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Convergence of a generalized MSSOR method for augmented systems[☆]

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ABSTRACT

Recently, Wu et al. [S.-L. Wu, T.-Z. Huang, X.-L. Zhao, A modified SSOR iterative method for augmented systems, J. Comput. Appl. Math. 228 (1) (2009) 424–433] introduced a modified SSOR (MSSOR) method for augmented systems. In this paper, we establish a generalized MSSOR (GMSSOR) method for solving the large sparse augmented systems of linear equations, which is the extension of the MSSOR method. Furthermore, the convergence of the GMSSOR method for augmented systems is analyzed and numerical experiments are carried out, which show that the GMSSOR method with appropriate parameters has a faster convergence rate than the MSSOR method with optimal parameters.

1. Introduction

For solving the large sparse augmented systems of linear equations

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}, \tag{1}$$

where $A \in R^{m\times m}$ is a symmetric and positive definite matrix and $B \in R^{m\times n}$ is a matrix of full column rank. It appears in many different applications of scientific computing, such as constrained optimization [1], the finite element method for solving the Navier–Stokes equation [2–4], and constrained least squares problems and generalized least squares problems [5–8]. There have been several recent papers for solving the augmented system (1). Santos et al. [6] studied preconditioned iterative methods for solving the augmented system (1) with $A = I$. Yuan and Iusem [7,8] proposed several variants of the SOR method and preconditioned conjugate gradient methods for solving general augmented system (1) arising from the generalized least squares problems where A can be symmetric and positive semidefinite and B can be rank deficient. The SOR-like method requires less arithmetic work per iteration step than other methods but it requires choosing an optimal iteration parameter in order to achieve a comparable rate of convergence. Golub et al. [9] presented SOR-like algorithms for solving system (1). Darvishi and Hessari [10] studied the SSOR method for solving the augmented systems. Bai et al. [11–14] presented the GSOR method, parameterized Uzawa (PU) and the inexact parameterized Uzawa (PIU) methods for solving systems (1). Zhang and Lu [15] showed the generalized symmetric SOR method for augmented systems. Peng and Li [16] studied unsymmetric block overrelaxation-type methods for saddle point. Bai and Golub [17–22] presented splitting iteration methods such as the Hermitian and the skew-Hermitian splitting (HSS) iteration scheme and its preconditioned

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variants, Krylov subspace methods such as preconditioned conjugate gradient (PCG), preconditioned MINRES (PMINRES) and restrictively preconditioned conjugate gradient (RPCG) iteration schemes, and preconditioning techniques related to Krylov subspace methods such as HSS, block-diagonal, block-triangular and constraint preconditioners and so on. Bai and Wang's 2009 LAA paper [22] and Chen and Jiang's 2008 AMC paper [14] studied some general approaches about the relaxed splitting iteration methods. Recently, Wu et al. [23] presented a modified SSOR (MSSOR) method for augmented systems (1).

In this paper, we establish a generalized MSSOR (GMSSOR) method for augmented systems and analyze convergence of the corresponding method. Moreover, numerical experiments show that the GMSSOR method with appropriate parameters has a faster convergence rate than the MSSOR method with optimal parameters for solving augmented linear systems. However, the relaxed parameters of the GMSSOR method are not optimal and only lie in the convergence region of the method.

2. Generalized MSSOR method

Recently, for the coefficient matrix of the augmented system (1), Wu et al. [23] make the following splitting

$$\mathcal{A} = \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = \mathcal{D} - \mathcal{L} - \mathcal{U}, \tag{2}$$

where

$$\mathcal{D} = \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} 0 & 0 \\ B^T & \frac{1}{2}Q \end{pmatrix}, \qquad \mathcal{U} = \begin{pmatrix} 0 & -B \\ 0 & \frac{1}{2}Q \end{pmatrix}, \tag{3}$$

and $Q \in R^{n\times n}$ is a nonsingular and symmetric matrix.

Let

$$L = \mathcal{D}^{-1}\mathcal{L} = \begin{pmatrix} 0 & 0 \\ Q^{-1}B^T & \frac{1}{2}I_n \end{pmatrix}, \qquad U = \mathcal{D}^{-1}\mathcal{U} = \begin{pmatrix} 0 & -A^{-1}B \\ 0 & \frac{1}{2}I_n \end{pmatrix}, \qquad \Omega = \begin{pmatrix} \omega I_m & 0 \\ 0 & \tau I_n \end{pmatrix}, \tag{4}$$

where ω and τ are two nonzero real numbers, $I_m \in R^{m\times m}$ and $I_n \in R^{n\times n}$ are $m \times m$ and $n \times n$ identity matrices, respectively. Then we can obtain following generalized MSSOR method:

$$z^{k+\frac{1}{2}} = \mathcal{F} z^k + (I - \Omega L)^{-1}\mathcal{D}^{-1}\Omega u, \tag{5}$$

