{-2/3}k^{-2/9}), \quad \gamma > 0; \tag{14}$$

$$\frac{x_k}{B} = 1 - O((\alpha k)^{-1/3}), \quad \gamma < -\frac{3(s+1)^{4/3}}{4(r^2-s^2)^{1/3}}; \tag{15}$$

$$\frac{x_k}{B} = 1 - O(|\gamma|^{-1} + |\gamma|^{-1/2} k^{-1/3}), \quad -\frac{3(s+1)^{4/3}}{4(r^2 - s^2)^{1/3}} \leqslant \gamma < 0; \tag{16}$$

$$|x_k| = O\left(\frac{1}{k^{1/6}\sqrt{k+\alpha}}\right), \quad |\gamma| \leqslant 1. \tag{17}$$

It is worth to compare the obtained inequalities with the classical results for the fixed values of the parameters. In particular, in the Laguerre case one has ([16, Theorem 6.32], see also [14] for a far-reaching generalization)

$$x_k < \left(\sqrt{4k + 2\alpha + 2} - 6^{-1/3}(4k + 2\alpha + 2)^{-1/6} i_{11}\right)^2,$$

where $6^{-1/3} i_{11} = 1.85575\ldots$, and i_{11} stands for the least positive zero of the Airy function. One can check that for a fixed α this differs from (2) only by the better factor $c = 2 \cdot 6^{-1/3} i_{11}$, instead of 3, before the second terms of (2). It is tempting to conjecture that asymptotically for $k \to \infty$, and uniformly in all the parameters involved, one should get the same constant c instead of 3 before the second terms in all the expressions (1)–(4).

The paper is organized as follows. In the next section we establish rather general inequalities being our main tool in the sequel. In Sections 3 and 4 we will prove Theorems 3 and 4, dealing with Laguerre and Jacobi polynomials, respectively. Section 4 also contains a proof of Theorem 5.

2. Bethe ansatz inequalities

In this section we will consider real polynomials $f = f(x)$ with only real simple zeros $x_1 < x_2 < \cdots < x_k$, satisfying a differential equation

$$f'' - 2af' + bf = 0. \tag{18}$$

We suppose here that $a = a(x)$ and $b = b(x)$ are meromorphic functions and none of x_i coincides with the singularities of a or b. For such an f we define the discriminant $\Delta(x) = b(x) - a^2(x)$, and consider the second negative moments of f at its zeros

$$S(f, x_i) = \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}.$$

Lemma 1.

$$S(f, x_i) = \sum_{j \neq i} \frac{1}{(x_i - x_j)^2} = \frac{\Delta(x_i) - 2a'(x_i)}{3}. \tag{19}$$

Proof. Using the logarithmic derivative and (18) we get

$$\sum \frac{1}{(x - x_j)^2} = -\left(\frac{f'}{f}\right)' = \frac{f'^2 - ff''}{f^2} = \frac{f'^2 - 2af'f + bf^2}{f^2}. \tag{20}$$

Thus

$$S(f, x_i) = \lim_{x \to x_i} \left(\frac{f'^2 - 2af'f + bf^2}{f^2} - \frac{1}{(x - x_i)^2} \right).$$

The result follows on applying four times L'Hôpital's rule and substituting f'' from (18) at each step. □

Remark 1. Results of this type are called Bethe ansatz equations and are known (or can be routinely established) in a more general situation and weaker smoothness assumptions. We refer to [1,6,12] and the references therein for a more detailed discussion.

Lemma 2.

$$D(f, x_i, x) = 1 + (x - x_i)^2 \left(\frac{\Delta(x_i) - 2a'(x_i)}{3} - \Delta(x) \right) > 0, \tag{21}$$

provided $x \notin [x_1, x_k]$. *In particular, if* $a'(x_i) \geqslant 0$, *then*

$$3 - 2(x - x_i)^2 \Delta(x_i) + 3(x - x_i)^2 (\Delta(x_i) - \Delta(x)) > 0. \tag{22}$$

Proof. From (20) we have

$$\frac{1}{(x - x_i)^2} + \sum_{j \neq i} \frac{1}{(x - x_j)^2} = \left(\frac{f'(x)}{f(x)} - a(x) \right)^2 + b(x) - a^2(x) \geqslant \Delta(x).$$

