



# The existence of multiple positive solutions for singular functional differential equations with sign-changing nonlinearity<sup>☆</sup>

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## ABSTRACT

In this paper, we study the existence of multiple positive solutions for boundary value problems based on second-order functional differential equations with the form

$$\begin{cases} y''(t) + f(t, y(t - \tau)) = 0, & \forall t \in (0, 1) \setminus \{\tau\}, \\ y(t) = \eta(t), & \forall t \in [-\tau, 0], \\ y(1) = 0 \end{cases}$$

where  $0 < \tau < 1$  and  $f : (0, 1) \times (0, +\infty) \rightarrow (-\infty, +\infty)$  is continuous, may be singular at  $t = 0, 1, y = 0$  and takes negative values. By applying the fixed point index theorem, we obtain the conditions for the existence of at least two and of three positive solutions. An example to illustrate our results is given.

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## 1. Introduction

The purpose of this paper is to establish the conditions for the existence of multiple positive solutions for the following singular second-order functional differential equation with sign-changing nonlinearity:

$$\begin{cases} y''(t) + f(t, y(t - \tau)) = 0, & \forall t \in (0, 1) \setminus \{\tau\}, \\ y(t) = \eta(t), & \forall t \in [-\tau, 0], \\ y(1) = 0 \end{cases} \quad (1.1)$$

where  $0 < \tau < 1$ ,  $\eta(t) \in C([-\tau, 0])$ ,  $\eta(t) > 0$  for  $t \in [-\tau, 0)$ ,  $\eta(0) = 0$ , and  $f : (0, 1) \times \mathbf{D} \rightarrow \mathbf{R}$  is continuous, may be singular at  $t = 0, 1, y = 0$  and takes negative values, where  $\mathbf{D} = C([-\tau, 1], \mathbf{R}_0^+)$ ,  $\mathbf{R}_0^+ = (0, +\infty)$ ,  $\mathbf{R}^+ = [0, +\infty)$ ,  $\mathbf{R} = (-\infty, +\infty)$ .

As pointed out by the authors of [1], the study of second-order functional differential equations is of significance since it arises and has applications in variational problems in control theory and other areas of applied mathematics. In recent years, there has been development of the theory of functional differential equations, and also many authors have paid attention to boundary value problems relating to second-order functional differential equations; for example, see [2–10] and the references therein.

In [4], Jiang and Zhang used a fixed point index theorem for cones to study the existence of at least one positive solution for the boundary value problem (1.1) with  $\eta(t) \equiv 0$ . In [6], Xu investigated the existence of a positive solution for the

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boundary value problem (1.1), in which the nonlinear term  $f$  may be singular at  $t = 0$ ,  $y = 0$ , and  $f(t, y) \geq -M$  ( $M$  is a positive constant). However, the work in [4,6,10] is on the existence of at least one solution for problem (1.1).

Motivated and inspired by [6,9,10], in the present paper we aim to establish some simple criteria for the existence of at least two, three, and  $2n + 1$  ( $n \in \mathbf{N}$ ) positive solutions for the problem (1.1). The main tool used in this paper is the theory of the fixed point index in a cone. Our results of this paper extend and supplement some results from [4,6,10].

In obtaining positive solutions of problem (1.1), we will need the following fixed point theorem for cones in the proofs of our results.

**Lemma A** ([11]). Let  $P$  be a cone in a Banach space  $E$ . Let  $\Omega$  be an open bounded subset of  $E$  with  $\Omega_P = \Omega \cap P \neq \emptyset$  and  $\bar{\Omega}_P \neq P$ . Assume that  $T : \bar{\Omega}_P \rightarrow P$  is a compact map such that  $x \neq Tx$  for  $x \in \partial\Omega_P$ . Then the following results hold:

- (i) If  $T(\bar{\Omega}_P) \subset \Omega_P$  for  $x \in \Omega_P$ , then  $i(T, \Omega_P, P) = 1$ .
- (ii) If there exists a  $\varphi \in P \setminus \{0\}$  such that  $x \neq Tx + \lambda\varphi$  for all  $x \in \partial\Omega_P$  and  $\lambda > 0$ , then  $i(T, \Omega_P, P) = 0$ .