where

$$\mathcal{F} = (I - \Omega L)^{-1}(I - \Omega + \Omega U) = \begin{pmatrix} (1-\omega)I_m & -\omega A^{-1}B \\ \dfrac{2\tau(1-\omega)}{2-\tau}Q^{-1}B^T & I_n - \dfrac{2\tau\omega}{2-\tau}Q^{-1}B^TA^{-1}B \end{pmatrix}, \tag{6}$$

and

$$u = \begin{pmatrix} b \\ -q \end{pmatrix}. \tag{7}$$

By backward generalized SOR we compute z^{k+1} from $z^{k+\frac{1}{2}}$ as

$$z^{k+1} = \mathcal{G} z^{k+\frac{1}{2}} + (I - \Omega U)^{-1}\mathcal{D}^{-1}\Omega u, \tag{8}$$

where

$$\mathcal{G} = (I - \Omega U)^{-1}(I - \Omega + \Omega L) = \begin{pmatrix} (1-\omega)I_m - \dfrac{2\omega\tau}{2-\tau}A^{-1}BQ^{-1}B^T & -\omega A^{-1}B \\ \dfrac{2\tau}{2-\tau}Q^{-1}B^T & I_n \end{pmatrix}. \tag{9}$$

We eliminate $z^{k+\frac{1}{2}}$ from (5) and (8), so we have generalized MSSOR (GMSSOR) method, which is as follows:

$$z^{k+1} = \mathcal{H} z^k + \mathcal{M}, \tag{10}$$

where

$$\begin{aligned} \mathcal{H} &= \mathcal{G}\mathcal{F} \\ &= \begin{pmatrix} (1-\omega)^2 I_m - \dfrac{4\omega\tau(1-\omega)}{2-\tau}A^{-1}BQ^{-1}B^T & \left[-\omega(2-\omega)I_m + \dfrac{4\omega^2\tau}{2-\tau}A^{-1}BQ^{-1}B^T\right]A^{-1}B \\ \dfrac{4\tau(1-\omega)}{2-\tau}Q^{-1}B^T & I_n - \dfrac{4\tau\omega}{2-\tau}Q^{-1}B^TA^{-1}B \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned}\mathcal{M} &= (I-\Omega U)^{-1}(2I-\Omega)(I-\Omega L)^{-1}\mathcal{D}^{-1}\Omega u\\ &= \begin{pmatrix} \omega(2-\omega)A^{-1}-\dfrac{4\omega^2\tau}{2-\tau}A^{-1}BQ^{-1}B^TA^{-1} & -\dfrac{4\omega\tau}{2-\tau}A^{-1}BQ^{-1}\\ \dfrac{4\omega\tau}{2-\tau}Q^{-1}B^TA^{-1} & \dfrac{4\tau}{2-\tau}Q^{-1}\end{pmatrix}.\end{aligned}$$

Generalized MSSOR method: Let $Q \in R^{n\times n}$ be a nonsingular and symmetric matrix. Given initial vectors $x^{(0)} \in R^m$ and $y^{(0)} \in R^n$, and two relaxed parameters $\omega > 0$ and $\tau > 0$. For $k = 0, 1, 2, \ldots$ until the iteration sequence $\{((x^k)^T, (y^k)^T)^T\}$ converges, compute

$$\begin{cases} y^{k+1} = y^k + \dfrac{4\tau}{2-\tau}Q^{-1}B^T[(1-\omega)x^k - \omega A^{-1}By^k + \omega A^{-1}b] - \dfrac{4\tau}{2-\tau}Q^{-1}q,\\ x^{k+1} = (1-\omega)^2x^k - \omega A^{-1}B[y^{k+1} + (1-\omega)y^k] + \omega(2-\omega)A^{-1}b\end{cases}$$

and Q is an approximate (preconditioning) matrix of the Schur complement matrix $B^TA^{-1}B$.

Remark 2.1. When the relaxed parameters $\tau = \omega$, the GMSSOR method reduces to the MSSOR method, so the GMSSOR method is the generalization of the MSSOR method. Furthermore, the GMSSOR method with appropriate parameters has a faster convergence rate than the MSSOR method with optimal parameters, which is shown by numerical experiments.

Remark 2.2. The GMSSOR iteration scheme is not suitable for solving large problems, as its computing cost is at least twice of that of SOR and GSOR iteration schemes due to solving two subsystems with respect to Q and three subsystems with respect to A at each step of MGSSOR.

3. Convergence of the GMSSOR method

Now, we will analyze convergence region for parameters τ and ω, in the generalized MSSOR (GMSSOR) method to solve augmented systems (1).