Since

$$\sum_{j \neq i} \frac{1}{(x - x_j)^2} < \sum_{j \neq i} \frac{1}{(x_i - x_j)^2} = S(f, x_i),$$

for $x \notin [x_1, x_k]$, we obtain

$$\frac{1}{(x - x_i)^2} + S(f, x_i) > \Delta(x),$$

and (21), (22) follow by Lemma 1. □

Remark 2. Similar arguments can be apply to $x \in [x_1, x_k]$, say $x_i < x < x_{i+1}$, giving an upper bound on $x_{i+1} - x_i$. Indeed,

$$\begin{aligned} \Delta(x) &\leqslant \frac{1}{(x - x_i)^2} + \frac{1}{(x - x_{i+1})^2} + \sum_{j<i} \frac{1}{(x - x_j)^2} + \sum_{j>i+1} \frac{1}{(x - x_j)^2} \\ &< \frac{1}{(x - x_i)^2} + \frac{1}{(x - x_{i+1})^2} + \sum_{j<i} \frac{1}{(x_j - x_i)^2} + \sum_{j>i+1} \frac{1}{(x_j - x_{i+1})^2} \\ &< \frac{1}{(x - x_i)^2} + \frac{1}{(x - x_{i+1})^2} - \frac{2}{(x_{i+1} - x_i)^2} + S_2(f, x_i) + S_2(f, x_{i+1}). \end{aligned}$$

By substituting here $x=(x_i+x_{i+1})/2$, one obtains

$$(x_{i+1}-x_i)^2 < \frac{18}{3\Delta((x_i+x_{i+1})/2)-\Delta(x_i)-\Delta(x_{i+1})+2a'(x_i)+2a'(x_{i+1})},$$

provided the denominator is positive.

We will solve inequality (21) for the Laguerre and Jacobi polynomials in the next section. This will require rather involved calculations but the following simple heuristic arguments show what type of bounds may be expected.

Suppose that $\Delta(x)$ has only two real zeros $y_1<y_2$. Neglecting the term $2a'(x)$, we obtain that all the zeros of f are in the interval (y_1, y_2). Let x_k be, say, the largest zero of f, we put $x_k=y_2-\varepsilon$, and choose $x=y_2-5\varepsilon/9$. Now, on omitting higher derivatives of Δ, that is putting $\Delta(y_2-\delta)\approx \Delta(y_2)-\delta\Delta'(y_2)=-\delta\Delta'(y_2)$, (21) can be rewritten as

$$0<1+\frac{16\varepsilon^2}{81}\left(\frac{\Delta(x_k)}{3}-\Delta(x)\right)\approx 1+\frac{32\varepsilon^3\Delta'(y_2)}{729}.$$

Thus we obtain $x_k>y_2+\frac{9}{2}(4\Delta'(y_2))^{-1/3}$. Notice that similar heuristic considerations given in [11] yield in the opposite direction $x_k<y_2+3(4\Delta'(y_2))^{-1/3}$, ($\Delta'(y_2)$ is negative as $\Delta(y_2)=0$).

3. Laguerre polynomials

The Laguerre polynomials $L_k^{(\alpha)}(x)$ are polynomials orthogonal on $[0,\infty)$ for $\alpha>-1$, with respect to the weight function $x^\alpha e^{-x}$. The corresponding ODE is

$$u''-(1-(\alpha+1)x^{-1})u'+kx^{-1}u=0, \quad u=L_k^{(\alpha)}(x).$$

We also need the explicit representation

$$L_k^{(\alpha)}(x)=\sum_{i-0}^{k}\binom{k+\alpha}{k-i}\frac{(-x)^i}{i!}. \tag{23}$$

Using the notation of Theorem 1 we get $k=(U-V)^2/4$, $\alpha=VU-1$, and the condition $\alpha>-1$, means $V>0$.