## 2. Main results and proofs

Let  $G(t, s)$  be the Green function of the problem  $-y'' = 0$ ,  $y(0) = y(1) = 0$ ; it is easy to verify that  $G(t, s)$  has the following property:

$$t(1-t)s(1-s) \leq G(t, s) \leq G(s, s) \leq 1. \quad (2.1)$$

In order to abbreviate our discussion, we give the following assumptions:

(C<sub>1</sub>) There exists  $p(t) \in C(0, 1)$ , such that

$$\phi_0(t)h_0(y) \leq f(t, y) + p(t) \leq \phi(t)(g(y) + h(y))$$

for all  $(t, y) \in (0, 1) \times \mathbf{R}_0^+$ , where  $\phi_0, \phi \in C((0, 1), \mathbf{R}_0^+)$ ,  $g \in C(\mathbf{R}_0^+, \mathbf{R}_0^+)$ , and  $g(y)$  is nonincreasing with respect to  $y$ ;  $h_0, h \in (\mathbf{R}^+, \mathbf{R}^+)$  and  $h_0(y), h(y)$  are nondecreasing with respect to  $y$ .

(C<sub>2</sub>) Let  $0 \leq \int_0^1 G(s, s)p(s)ds < \infty$ , and let there exist a  $k_0 > 0$  such that

$$a := \int_{\tau}^1 s(1-s)[\phi(s)g(k_0s(1-s)) + \phi_0(s)]ds < \infty,$$

and

$$b := \int_0^{\tau} s(1-s)[\phi(s)g(\eta(s-\tau)) + \phi_0(s)]ds < \infty.$$

(C<sub>3</sub>) There exists an  $R_0 > 2a_0$  such that

$$\frac{R_0}{\Delta + \int_{\tau}^1 s(1-s)\phi(s) \left[ g\left(\frac{1}{2}R_0s(1-s)\right) + h(R_0 + 1) \right] ds} > 1$$

where  $a_0 := \int_0^1 p(s)ds$ ,  $\Delta := 1 + \int_0^{\tau} s(1-s)\phi(s)[g(\eta(s-\tau)) + h(\eta(s-\tau))]ds$ .

(C<sub>4</sub>) For any  $[\alpha, \beta] \in (\tau, 1)$ , there exist  $a_1, a_2$  with  $a_2 > a_1 > R_0$  ( $R_0$  as in (C<sub>3</sub>)) such that

$$\sigma h_0 \left( \frac{1}{2}a_i \right) \int_{\alpha}^{\beta} s(1-s)\phi_0(s)ds > a_i, \quad (i = 1, 2),$$

where  $\sigma := \min\{\alpha(1-\alpha), \beta(1-\beta)\}$ .

A function  $y \in C[-\tau, 1] \cap C^2((0, 1) \setminus \{\tau\})$ , and with  $y(t) > 0$ ,  $t \in [-\tau, 0) \cup (0, 1)$ , is called a positive solution of problem (1.1) if it satisfies problem (1.1).

Let  $\Gamma = C[-\tau, 1, \mathbf{R})$  be a space with a norm  $\|x\| = \max_{t \in [-\tau, 1]} |x(t)|$  for all  $x \in \Gamma$ , and the set  $P, Q$  be two cones in  $\Gamma$  defined by

$$P = \{x \in \Gamma : x(t) \geq 0, t \in [-\tau, 1]\}, \quad Q = \{x \in P : x(t) \geq t(1-t)\|x\|, t \in [0, 1]\}.$$

Let

$$\begin{aligned} x_0(t) &= \begin{cases} \eta(t), & t \in [-\tau, 0], \\ 0, & t \in (0, 1), \end{cases} \\ w(t) &= \begin{cases} 0, & t \in [-\tau, 0], \\ \int_0^1 G(t, s)p(s)ds, & t \in (0, 1). \end{cases} \end{aligned} \quad (2.2)$$

Setting  $F(t, y) = f(t, y) + p(t)$ , and for any  $x \in P, j \in \mathbf{N}$ , define an operator  $T_j$  as follows:

$$T_j x(t) = \begin{cases} j^{-1}, & t \in [-\tau, 0], \\ j^{-1} + \int_0^1 G(t, s)F(s, [x(s - \tau)]^* + j^{-1})ds, & t \in (0, 1) \end{cases} \quad (2.3)$$