Theorem 3.1. *Suppose that μ is an eigenvalue of $Q^{-1}B^TA^{-1}B$, if λ satisfies*

$$[\lambda-(\omega-1)^2](1-\lambda)(2-\tau) = 4\tau\omega\lambda(2-\omega)\mu, \tag{11}$$

then λ is an eigenvalue of $\mathcal{H}$. Conversely, if λ is an eigenvalue of $\mathcal{H}$ such that $\lambda \neq 1$ and $\lambda \neq (1-\omega)^2$, and μ satisfies (11), then μ is a nonzero eigenvalue of $Q^{-1}B^TA^{-1}B$.

Proof. Suppose that λ and u are the eigenvalue and eigenvector of $\mathcal{H}$, respectively. Then we can obtain

$$\mathcal{H}u = \lambda u$$

or

$$[I-(I-\Omega U)^{-1}(2I-\Omega)(I-\Omega L)^{-1}\Omega\mathcal{D}^{-1}\mathcal{A}]u = \lambda u$$

hence

$$(1-\lambda)(I-\Omega U)u = (2I-\Omega)(I-\Omega L)^{-1}\Omega\mathcal{D}^{-1}\mathcal{A}u$$

so

$$(1-\lambda)\begin{pmatrix} I_m & \omega A^{-1}B\\ 0 & \dfrac{2-\tau}{2}I_n\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}\omega(2-\omega)I_m & \omega(2-\omega)A^{-1}B\\ (2\omega\tau-2\tau)Q^{-1}B^T & 2\omega\tau Q^{-1}B^TA^{-1}B\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

from this we have the following system of two equations

$$\begin{cases}[(\omega-1)^2-\lambda]x_1 = \omega(\lambda+1-\omega)A^{-1}Bx_2,\\ (1-\lambda)\dfrac{2-\tau}{2}x_2 = (2\omega\tau-2\tau)Q^{-1}B^Tx_1 + 2\omega\tau Q^{-1}B^TA^{-1}Bx_2\end{cases}$$

so from the first equation, we can obtain

$$x_1 = \frac{\omega(\lambda+1-\omega)}{(\omega-1)^2-\lambda}A^{-1}Bx_2$$

setting x_1 in the second equation, yields

$$(1-\lambda)\frac{2-\tau}{2}x_2 - 2\omega\tau Q^{-1}B^TA^{-1}Bx_2 = 2\tau(\omega-1)Q^{-1}B^T\frac{\omega(\lambda+1-\omega)}{(\omega-1)^2-\lambda}A^{-1}Bx_2,$$

equivalently

$$\frac{(1-\lambda)(2-\tau)}{2}x_2 = \left[2\omega\tau - \frac{2\tau\omega(\omega-1)(\omega-\lambda-1)}{(\omega-1)^2-\lambda}\right]Q^{-1}B^TA^{-1}Bx_2.$$

Since μ is an eigenvalue of $Q^{-1}B^TA^{-1}B$, then we have

$$(1-\lambda)(2-\tau)[(\omega-1)^2-\lambda] = \{4\omega\tau[(\omega-1)^2-\lambda] - 4\tau\omega(\omega-1)(\omega-\lambda-1)\}\mu$$

so

$$[\lambda-(\omega-1)^2](1-\lambda)(2-\tau) = 4\tau\omega\lambda(2-\omega)\mu.$$

We can prove the second assertion by reversing the process. □

Lemma 3.2 ([24]). *Consider the quadratic equation $x^2 - bx + c = 0$, where b and c are real numbers. Both roots of the equation are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

Theorem 3.3. *Suppose that B has full rank, and A and Q are both symmetric and positive definite. Assume that all eigenvalues μ of $Q^{-1}B^TA^{-1}B$ are real. Then if $\mu > 0$, the generalized MSSOR (GMSSOR) method converges if the parameters ω satisfies $0 < \omega < 2$ and two relaxed parameters ω and τ satisfy the following condition*

$$0 < \tau < \frac{2+2(\omega-1)^2}{2\omega(2-\omega)\mu + 1 + (\omega-1)^2}.$$

Proof. After some manipulations on Theorem 3.1, we have

$$\lambda^2 - \left[1 + (\omega-1)^2 - \frac{4\tau\omega(2-\omega)}{2-\tau}\mu\right]\lambda + (\omega-1)^2 = 0.$$

By setting

$$b = 1 + (\omega-1)^2 - \frac{4\tau\omega(2-\omega)}{2-\tau}\mu$$

and

$$c = (\omega-1)^2.$$

By Lemma 3.2, $|\lambda| < 1$ if and only if

$$|(\omega-1)^2| < 1 \tag{12}$$

and

$$\left|1 + (\omega-1)^2 - \frac{4\tau\omega(2-\omega)}{2-\tau}\mu\right| < 1 + (\omega-1)^2. \tag{13}$$

From (12) we have

$$0 < \omega < 2, \tag{14}$$

and the relation (13) changes to the following inequalities:

$$-1-(\omega-1)^2 < 1 + (\omega-1)^2 - \frac{4\tau\omega(2-\omega)}{2-\tau}\mu < 1 + (\omega-1)^2 \tag{15}$$

it follows that

$$\frac{4\tau\omega(2-\omega)}{2-\tau}\mu > 0, \tag{16}$$

and

$$2 + 2(\omega-1)^2 - \frac{4\tau\omega(2-\omega)}{2-\tau}\mu > 0. \tag{17}$$

Table 1
Choices of matrix Q.