We have $a(x)=(x-VU)/2x$, $a'(x)=VU/2x^2>0$, and also

$$\Delta(x)=\frac{(U^2-x)(x-V^2)}{4x^2}, \tag{24}$$

Let x_1 and x_k be the least and the largest zeros of $L_k^{(\alpha)}(x)$, respectively. We need the following (rather weak) bound on x_1. ([16, Section 6.31]).

$$x_1\leqslant\frac{(\alpha+1)(\alpha+3)}{2k+\alpha+1}=\frac{2VU(VU+2)}{V^2+U^2}. \tag{25}$$

By (23) we have $\sum_{i=0}^{k} x_i = k(k+\alpha)$, implying $x_1 < k+\alpha = (U+V)^2/4 < x_k$. Moreover, as $0 < S(L_k^{(\alpha)}, x_i) < \Delta(x_i)$, we get that all the zeros satisfy $V^2 < x_i < U^2$, hence

$$V^2 < x_1 < \frac{(\alpha+1)(\alpha+3)}{2k+\alpha+1} < x_k < U^2. \tag{26}$$

Lemma 3. *For* $V^2 < x < x_1$,

$$\Delta(x_1) - \Delta(x) < \frac{U^2 - V^2}{4V^4}(x_1 - x). \tag{27}$$

For $x_k < x < U^2$,

$$\Delta(x_k) - \Delta(x) < \frac{U^2 - V^2}{4x_k^2}(x - x_k). \tag{28}$$

Proof. Using that $((V^2+U^2)xy - V^2U^2(x+y))/xy$ is an increasing function in x and y we obtain

$$\frac{\Delta(x_1) - \Delta(x)}{x_1 - x} = \frac{V^2U^2(x+x_1) - (V^2+U^2)xx_1}{4x^2x_1^2} < \frac{U^2 - V^2}{4xx_1} < \frac{U^2 - V^2}{4V^4};$$

$$\frac{\Delta(x_k) - \Delta(x)}{x - x_k} = \frac{(V^2+U^2)xx_k - V^2U^2(x+x_k)}{4x^2x_k^2} < \frac{U^2 - V^2}{4xx_k} < \frac{U^2 - V^2}{4x_k^2}.$$

and the result follows. □

Proof of Theorem 3. (i) We choose $x = x_1 - \varepsilon$, where $\varepsilon = 2V^{4/3}/(U^2 - V^2)^{1/3}$. Then (22) and (3) give

$$0 < 3 - \frac{\varepsilon^2(U^2 - x_1)(x_1 - V^2)}{2x_1^2} + \frac{3\varepsilon^3(U^2 - V^2)}{4V^4} = 9 - \frac{\varepsilon^2(U^2 - x_1)(x_1 - V^2)}{2x_1^2} := F(x_1).$$

We claim that under our assumptions $F(x)$ has two zeros $y_1 < y_2$, and $x_1 < y_1$. As $x_1 < x_0 = ((\alpha+1)(\alpha+3))/(2k+\alpha+1)$, it is enough to show that $F(x_0) < 0$. Putting $b = \alpha + 1$, we have

$$F(x_0) = 9 + \frac{2\varepsilon^2}{(b+2)^2} + \frac{8\varepsilon^2 k(k+b)}{b^2(b+2)^2} - \frac{2\varepsilon^2 k(k+b)}{b^2}.$$

Here

$$\frac{2\varepsilon^2}{(b+2)^2} + \frac{8\varepsilon^2 k(k+b)}{b^2(b+2)^2} < \frac{8\varepsilon^2(k+b)^2}{b^4} < \frac{16}{b}\delta^{1/3} < 16\delta^{4/3},$$

and

$$\frac{2\varepsilon^2 k(k+b)}{b^2} = \left(\frac{16bk(b+k)}{U^4}\right)^{2/3} > \left(\frac{bk}{k+b}\right)^{2/3} = \delta^{-2/3}.$$

Now it is left to check that $9 + 16\delta^{4/3} - \delta^{-2/3} < 0$, for $\delta < \frac{1}{50}$, proving the claim.