where

$$[x(t - \tau)]^* = \max\{x(t - \tau) + x_0(t - \tau) - w(t - \tau), 0\}.$$

**Lemma 2.1.** For any  $j \in \mathbf{N}, T_j : P \rightarrow Q$  is completely continuous.

**Proof.** For any  $x \in P, j \in \mathbf{N}$ , by virtue of (2.3), we have  $j^{-1} \leq T_j x(s) \leq T_j x(t)$  for  $t \in [-\tau, 1], s \in [-\tau, 0]$ . Thus  $\|T_j x\|_{[-\tau, 1]} = \|T_j x\|_{[0, 1]}$ . For  $t \in (0, 1)$ , we have

$$\begin{aligned} T_j x(t) &= j^{-1} + \int_0^1 G(t, s)F(s, [x(s - \tau)]^* + j^{-1})ds \\ &\geq j^{-1} + t(1 - t) \int_0^1 G(s, s)F(s, [x(s - \tau)]^* + j^{-1})ds \\ &\geq t(1 - t) \left\{ j^{-1} + \int_0^1 G(s, s)F(s, [x(s - \tau)]^* + j^{-1}) \right\} ds \\ &\geq t(1 - t) \|T_j x\|_{[0, 1]} = t(1 - t) \|T_j x\|_{[-\tau, 1]}, \end{aligned}$$

which implies  $T_j(Q) \subset Q$ . Thus  $T_j : P \rightarrow Q$ .

It is easy to show that  $T_j : P \rightarrow Q$  is continuous and bounded. Next, we show that  $T_j$  is equicontinuous.

Suppose  $B \subset Q$  is any bounded set; then, for any  $x \in B$ , there exists  $M_0 > 0$  such that  $\|x(t)\| \leq M_0$ . We write

$$L(j^{-1}, M_0) = g(j^{-1}) + h(M_0 + \|\eta\| + \|w\| + 1).$$

For any  $\varepsilon > 0$ , from (C<sub>2</sub>), there exists  $\delta_1 > 0$  such that

$$\begin{aligned} \int_0^{\delta_1} s(1 - s)\phi(s)ds &< \int_0^{\delta_1} \phi(s)ds < \frac{\varepsilon}{6L(j^{-1}, M_0)}, \\ \int_{1-\delta_1}^1 s(1 - s)\phi(s)ds &< \int_{1-\delta_1}^1 \phi(s)ds < \frac{\varepsilon}{6L(j^{-1}, M_0)}. \end{aligned}$$

By the property of uniform continuity of  $G(t, s)$ , there exists  $\delta : \delta_1 > \delta > 0$  such that for any  $t, t' \in [0, 1], s \in [0, 1], |t - t'| < \delta$ , we have

$$|G(t, s) - G(t', s)| < \frac{\varepsilon}{3c_0 L(j^{-1}, M_0)},$$

where  $c_0 := \max_{t \in [\delta_1, 1-\delta_1]} \phi(t)$ . Then

$$\begin{aligned} |T_j x(t) - T_j x(t')| &\leq \int_0^1 |G(t, s) - G(t', s)| \phi(s) L(j^{-1}, M_0) ds \\ &\leq 2L(j^{-1}, M_0) \left\{ \int_0^{\delta_1} s(1 - s)\phi(s)ds + \int_{1-\delta_1}^1 s(1 - s)\phi(s)ds \right\} \\ &\quad + c_0 L(j^{-1}, M_0) \int_{\delta_1}^{1-\delta_1} |G(t, s) - G(t', s)| ds < \varepsilon. \end{aligned}$$

Thus,  $T_j(B)$  is equicontinuous in  $[0, 1]$ . It is easy to see that  $T_j(B)$  is also equicontinuous in  $[-\tau, 0]$ . By the Arzela–Ascoli theorem we conclude that  $T_j : P \rightarrow Q$  is compact in  $[-\tau, 1]$ . So  $T_j : P \rightarrow Q$  is completely continuous.  $\square$

Now, we can state and prove our main results.

**Theorem 2.2.** Suppose that (C<sub>1</sub>)–(C<sub>4</sub>) hold, and

$$\lim_{y \rightarrow +\infty} \frac{h(y)}{y} = 0. \quad (H_1)$$

Then the problem (1.1) has at least two positive solutions  $y_1$  and  $y_2$  with  $R_0 \leq \|y_1(t)\| \leq a_1 < a_2 \leq \|y_2(t)\|$ .

**Proof.** By virtue of the definition of  $T_j$  and Lemma 2.1,  $T_j$  is a completely continuous operator. First, we show that

$$i(T_j, \Omega_0, Q) = 1, \quad (2.4)$$

where  $\Omega_0 = \{x \in Q : \|x\| < R_0\}$ .