Case no.	Matrix Q	Description
I	$B^T\hat{A}^{-1}B$	$\hat{A} = \mathrm{tridiag}(A)$
II	$B^T\hat{A}^{-1}B$	$\hat{A} = \mathrm{diag}(A)$

We see that the inequality (16) is true if $\frac{\tau}{2-\tau} > 0$, so

$$0 < \tau < 2.$$

And the inequality (17) is equal to

$$0 < \tau < \frac{2+2(\omega-1)^2}{2\omega(2-\omega)\mu+1+(\omega-1)^2}.$$

Obviously

$$0 < \tau < \frac{2+2(\omega-1)^2}{2\omega(2-\omega)\mu+1+(\omega-1)^2} = 2\times\frac{1+(\omega-1)^2}{2\omega(2-\omega)\mu+1+(\omega-1)^2} < 2.$$

Then, we have

$$0 < \tau < \frac{2+2(\omega-1)^2}{2\omega(2-\omega)\mu+1+(\omega-1)^2}.$$

This completes the proof. □

Remark 3.1. Obviously, Theorem 3 in [25] is included in Theorem 3.3 with the new conditions.

Corollary 3.4 ([23]). *Suppose that μ is an eigenvalue of $Q^{-1}B^TA^{-1}B$, if λ satisfies*

$$[\lambda-(\omega-1)^2](1-\lambda) = 4\tau\omega\lambda\mu, \tag{18}$$

then λ is an eigenvalue of $\mathcal{H}$ with $\tau=\omega$. Conversely, if λ is an eigenvalue of $\mathcal{H}$ such that $\lambda \neq 1$ and $\lambda \neq (1-\omega)^2$, and μ satisfies (18), then μ is a nonzero eigenvalue of $Q^{-1}B^TA^{-1}B$.

4. Numerical examples

In this section, we give two examples to compare the performance of the GMSSOR method and MSSOR method. All numerical examples are carried out in Matlab 7.0. We report the number of iterations and norm of absolution residual vectors. Here, RES is defined as

$$\mathrm{RES} = \sqrt{\|b-Ax^k-By^k\|_2^2+\|q-B^Tx^k\|_2^2}$$

with $\{((x^k)^T,(y^k)^T)^T\}$ the final approximate solution. Here, we choose the right-hand vector $(b^T,q^T)^T \in R^{m+n}$ such that the exact solution of the augmented linear system (1) is $((x^*)^T,(y^*)^T)^T = (1,1,\ldots,1)^T \in R^{m+n}$. All numerical results show that the GMSSOR method with appropriate parameters has a faster convergence rate than the MSSOR method with optimal parameters. Furthermore, the relaxed parameters of the GMSSOR method is not optimal and only lies in the convergence region of the method.

Example 4.1 ([11]). Let the augmented system (1) in which

$$A = \begin{pmatrix} I\otimes T+T\otimes I & 0 \\ 0 & I\otimes T+T\otimes I\end{pmatrix} \in R^{2p^2\times 2p^2}, \qquad B = \begin{pmatrix} I\otimes F \\ F\otimes I\end{pmatrix} \in R^{2p^2\times 2p^2}$$

and

$$T = \frac{1}{h^2}\mathrm{tridiag}(-1,2,-1) \in R^{p\times p}, \qquad F = \frac{1}{h}\mathrm{tridiag}(-1,1,0) \in R^{p\times p},$$

with $\otimes$ is the Kronecker product symbol and $h = \frac{1}{1+p}$ and $S = \mathrm{tridiag}(a,b,c)$ is a tridiagonal matrix with $S_{i,i} = b$, $S_{i-1,i} = a$, $S_{i,i+1} = c$ for appropriate i.

For this example, we set $m = 2p^2$ and $n = p^2$. Hence, the total number of variables is $m+n = 3p^2$. We choose the matrix Q as an approximation to the matrix $B^TA^{-1}B$, according to the cases listed in Table 1. In our experiments, all runs with respect to both the MSSOR method and GMSSOR method are started from initial vector $((x^{(0)})^T,(y^{(0)})^T)^T = 0$, and terminated if the current iteration satisfies RES $< 10^{-6}$.