For y_1 we get

$$y_1 = V^2 + \frac{9V^2}{h\left(1+\sqrt{1-18V^2U^2h^{-4}}\right)-9},$$

where $h = V^{2/3}(U^2-V^2)^{1/3}$, and

$$\begin{aligned} h\left(1+\sqrt{1-18V^2U^2h^{-4}}\right)-9 &> 2h - \frac{18V^2U^2}{h^3} - 9 < 2h - \frac{27U^2}{U^2-V^2} \\ &= 2h\left(1-\frac{27U^{8/3}}{2b^{2/3}(U^2-V^2)^{4/3}}\right) < h(2-27\delta^{2/3}). \end{aligned}$$

As $2-27\cdot 50^{-2/3} > 0$, the result follows.

(ii) We choose $x = x_k - \varepsilon$, where $\varepsilon = 2U^{4/3}/(U^2-V^2)^{1/3}$. By (22) and (3) we have

$$\begin{aligned} 0 &< 3 - \frac{\varepsilon^2(U^2-x_k)(x_k-V^2)}{2x_k^2} + \frac{3\varepsilon^3(U^2-V^2)}{4x_k^4} \\ &= 3 + \frac{6U^4}{x_k^2} - \frac{\varepsilon^2(U^2-x_k)(x_k-V^2)}{2x_k^2} \\ &< \frac{9U^4}{x_k^2} - \frac{\varepsilon^2(U^2-x_k)(x_k-V^2)}{2x_k^2}. \end{aligned}$$

Thus

$$F(x_k) := 18x_k^2 - \varepsilon^2(U^2-x_k)(x_k-V^2) > 0.$$

The equation $F(x)=0$, has two zeros, $y_1 < y_2$, and $x_k > y_2$. Indeed, as $x_k > x_0 = (V+U)^2/4$, it is enough to check $F(x_0) < 0$. We have

$$\begin{aligned} 4F(x_0) &= 72U^4 - \frac{\varepsilon^2(U-V)^2(3U^2+10VU+3V^2)}{4} \leqslant 72U^4 - 3U^{8/3}(U^2-V^2)^{4/3} \\ &\leqslant 3U^{8/3}(U+V)^{4/3}(24-(U-V)^{4/3}) = 3U^{8/3}(U+V)^{4/3}(24-(4k)^{2/3}) < 0, \end{aligned}$$

for $k \geqslant 30$. Thus,

$$\begin{aligned} x_k > y_2 &= U^2 - \frac{9U^2}{9+U^{2/3}(U^2-V^2)^{1/3}\left(1+\sqrt{1-18V^2U^{-2/3}(V^2+U^2)^{-4/3}}\right)} \\ &> U^2 - \frac{9U^2}{2U^{2/3}(U^2-V^2)^{1/3}+9-18V^2/(U^2-V^2)}. \end{aligned}$$

Finally, $9-18V^2/(U^2-V^2)\geqslant 0$, if $\alpha\leqslant 2\left(3+2\sqrt{3}\right)k-1$, proving (6). Otherwise,

$$2U^{2/3}(U^2-V^2)^{1/3}-\frac{18V^2}{U^2-V^2}=U^{2/3}(U^2-V^2)^{1/3}\left(2-\frac{18V^2}{U^{2/3}(U^2-V^2)^{4/3}}\right)$$
$$>U^{2/3}(U^2-V^2)^{1/3}\left(2-\frac{9b^2}{2^{5/3}k^{2/3}(k+b)^2}\right)$$
$$>U^{2/3}(U^2-V^2)^{1/3}(2-3k^{-2/3}),$$

and (7) follows. □

4. Jacobi polynomials

The Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$ are polynomials orthogonal on $[-1,1]$ for $\alpha,\beta>-1$, with respect to the weight function $(1-x)^\alpha(1+x)^\beta$. The corresponding ODE is

$$u''-\frac{(\alpha+\beta+2)x+\alpha-\beta}{1-x^2}u'+\frac{k(k+\alpha+\beta+1)}{1-x^2}u=0,\quad u=P_k^{(\alpha,\beta)}(x).$$

We will use the notation of Theorem 2 and put $p=r^2+2s+1$ throughout this section.