By virtue of (2.2), for any  $t \in (0, 1)$ , we get

$$w(t) = \int_0^1 G(t, s)p(s)ds \leq t(1-t) \int_0^1 p(s)ds = a_0 t(1-t).$$

Then

$$x(t) - w(t) \geq x(t) - a_0 t(1-t) \geq \frac{1}{2}x(t) \geq \frac{1}{2}R_0 t(1-t), \quad t \in (0, 1). \quad (2.5)$$

Thus

$$\begin{aligned} T_j x(t) &\leq j^{-1} + \int_0^1 G(t, s)(f(s, [x(s-\tau)]^* + j^{-1}) + p(s))ds \\ &\leq 1 + \int_0^\tau s(1-s)\phi(s)(g(\eta(s-\tau)) + h(\eta(s-\tau) + 1))ds \\ &\quad + \int_\tau^1 s(1-s)\phi(s) \left( g \left( \frac{1}{2}R_0 s(1-s) \right) + h(R_0 + 1) \right) ds < R_0 = \|x\|_{[-\tau, 1]}, \end{aligned}$$

which implies  $T_j(\bar{\Omega}_0) \subset \Omega_0$ . From Lemma A(i), we obtain  $i(T_j, \Omega_0, Q) = 1$ .

From (C<sub>4</sub>) and (H<sub>1</sub>), there exist  $R^* > a_2 > a_1 > R_0$ ,  $0 < k < \frac{1}{2}$  such that

$$h(x) \leq kx, \quad \forall x \geq R^*. \quad (2.6)$$

Let

$$R_1 > \max\{2R^*, 2\rho\}, \quad (2.7)$$

where

$$\rho := 1 + \int_0^\tau s(1-s)\phi(s)(g(\eta(s-\tau)) + \|\eta\| + 1)ds + \int_\tau^1 s(1-s)\phi(s) \left( g \left( \frac{R_0}{2}s(1-s) \right) + 1 \right) ds.$$

In the following, let

$$\begin{aligned} \Omega_1 &= \{x \in Q : \|x\| < R_1\}, \\ \Omega_{10} &= \{x \in Q : \|x\| < R_1, \min_{t \in [\alpha, \beta]} x(t) > a_1\}, \\ \Omega_{11} &= \{x \in Q : \|x\| < R_1, \min_{t \in [\alpha, \beta]} x(t) > a_2\}. \end{aligned}$$

It is easy to see that  $\Omega_i, \Omega_{1i}$  ( $i = 0, 1$ ) are bounded sets, satisfying

$$\Omega_0 \subset \Omega_1, \quad \Omega_{10} \subset \Omega_1, \quad \Omega_{11} \subset \Omega_{10}, \quad \Omega_{10} \cap \Omega_0 = \emptyset.$$

For any  $x \in \bar{\Omega}_1$ , from (2.6) and (2.7), we get

$$\begin{aligned} T_j x(t) &\leq j^{-1} + \int_0^1 G(t, s)F(s, [x(s-\tau)]^* + j^{-1})ds \\ &\leq 1 + \int_0^1 s(1-s)\phi(s)(g([x(s-\tau)]^* + j^{-1}) + h([x(s-\tau)]^* + j^{-1}))ds \\ &\leq 1 + \int_0^\tau s(1-s)\phi(s)(g(\eta(s-\tau)) + h(\eta(s-\tau) + 1))ds \\ &\quad + \int_\tau^1 s(1-s)\phi(s) \left( g \left( \frac{R_1}{2}s(1-s) \right) + h(R_1 + 1) \right) ds \\ &\leq 1 + \int_0^\tau s(1-s)\phi(s)(g(\eta(s-\tau)) + k(\|\eta\| + 1))ds \\ &\quad + \int_\tau^1 s(1-s)\phi(s) \left( g \left( \frac{R_0}{2}s(1-s) \right) + k(R_1 + 1) \right) ds < R_1 \end{aligned}$$

which implies  $T_j(\bar{\Omega}_1) \subset \Omega_1$  for  $x \in \bar{\Omega}_1$ . Thus

$$i(T_j, \Omega_1, Q) = 1. \quad (2.8)$$

For any  $x \in \bar{\Omega}_{10}$ , by (C<sub>4</sub>),

$$x(t) - w(t) \geq \frac{1}{2}x(t) \geq \frac{1}{2}a_1, \quad \forall t \in [\alpha, \beta] \subset (\tau, 1).$$