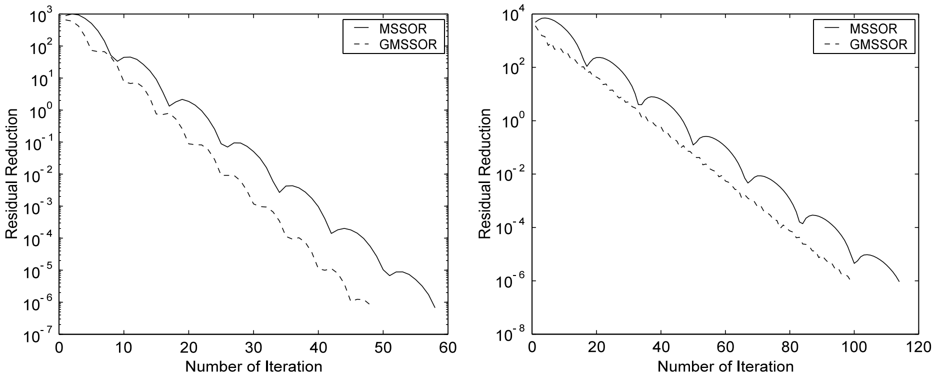

Fig. 1. Reduction of residual 2-norm with Case I, $m + n = 192$ and Case I, $m + n = 768$.

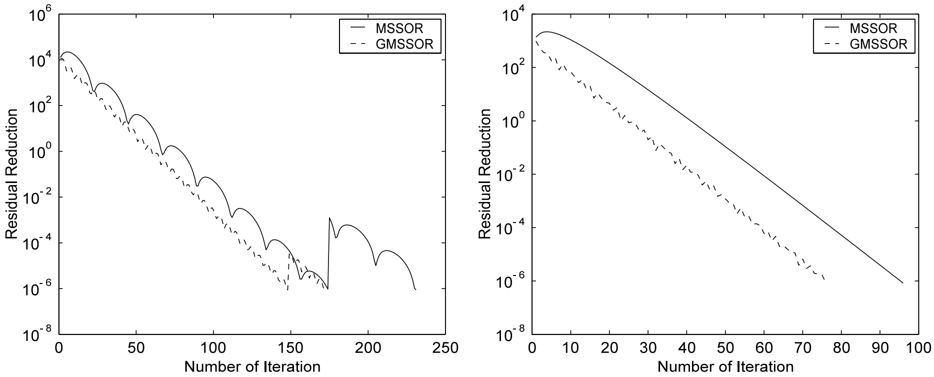

Fig. 2. Reduction of residual 2-norm with Case I, $m + n = 1728$ and Case II, $m + n = 192$.

Table 2
Spectral radius, IT and RES for Example 4.1.

m			128	512	152
n			64	256	576
$m + n$			192	768	1728
Case I	MSSOR	ω_{opt}	0.3081	0.1848	0.1316
		$\rho(\mathcal{H}(\omega_{opt}))$	0.6919	0.8152	0.8684
		IT	58	114	174
		RES	6.7036×10^{-7}	9.2395×10^{-7}	9.4418×10^{-7}
	GMSSOR	τ	0.2489	0.1140	0.1106
		ω	0.3600	0.1980	0.1436
		$\rho(\mathcal{H}(\tau, \omega))$	0.6400	0.8020	0.8564
		IT	48	99	148
		RES	8.3087×10^{-7}	9.5781×10^{-7}	8.6868×10^{-7}
Case II	MSSOR	ω_{opt}	0.2375	0.1367	0.0960
		$\rho(\mathcal{H}(\omega_{opt}))$	0.7625	0.8633	0.9040
		IT	96	172	231
		RES	8.2794×10^{-7}	9.2069×10^{-7}	8.8999×10^{-7}
	GMSSOR	τ	0.1510	0.1057	0.0660
		ω	1.7160	0.1727	0.1250
		IT	76	119	172
		RES	8.4689×10^{-7}	8.2134×10^{-7}	9.0700×10^{-7}

In Table 2, we list ω_{opt} and (τ, ω), the corresponding $\rho(\mathcal{H}(\omega_{opt}))$ and $\rho(\mathcal{H}(\tau, \omega))$ of the MSSOR method and GMSSOR method for various problem sizes (m, n), respectively. We also list the numerical results with respect to IT and RES for the testing methods for varying m and n, where $\rho(\mathcal{H}(\omega_{opt}))$ denotes the spectral radius of iterative matrix of the MSSOR method when choosing optimal parameter and $\rho(\mathcal{H}(\tau, \omega))$ denotes the spectral radius of iterative matrix of the GMSSOR method when choosing general parameters which are not optimal. Furthermore, Figs. 1–3 also show the history of residual reduction. Since two methods have the same computational complexity, we do not report the computing time. They clearly show that the GMSSOR method has a faster convergence rate than the MSSOR method. However, the relaxed parameters of GMSSOR method are not optimal and only lie in the convergence region of the method.