We have

$$\Delta(x)=-\frac{px^2+2q(s+1)x+s^2+q^2-r^2}{4(1-x^2)^2}=\frac{p(x-A)(B-x)}{4(1-x^2)^2}. \tag{29}$$

As

$$a'(x)=\frac{((\alpha+\beta+2)x+\alpha-\beta)^2+4(\alpha+1)(\beta+1)}{2(\alpha+\beta+2)(1-x^2)^2}>0,$$

we can use (22) and moreover, as $\Delta(x_i)>0$, we obtain

$$A<x_i<B \tag{30}$$

In the opposite direction it is known ([16, Section 6.2])

$$x_1<-\frac{2k+\alpha-\beta-2}{2k+\alpha+\beta}<\frac{2k+\beta-\alpha-2}{2k+\alpha+\beta}<x_k. \tag{31}$$

It is also easy to show that $x_1<0$, for $\alpha\geqslant\beta$, (see e.g. [11]).

Lemma 4. *For $A<x<x_1$,*

$$\Delta(x_1)-\Delta(x)<\frac{R(x_1-x)}{2(1-x_1^2)^2}.$$

For $x_k<x<B$,

$$\Delta(x_k)-\Delta(x)<\frac{R(x-x_k)}{2(1-x_k^2)^2}.$$

Proof. We have

$$\Delta(x_1) - \Delta(x) < \frac{p}{4(1-x_1^2)^2}\,((x_1 - A)(B - x_1) - (x - A)(B - x))$$
$$< \frac{p}{4(1-x_1^2)^2}\,(B - A)(x_1 - x) = \frac{R(x_1 - x)}{2(1-x_1^2)^2}.$$
$$\Delta(x_k) - \Delta(x) < \frac{p}{4(1-x_k^2)^2}((x_k - A)(B - x_k) - (x - A)(B - x))$$
$$< \frac{p}{4(1-x_k^2)^2}\,(B - A)(x - x_k) = \frac{R(x - x_k)}{2(1-x_k^2)^2}. \qquad \square$$

Proof of Theorem 4. (i) Choose $\varepsilon = (2-2A^2)^{2/3}/R^{1/3}$, and put $x = x_1 - \varepsilon$. Then $x > A$, otherwise there is nothing to prove. Using the previous lemma and (22) we obtain

$$0 < 3 - \frac{\varepsilon^2 p(x_1 - A)(B - x_1)}{2(1-x_1^2)^2} + \frac{3\varepsilon^3 R}{2(1-x_1^2)^2}$$
$$< \frac{3(1-A^2)^2}{(1-x_1^2)^2} - \frac{\varepsilon^2 p(x_1 - A)(B - x_1)}{2(1-x_1^2)^2} + \frac{3\varepsilon^3 R}{2(1-x_1^2)^2}.$$

Thus, we get

$$18(1-A^2)^2 - \varepsilon^2 p(B - x_1)(x_1 - A) := F(x_1) > 0. \tag{32}$$

We shall show that this quadratic has two real zeros $z_1 < z_2$, and $x_1 < z_1$. For, it is enough to prove $F((A+B)/2) < 0$, and $x_1 < (A+B)/2$. The last claim follows from (31), as

$$x_1 < -\frac{2k + \alpha - \beta - 2}{2k + \alpha + \beta} < \frac{A+B}{2}.$$

Indeed, $\alpha, \beta > -1$, and we obtain

$$\frac{A+B}{2} - x_1 > \frac{A+B}{2} + \frac{2k + \alpha - \beta - 2}{2k + \alpha + \beta}$$
$$= \frac{4(2k^3 + (3\alpha + \beta + 4)k^2 + (\alpha^2 + \alpha\beta + 4\alpha + 4\beta + 4)k + (\alpha + 1)(\alpha + \beta + 2))}{(r-1)p} > 0.$$

Now we have

$$F\left(\frac{A+B}{2}\right) = 72(1-A^2)^2 - \varepsilon^2 p(B - A)^2,$$

and it is negative whenever

$$2R^4 > 729p^3(1-A^2)^2. \tag{33}$$

As

$$\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{1-A^2}{R^2}\right) = -\frac{2((s+1)^2 - q^2)}{(p-q^2)R(qR + (s+1)(p-q^2))} < 0,$$

and for $q=0$,

$$\frac{1-A^2}{R^2}=\frac{(s+1)^2}{p^2(r^2-s^2)}.$$

We have

$$\frac{p^3(1-A^2)^2}{R^4}<\frac{(s+1)^4}{p(r^2-s^2)^2}<\frac{(s+1)^4}{16k^2(k+s)^2(2k+s)^2}<\frac{1}{16k^2}<\frac{2}{729},$$

provided $k \geqslant 5$. This proves (33) and, thus, $x_1<z_1$.