Thus

$$\begin{aligned} \min_{t \in [\alpha, \beta]} T_j x(t) &= \min_{t \in [\alpha, \beta]} \left\{ j^{-1} + \int_0^1 G(t, s) F(s, [x(s - \tau)]^* + j^{-1}) ds \right\} \\ &\geq \min_{t \in [\alpha, \beta]} \int_0^1 G(t, s) \phi_0 h_0([x(s - \tau)]^* + j^{-1}) ds \\ &\geq \sigma h_0 \left( \frac{1}{2} a_1 \right) \int_\alpha^\beta s(1-s) \phi_0(s) ds > a_1, \end{aligned}$$

which implies that  $T_j(\bar{\Omega}_{10}) \subset \Omega_{10}$  for  $x \in \bar{\Omega}_{10}$ . Thus, It follows from Lemma A(i) that

$$i(T_j, \Omega_{10}, Q) = 1. \quad (2.9)$$

Similarly, we can prove that, for any  $x \in \bar{\Omega}_{11}$ ,

$$i(T_j, \Omega_{11}, Q) = 1. \quad (2.10)$$

Thus, using (2.8) and (2.9), we obtain

$$i(T_j, \Omega_1 \setminus (\bar{\Omega}_0 \cap \bar{\Omega}_{10}), Q) = i(T_j, \Omega_1, Q) - i(T_j, \Omega_0, Q) - i(T_j, \Omega_{10}, Q) = -1$$

which implies that  $T_j$  has at least one fixed point  $x_{j1} \in \Omega_1 \setminus (\bar{\Omega}_0 \cap \bar{\Omega}_{10})$ , satisfying

$$T_j x_{j1}(t) = \begin{cases} j^{-1}, & t \in [-\tau, 0], \\ j^{-1} + \int_0^1 G(t, s) F(s, [x_{j1}(s - \tau)]^* + j^{-1}) ds, & t \in (0, 1) \end{cases}$$

and  $R_0 \leq \|x_{j1}(t)\| \leq a_1$ .

By direct computation, we have

$$\begin{cases} x_{j1}''(t) + f(t, [x_{j1}(t - \tau)]^* + j^{-1}) + p(t) = 0, & \forall t \in (0, 1) \setminus \{\tau\}, \\ x_{j1}(t) = j^{-1}, & \forall t \in [-\tau, 0], \\ x_{j1}(1) = j^{-1}. \end{cases} \quad (2.11)$$

Obviously, the sequence  $\{x_{j1}\}_{j=1}^\infty$  is uniformly bounded in  $C[-\tau, 1]$ . Like in the proof of Lemma 2.1, we can prove that the sequence  $\{x_{j1}\}_{j=1}^\infty$  is equicontinuous in  $[-\tau, 1]$ . By the Ascoli–Arzela theorem,  $\{x_{j1}\}_{j=1}^\infty$  is relatively compact. Thus, there exists a subsequence  $\{x_{j_m, 1}\}_{m=1}^\infty \subseteq \{x_{j1}\}_{j=1}^\infty$  ( $m \in \mathbb{N}$ ) such that

$$\lim_{m \rightarrow +\infty} x_{j_m, 1}(t) = x_1(t).$$

This, together with (2.11) and the Lebesgue dominated convergence theorem, implies that

$$\begin{cases} x_1''(t) + f(t, [x_1(t - \tau)]^*) + p(t) = 0, & \forall t \in (0, 1) \setminus \{\tau\}, \\ x_1(t) = 0, & \forall t \in [-\tau, 0], \\ x_1(1) = 0. \end{cases} \quad (2.12)$$

Like for (2.5), we have

$$x_1(t - \tau) + x_0(t - \tau) - w(t - \tau) \geq 0.$$

Let  $y_1(t) = x_1(t) + x_0(t) - w(t)$ ; then  $y_1(t)$  is a positive solution of problem (1.1).

Similarly, there exists a subsequence  $\{x_{j_m, 2}\}_{m=1}^\infty \subseteq \{x_{j2}\}_{j=1}^\infty$  ( $m \in \mathbb{N}$ ) such that

$$\lim_{m \rightarrow +\infty} x_{j_m, 2}(t) = x_2(t) \in \bar{\Omega}_{11},$$

and  $a_2 \leq \|x_2(t)\| \leq R_1$ ,  $\forall t \in [\alpha, \beta]$ . Let  $y_2(t) = x_2(t) + x_0(t) - w(t)$ ; then  $y_2(t)$  is also a positive solution of the problem (1.1).