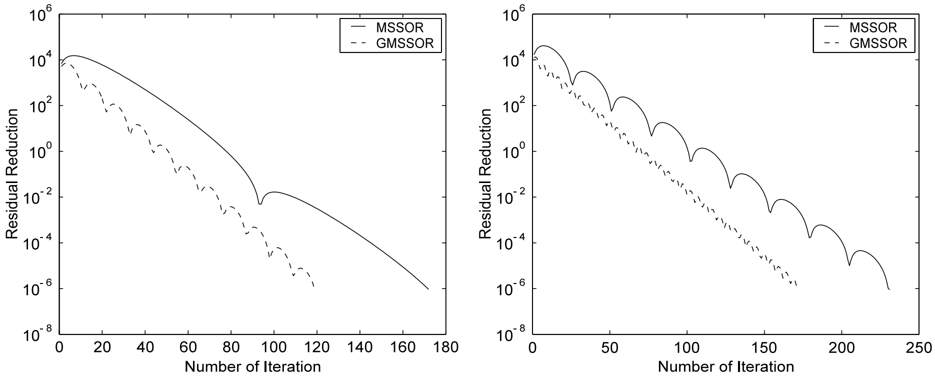

Fig. 3. Reduction of residual 2-norm with Case II, $m + n = 768$ and Case II, $m + n = 1728$.

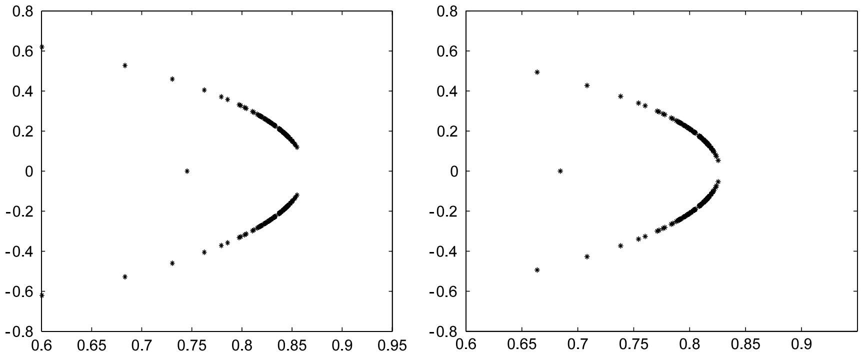

Fig. 4. Eigenvalues distributions of the MSSOR method with respect to $\omega_{opt} = 0.1367$ (the left) and the GMSSOR method with respect to $\tau = 0.1057$, $\omega = 0.1727$ (the right) with Case I, $m + n = 768$ for Example 4.1.

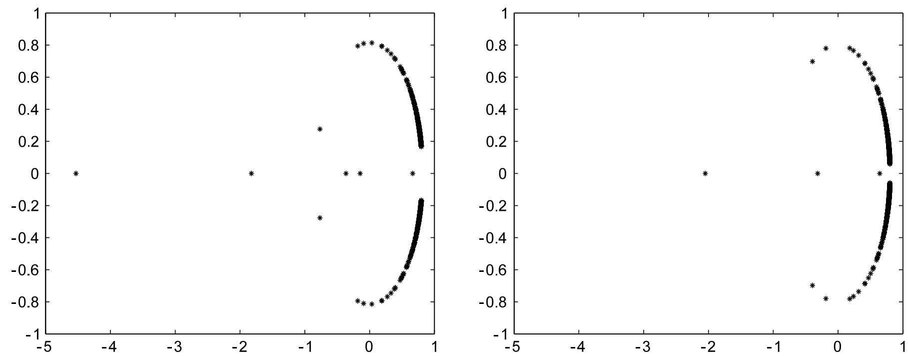

Fig. 5. Eigenvalues distributions of the MSSOR method with respect to $\omega_{opt} = 0.1848$ (the left) and the GMSSOR method with respect to $\tau = 0.1140$, $\omega = 0.1980$ (the right) with Case II, $m + n = 768$ for Example 4.1.

We also report the eigenvalues distributions of the MSSOR method with respect to $\omega_{opt} = 0.1367$ and the GMSSOR method with respect to $\tau = 0.1057$, $\omega = 0.1727$ with $\hat{A} = \text{tridiag}(A)$, $m + n = 768$, and the eigenvalues distributions of the MSSOR method with respect to $\omega_{opt} = 0.1848$ and the GMSSOR method with respect to $\tau = 0.1140$, $\omega = 0.1980$ with $\hat{A} = \text{diag}(A)$, $m + n = 768$, please see Figs. 4 and 5. These figures show that the eigenvalues distributions of the GMSSOR method with appropriate parameters are the same clustered as those of the MSSOR method with optimal parameters.