Finally, solving $F(x)=0$, we obtain

$$x_1<A+\frac{18(1-A^2)^2}{\varepsilon^2 R\left(1+\sqrt{1-(18p(1-A^2)^2)/\varepsilon^2R^2}\right)}<A+\frac{18(1-A^2)^2}{\varepsilon^2 R}=A+\frac{9(1-A^2)^{2/3}}{(2R)^{1/3}}.$$

(ii) Choose $\varepsilon=(2-2B^2)^{2/3}/R^{1/3}$, and put $x=x_k+\varepsilon$. Similar to the previous case we get

$$18(1-B^2)^2-\varepsilon^2 p(B-x_k)(x_k-A):=F(x_k)>0. \tag{34}$$

We shall show that x_k is greater than the largest zero of $F(x)=0$. To prove this we establish $F(x_0)<0$, where $x_0=(2k+\beta-\alpha-2)/(2k+\alpha+\beta)<x_k$, by Lemma 31. For it is enough to show

$$G=\left(\frac{18(1-B^2)^2}{\varepsilon^2 p(B-x_k)(x_k-A)}\right)^3=\frac{729R^2(1-B^2)^2}{2(p(B-x_k)(x_k-A))^3}<1.$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{1-B^2}{R^2}\right)=\frac{2(r^2-s^2)(q(r^2-s^2)+(s+1)R)}{pR^4}>0.$$

As $q=\alpha-\beta<\alpha+\beta+2=s+1$, we obtain

$$\frac{1-B^2}{R^2}<\frac{4(s+1)^2}{p^2(r^2-s^2)},$$

that is

$$1-B^2<\frac{4(s+1)^2R^2}{p^2(r^2-s^2)}.$$

We also have

$$\begin{aligned}p(B-x_0)(x_0-A)&=\frac{16(\alpha+1)((k-1)(\alpha+1)+k(k+\beta)(2k+\alpha+\beta))}{(2k+\alpha+\beta)^2}\\&\geqslant\frac{16k(\alpha+1)(k+\beta)}{2k+\alpha+\beta+1}>\frac{2(r-q-1)(s+q+1)(r-s)}{r}.\end{aligned}$$

Therefore we obtain

$$G<\frac{729r^3(s+1)^4(r+s)(p-q^2)^3}{p^4(r-q-1)^3(r-s)^2(s+q+1)^3}<\frac{729(s+1)(r+s)(p-q^2)^3}{r^5(r-q-1)^3(r-s)^2}.$$

The last expression is an increasing function in q, and substituting $q=s+1$, we get

$$G<\frac{729(s+1)(r-s)(r+s)^4}{r^5(r-s-2)^3}=\frac{2916k(s+1)(k+s)^4}{(k-1)^3(2k+s)^5}<\frac{2916k}{(k-1)^3}<1$$

for $k \geqslant 56$. Finally, solving $F(x_0)=0$, we obtain

$$\begin{aligned}x_k &> B-\frac{18(1-B^2)^2}{\varepsilon^2 R\left(1+\sqrt{1-18p(1-B^2)^2/\varepsilon^2R^2}\right)}\\ &> B-\frac{18(1-B^2)^2}{\varepsilon^2 R}=B-\frac{9(1-B^2)^{2/3}}{(2R)^1/3}. \qquad \square\end{aligned}$$

Remark 3. More accurate calculations show that in fact (9) holds for $k \geqslant 20$, instead of 56. It is also easy to improve the constant 9 in (8), (9) to $9/(2-o(1))$, similar to the Laguerre case.

Proof of Theorem 5. The asymptotics for the Laguerre case is an easy exercise, here we will establish (12)–(17).