We have

$$R_0 \leq \|x_1(t)\| \leq a_1 < a_2 \leq \|x_2(t)\| \leq R_1, \quad \forall t \in [\alpha, \beta] \subset (\tau, 1),$$

which implies that  $y_1(t), y_2(t)$  are two different positive solutions of the problem (1.1).  $\square$

**Theorem 2.3.** Assume that  $(C_1)$ – $(C_4)$  hold, and further there exists an  $R_1 > a_2$  such that

$$\Delta + \int_{\tau}^1 s(1-s)\phi(s) \left\{ g\left(\frac{1}{2}R_1s(1-s)\right) + h(R_1+1) \right\} ds < R_1, \quad (H_2)$$

where  $\Delta$  is given by  $(C_3)$ , and

$$\lim_{y \rightarrow +\infty} \frac{h_0(y)}{y} = +\infty. \quad (H_3)$$

Then the problem (1.1) has at least three different positive solutions.

**Proof.** We first note that assumption  $(H_2)$  is equal to the condition  $(H_1)$ . As a result, the problem (1.1) has at least two positive solutions  $y_1(t), y_2(t)$  with  $\|y_1(t)\| < \|y_2(t)\| < R_1$ . Choose

$$M_* > 2 \left[ (\alpha - \tau)(1 - \beta + \tau) \min_{t \in [0, 1]} \int_{\alpha}^{\beta} G(t, s)\phi_0(s)ds \right]^{-1}. \quad (2.13)$$

From  $(H_3)$ , there exists an  $R_* > R_1$  such that

$$h_0(x) \geq M_*x, \quad \forall t \in [\alpha, \beta], \quad \forall x \geq R_*. \quad (2.14)$$

Let

$$R > 2R_*[(\alpha - \tau)(1 - \beta + \tau)]^{-1}$$

with  $\varphi \in Q \setminus \{\theta\}$ ,  $\Omega_2 = \{x \in Q : \|x\| < R\}$ .

In the following, we will claim that

$$x \neq T_jx + \lambda\varphi, \quad \forall x \in \partial Q_{(R)}, \lambda \in [0, 1], \forall j \in \mathbf{N}. \quad (2.15)$$

Suppose that this is false; then there exist  $\lambda_0 \in [0, 1]$ ,  $x^+ \in \partial Q_{(R)}$  such that  $x^+ = T_jx^+ + \lambda_0\varphi$ .

Like for (2.5), we get

$$x^+(t) - w(t) \geq \frac{1}{2}x^+(t) \geq t(1-t)\|x^+\|, \quad \forall t \in [0, 1].$$

Thus, for any  $t \in [\alpha, \beta]$ ,

$$[x^+(t)]^* = x^+(t - \tau) + x_0(t - \tau) - w(t - \tau) \geq \frac{1}{2}(\alpha - \tau)(1 - \beta + \tau)\|x^+\| \geq R_*.$$

It follows from (2.6) that we have

$$\begin{aligned} R &= \|x^+\|_{[-\tau, 1]} = \|x^+\|_{[0, 1]} \\ &\geq \int_0^1 G(t, s)(f(s, [x^+(s)]^* + j) + p(s))ds \\ &\geq \int_{\alpha}^{\beta} \min_{t \in [\alpha, \beta]} G(t, s)\phi_0(s)M_*(x^+(t - \tau) + x_0(t - \tau) - w(t - \tau) + j_{-1})ds \\ &\geq \frac{R}{2}M_*(\alpha - \tau)(1 - \beta + \tau) \min_{t \in [\alpha, \beta]} \int_{\alpha}^{\beta} G(t, s)\phi_0(s)ds > R \end{aligned}$$

which is a contradiction. Thus (2.15) is true; by Lemma A(ii), we obtain

$$i(T_j, \Omega_2, Q) = 0, \quad \forall j \in \mathbf{N}, \quad (2.16)$$

and this and (2.8) imply

$$i(T_j, \Omega_2 \setminus \bar{\Omega}_1, Q) = i(T_j, \Omega_1, Q) - i(T_j, \Omega_2, Q) = -1,$$

which implies that  $T_j$  has a fixed point  $x_{j3} \in \Omega_2 \setminus \bar{\Omega}_1$  with  $R_1 \leq \|x_{j3}\| \leq R_2$ . Similarly, there exists a subsequence  $\{x_{j_m, 3}\}_{m=1}^{\infty} \subset \{x_{j3}\}_{j=1}^{\infty}$  ( $m \in \mathbf{N}$ ) such that the subsequence uniformly converges to  $x_3(t) \in \Omega_2 \setminus \bar{\Omega}_1$ . Let  $y_3(t) = x_3(t) + x_0(t) - w(t)$ ; then  $y_3(t)$  is also a positive solution of the problem (1.1).  $\square$

Further we can establish the following multiplicity results for problem (1.1).