Example 4.2 (*[11]*)**.** Consider the augmented linear system (1) in which

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in R^{2p^2 \times 2p^2}, \qquad B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in R^{2p^2 \times 2p^2}$$

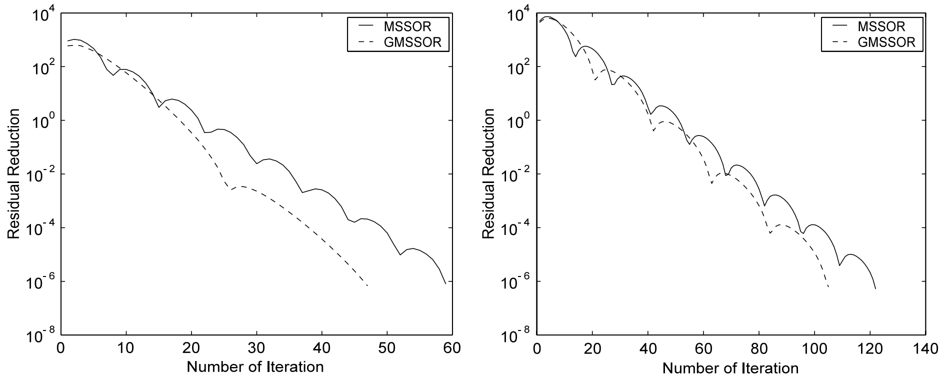

Fig. 6. Reduction of residual 2-norm with Case I, $m+n=192$ and Case I, $m+n=768$.

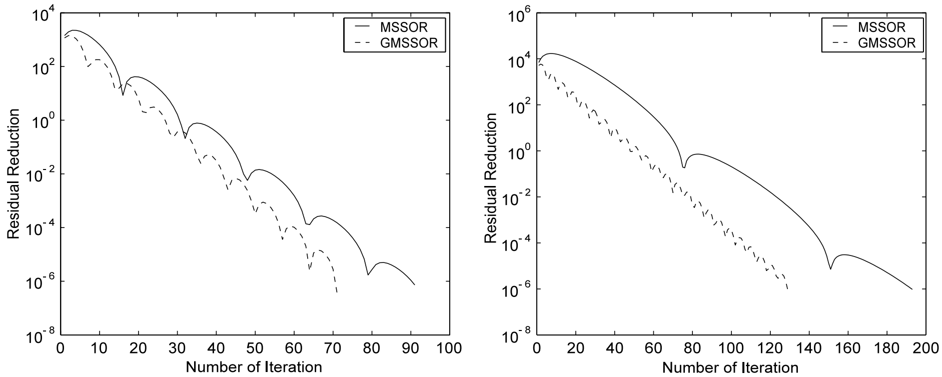

Fig. 7. Reduction of residual 2-norm with Case II, $m+n=192$ and Case II, $m+n=768$.

and

$$T=\frac{1}{h^2}\text{tridiag}(-1,2,-1)\in R^{p\times p},\qquad F=\frac{1}{h}K\in R^{p\times p},$$

with

$$K=(k_{i,j})\in R^{p\times p},\quad k_{i,j}=\frac{1}{2\sqrt{2\pi}}e^{\frac{-|i-j|^2}{8}},\ i,j=1,2,\ldots,p,$$

where $\otimes$ denotes the Kronecker product symbol and $\frac{1}{p+1}$ the discretization mesh-size. For this example, we set $m=2p^2$ and $n=p^2$. Hence, the total number of variables is $m+n=3p^2$. We choose the matrix Q as an approximation to the matrix $B^TA^{-1}B$, according to the cases listed in Table 1. In our experiments, all runs with respect to both the MSSOR method and GMSSOR method are started from initial vector $((x^{(0)})^T,(y^{(0)})^T)^T=0$, and terminated if the current iteration satisfies RES $<10^{-6}$.

Similar to Example 4.1, in Table 3, we list ω_{opt} and (τ,ω), the corresponding $\rho(\mathcal{H}(\omega_{opt}))$ and $\rho(\mathcal{H}(\tau,\omega))$ of the MSSOR method and GMSSOR method for various problem sizes (m,n), respectively. We also list the numerical results with respect to IT and RES for the testing methods for varying m and n. Furthermore, Figs. 6 and 7 also show the history of residual reduction. They clearly show that the GMSSOR method with appropriate parameters has a faster convergence rate than the MSSOR method with optimal parameters. However, the relaxed parameters of the GMSSOR method are not optimal and only lie in the convergence region of the method.