Notice that the inequality $r^2 \geqslant s^2+q^2$ is equivalent to $R \geqslant q(s+1)$. We also observe that the last term in (4) may be ignored. Indeed,

$$1-B^2=\frac{(q+s+1)^2(R+p-q(s+1))}{p(R+p+q(s+1))},$$

and this is an increasing function on R. As $q<s+1$, we get $R>r^2-s^2$, what implies

$$\begin{aligned}1-B^2 &> \frac{(q+s+1)^2(2r^2-s^2-q(s+1))}{p(2r^2-s^2+q(s+1))}>\frac{2(\alpha+1)^2(2r^2-s^2-q(s+1))}{p^2}\\ &> \frac{(\alpha+1)^2(k+\alpha)(k+\beta)}{p^2}.\end{aligned}$$

Now calculations yield

$$\left(\frac{q(s+1)R^{1/3}}{p^{3/2}(1-B^2)^{2/3}}\right)^6<c\frac{q^6(s+1)^6k}{(\alpha+1)^8(k+\alpha)^4(k+\beta)^3}$$

for some positive constant c. This expression is a decreasing function in β and for $\beta=-1$, is $O((\alpha+1)^4/k^2(k+\alpha)^4)$. Thus the last term in (4) is negligible whenever $k\to\infty$.

Proof of (12). As $R \geqslant q(s+1)$, we have $|A|>R/2r^2$, and

$$1-A^2<2(1+A)=\frac{2(s+1-q)^2}{R+p-q(s+1)}<\frac{8(\beta+1)^2}{p}<\frac{16(\beta+1)^2}{r^2}.$$

Therefore,

$$\left(\frac{(1-A^2)^{2/3}}{|A|R^{1/3}}\right)^{3/2} < \frac{128(\beta+1)^2 r}{R^2} < \frac{256(\beta+1)^2 r}{(r^2-q^2)(r^2-s^2)} < \frac{32(\beta+1)^2}{k(k+\alpha)(k+\beta)},$$

and (12) follows.

Proof of (13). As $R<q(s+1)$, we get $q^2>r^2-s^2$. This yields $-1<\beta<2k+\alpha-2\sqrt{k(2k+2\alpha+1)}$, $\alpha>2k-1+2\sqrt{k(2k-1)}$, and $k<\alpha/2$. Thus, s is a large positive number and $|A|>qs/r^2$. Now, using $R>r^2-s^2$, we obtain

$$1-A^2<\frac{2(s+1-q)^2}{R+p-q(s+1)}<\frac{8(\beta+1)^2}{2r^2-s^2-q(s+1)}<\frac{4(\beta+1)^2}{\alpha(k+\beta)}.$$

This yields

$$\left(\frac{(1-A^2)^{2/3}}{|A|R^{1/3}}\right)^{6} < \frac{256r^{12}(\beta+1)^8}{\alpha^4(k+\beta)^4q^6s^6(r^2-q^2)(r^2-s^2)} < \frac{10^8(\beta+1)^8}{k^4(k+\alpha)^3(k+\beta)^5},$$

and the result follows.

Proof of (14). The condition $q^2=r^2-s^2-\gamma(s+1)^{2/3}(r^2-s^2)^{1/3}$, $\gamma>0$, implies that $R>q(s+1)$, and $B>0$. Rewriting B as $(r^2-s^2-q^2)/(R+q(s+1))$ we obtain $B>(r^2-s^2-q^2)/2R$. We also have

$$1-B^2<2(1-B)=\frac{8(\alpha+1)^2}{R+p+q(s+1)}<\frac{8(\alpha+1)^2}{r^2}.$$

Hence

$$\begin{aligned}\left(\frac{(1-B^2)^{2/3}}{BR^{1/3}}\right)^{3} &= \frac{512(\alpha+1)^4}{r^4}\left(\gamma^{-3}+\frac{(r^2-s^2)^{1/3}}{\gamma^2(s+1)^{4/3}}\right)\\ &< 512\gamma^{-3}+\frac{900k^{1/3}(\alpha+1)^4(k+\alpha+\beta+1)^{1/3}}{\gamma^2(\alpha+\beta+2)^{4/3}(2k+\alpha+\beta+1)^4}.\end{aligned}$$

The second term here is a decreasing function in $\beta>-1$, and does not exceed

$$\frac{900k^{1/3}(\alpha+1)^{8/3}(k+\alpha)^{1/3}}{\gamma^2(2k+\alpha)^4}<900k^{-2/3}\gamma^{-2},$$

and the result follows.