**Corollary 2.4.** Assume that  $(C_1)$ – $(C_3)$  hold, and further there exist  $[\alpha, \beta] \subset (\tau, 1)$ , and  $0 < R_0 < a_{11} < a_{12} < R_1 < a_{21} < a_{22} < R_2 < \dots < a_{n1} < a_{n2} < R_n$  such that

$$\Delta + \int_{\tau}^1 s(1-s)\phi(s) \left( g\left(\frac{1}{2}R_i s(1-s)\right) + h(R_i + 1) \right) ds < R_i, \quad i = 1, 2, \dots, n;$$

$$\sigma h_0\left(\frac{1}{2}a_{ij}\right) \int_{\alpha}^{\beta} s(1-s)\phi_0(s)ds > a_{ij}, \quad i = 1, 2, \dots, n; j = 1, 2;$$

and

$$\lim_{y \rightarrow +\infty} \frac{h_0(y)}{y} = +\infty$$

where  $\Delta$  is given by  $(C_3)$ , and  $\sigma = \min\{\alpha(1-\alpha), \beta(1-\beta)\}$ . Then the problem (1.1) has at least  $2n + 1$  positive solutions.

Now we present an example to illustrate our results.

**Example 2.5.** Consider the following singular functional differential equation:

$$\begin{cases} y''(t) + f\left(t, y\left(t - \frac{1}{4}\right)\right) = 0, & 0 < t < 1, t \neq \frac{1}{4}, \\ y(t) = -t, & \forall t \in \left[-\frac{1}{4}, 0\right], \\ y(1) = 0, \end{cases} \quad (*)$$

where

$$f\left(t, y\left(t - \frac{1}{4}\right)\right) = \frac{1}{\sqrt{t(1-t)}} \left( \frac{1}{\sqrt[4]{y\left(t - \frac{1}{4}\right)}} + h\left(y\left(t - \frac{1}{4}\right)\right) \right) - \frac{1}{4\sqrt{t}}.$$

Let  $p(t) = \frac{1}{4\sqrt{t}}$ ,  $\phi_0(t) = \phi(t) = \frac{1}{\sqrt{t(1-t)}}$ , and  $g(y) = \frac{1}{\sqrt[4]{y}}$ ,

$$h_0(y) = h(y) = \begin{cases} \frac{y^2}{100}, & \forall y \in [0, 9 \times 10^4], \\ 2.7 \times 10^5 \sqrt{y}, & \forall y \in [9 \times 10^4, +\infty). \end{cases}$$

Then,  $a_0 = \int_0^1 p(s)ds = \frac{1}{2}$ , and it is easy to see that the assumptions  $(C_1)$ ,  $(C_2)$  and  $(H_1)$  hold.

Choose  $[\alpha, \beta] = [\frac{1}{2}, \frac{3}{4}]$ ,  $R_0 = 4$ ,  $a_1 = 1.8 \times 10^4$ ,  $a_2 = 2 \times 10^4$ . Then we can see that the assumptions  $(C_3)$  and  $(C_4)$  are satisfied. So by applying Theorem 2.2, we obtain that the problem (\*) has at least two positive solutions.

**Remark 2.1.** Suppose we let

$$h_0(y) = h(y) = \begin{cases} \frac{y^2}{100}, & \forall y \in [0, 9 \times 10^4], \\ 2.7 \times 10^5 \sqrt{y}, & \forall y \in [9 \times 10^4, 1 \times 10^{10}], \\ 2.7 \times 10^{-10} y^2, & \forall y \in [1 \times 10^{10}, +\infty), \end{cases}$$

and the other conditions of the problem (\*) do not change. Further choose  $R_1 = 8 \times 10^8$ ; then we can see that the assumptions of Theorem 2.3 are satisfied. So by applying Theorem 2.3 we obtain that the problem (\*) has at least three positive solutions.

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