We also report the eigenvalues distributions of the MSSOR method with respect to $\omega_{opt}=0.1706$ and the GMSSOR method with respect to $\tau=0.1536$, $\omega=0.1906$ with $\hat{A}=\text{tridiag}(A)$, $m+n=768$, and the eigenvalues distributions of the MSSOR method with respect to $\omega_{opt}=0.1250$ and the GMSSOR method with respect to $\tau=0.0920$, $\omega=0.1560$ with $\hat{A}=\text{diag}(A)$, $m+n=768$, please see Figs. 8 and 9. These figures show that the eigenvalues distributions of the GMSSOR method with appropriate parameters are identically clustered as those of the MSSOR method with optimal parameters. The determination of optimum values of the parameters needs further studies.

Table 3
Spectral radius, IT and RES for Example 4.2.

m			128	512
n			64	256
$m+n$			192	768
Case I	MSSOR	ω_{opt}	0.2933	0.1706
		$\rho(\mathcal{H}(\omega_{opt}))$	0.7067	0.8294
		IT	59	122
		RES	7.9432×10^{-7}	5.1633×10^{-7}
	GMSSOR	τ	0.2323	0.1536
		ω	0.3793	0.1906
		$\rho(\mathcal{H}(\tau, \omega))$	0.6207	0.8094
		IT	47	105
		RES	6.7956×10^{-7}	6.1249×10^{-7}
Case II	MSSOR	ω_{opt}	0.2235	0.1250
		$\rho(\mathcal{H}(\omega_{opt}))$	0.7765	0.8750
		IT	91	193
		RES	7.2396×10^{-7}	9.5194×10^{-7}
	GMSSOR	τ	0.1905	0.0920
		ω	0.2505	0.1560
		IT	71	129
		RES	3.7897×10^{-7}	8.7846×10^{-7}

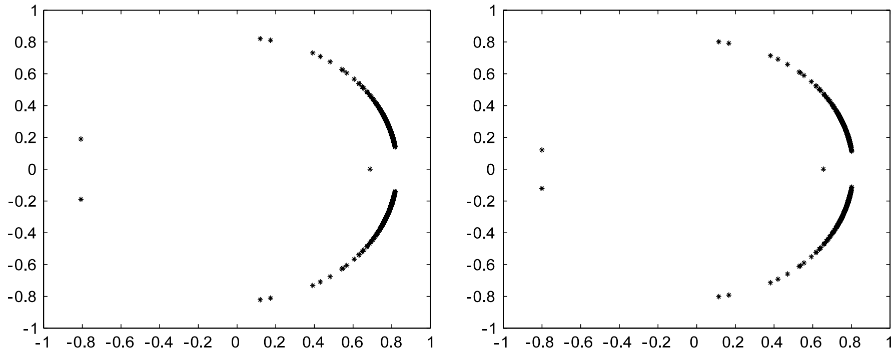

Fig. 8. Eigenvalues distributions of the MSSOR method with respect to $\omega_{opt} = 0.1706$ (the left) and the GMSSOR method with respect to $\tau = 0.1536$, $\omega = 0.1906$ (the right) with Case I, $m+n = 768$ for Example 4.2.

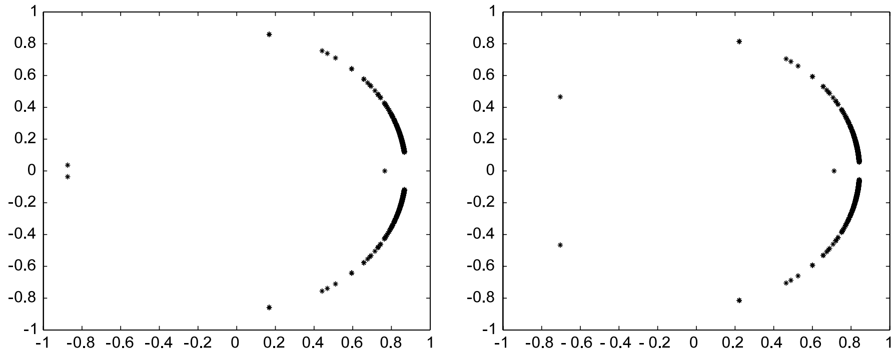

Fig. 9. Eigenvalues distributions of the MSSOR method with respect to $\omega_{opt} = 0.1250$ (the left) and the GMSSOR method with respect to $\tau = 0.0920$, $\omega = 0.1560$ (the right) with Case II, $m+n = 768$ for Example 4.2.

5. Conclusions

In this paper, we establish a generalized MSSOR (GMSSOR) method for solving the large sparse augmented systems of linear equations, which is the extension of the MSSOR method. Furthermore, the convergence of the GMSSOR method for augmented systems is analyzed and numerical experiments are carried out, which show that the GMSSOR method with appropriate parameters has a faster convergence rate than the MSSOR method with optimal parameters. However, the relaxed parameters of the GMSSOR method are not optimal and only lie in the convergence region of the method. Furthermore, the determination of optimum values of the parameters needs further studies.

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