Proof of (15), (16). In those case $k<\left(\sqrt{2\alpha^2+2\alpha+1}-\alpha\right)/2$, $\alpha>2k-1+2\sqrt{k(2k-1)}$, and so α is large. Therefore,

$$0>r^2-q^2-s^2>r^2-2(s+1)^2>r^2-4s^2,$$

and hence $s<r<2s$. By $q<s+1$, it follows

$$-\gamma<\frac{2s^2+2s+1-r^2}{(s+1)^{2/3}(r^2-s^2)^{1/3}}<\frac{(s+1)^{4/3}}{(r^2-s^2)^{1/3}}.$$

Rewriting B as $(r^2-q^2-s^2)/(R+q(s+1))$, and using $R<q(s+1)$, we have $B>(r^2-q^2-s^2)/2q(s+1)$. Now

$$\begin{aligned}\left(\frac{(1-B^2)^{2/3}}{BR^{1/3}}\right)^6 &\leqslant B^{-6}R^{-2}<\frac{64q^2(s+1)^6}{R^2(r^2-q^2-s^2)^6}\\ &=\frac{64(s+1)^{4/3}((r^2-s^2)^{2/3}-\gamma(s+1)^{2/3})^3}{\gamma^6(r^2-s^2)^2((s+1)^{4/3}+\gamma(r^2-s^2)^{1/3})}\\ &<\frac{-256(s+1)^{10/3}}{\gamma^3(r^2-s^2)^2((s+1)^{4/3}+\gamma(r^2-s^2)^{1/3})}+\frac{256(s+1)^{4/3}}{\gamma^6((s+1)^{4/3}+\gamma(r^2-s^2)^{1/3})}\\ &=I_1+I_2.\end{aligned}$$

Now we shall consider two cases corresponding to the restrictions in (16) and (15). If $-\gamma\leqslant 3(s+1)^{4/3}/4(r^2-s^2)^{1/3}$, that is $q^2<r^2-(s^2-6s-3)/4$, then

$$I_1<\frac{-1024(s+1)^2}{\gamma^3(r^2-s^2)^2}<\frac{-64}{\gamma^3k^2}.$$

Otherwise, using $-\gamma<(2s^2+2s+1-r^2)/(s+1)^{2/3}(r^2-s^2)^{1/3}$, and $k<\alpha/4$, for large α, we get

$$I_1<\frac{256(s+1)^6}{(r^2-s^2)^2(2s^2+2s+1-r^2)}<\frac{128(s+1)^6}{\alpha^6k^2(k+s)^2}=O(\alpha^{-2}k^{-2}).$$

Similarly, $I_2=O(\gamma^{-6})$, if $-\gamma\leqslant 3(s+1)^{4/3}/4(r^2-s^2)^{1/3}$,

$$I_2=O\left(\frac{(r^2-s^2)^2}{(s+1)^8}\right)=O\left(k^2a^{-6}\right)$$

if $\frac{3}{4}<-\gamma(r^2-s^2)^{1/3}/(s+1)^{4/3}\leqslant\frac{6}{7}$, and $I_2=\left(k\alpha^{-5}\right)$, otherwise. These readily yield (15), (16).

Proof of (17). In this case

$$\begin{aligned}|B|&=\left|\frac{r^2-q^2-s^2}{R+q(s+1)}\right|<\frac{|\gamma|(s+1)^{1/3}}{(r^2-s^2)^{1/6}\sqrt{(s+1)^{4/3}+\gamma(r^2-s^2)^{1/3}}}\\ &=O\left(\frac{|\gamma|}{(s+1)^{1/3}(r^2-s^2)^{1/6}}\right)=O\left(\frac{1}{k^{1/6}\sqrt{k+\alpha}}\right),\end{aligned}$$

and $R=s\sqrt{r^2-s^2}(1+o(1))$. Thus,

$$(1-B^2)^{2/3}R^{-1/3}=\frac{(1-o(1))}{s^{1/3}(r^2-s^2)^{1/6}}=O\left(\frac{1}{k^{1/6}\sqrt{k+\alpha}}\right),$$

and (17) follows. $\square$

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