



Oscillatory behavior of third-order nonlinear delay dynamic equations on time scales



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ABSTRACT

In this paper, a class of third-order nonlinear delay dynamic equations on time scales is studied. By using the generalized Riccati transformation and the integral averaging technique, three new sufficient conditions which ensure that every solution is oscillatory or converges to zero are established. The results obtained essentially generalize and improve earlier ones.

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1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [1], in order to unify continuous and discrete analysis. Several authors have expounded on various aspects of this new theory; see [2–6]. A time scale \mathbb{T} is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models which are discrete in season (and may follow a difference scheme with variable step-size or be modeled by continuous dynamic systems); they die out, say in winter, while their eggs are incubating or dormant, and then, in season again, hatching gives rise to a nonoverlapping population; see [4]. Not only does the new theory of so-called “dynamic equations” unify the theories of differential equations and difference equations, but it also extends these classical cases to cases “in between”, e.g., to so-called q -difference equations when $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ for some $q > 1$ (which has important applications in quantum theory) and can be applied on different types of time scales such as $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$ and the space of the harmonic numbers.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to the studies by Bohner et al. [7] and Erbe et al. [8,9]. And there are some results dealing with oscillatory behavior of second-order delay dynamic equations on time scales [10–15]. However, there are few results dealing with the oscillation of the solutions of third-order delay dynamic equations on time scales; we refer the reader to the papers [16–18].

In this paper, we consider oscillatory behavior of all solutions of the third-order nonlinear delay dynamic equation

$$\left(r_2(t) \left[(r_1(t)x^\Delta(t))^\Delta \right]^\alpha \right)^\Delta + q(t)f(x[\tau(t)]) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0, \quad (1.1)$$

where $\alpha \geq 1$ is the ratio of two positive odd integers.

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Throughout this paper, we will assume the following hypotheses.

- (H₁) \mathbb{T} is a time scale (i.e., a nonempty closed subset of the real numbers \mathbb{R}) which is unbounded above, and $t_0 \in \mathbb{T}$ with $t_0 > 0$. We define the time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$.
 (H₂) $r_1(t), r_2(t), q(t)$ are positive, real-valued rd-continuous functions (i.e., functions are said to be rd-continuous if they are continuous at each right-dense point and if there exists a finite left limit at all left-dense points) defined on \mathbb{T} , and $r_1(t), r_2(t)$ satisfy

$$\int_{t_0}^{\infty} \frac{1}{r_1(s)} \Delta s = \infty, \quad \int_{t_0}^{\infty} \left(\frac{1}{r_2(s)} \right)^{\frac{1}{\alpha}} \Delta s = \infty.$$

- (H₃) $\tau : \mathbb{T} \rightarrow \mathbb{T}$ is a strictly increasing and differentiable function such that

$$\tau(t) \leq t, \quad \lim_{t \rightarrow \infty} \tau(t) = \infty, \quad \text{and} \quad \tau(\mathbb{T}) = \mathbb{T}.$$

- (H₄) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $\frac{f(x)}{x^\alpha} \geq K > 0$ for $x \neq 0$.

By a solution of (1.1), we mean a nontrivial function $x(t)$ satisfying (1.1) which has the properties $x(t) \in C_{\text{rd}}^1([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$ for $T_x \geq t_0$, and $r_2(t) [(r_1(t)x^\Delta(t))^\Delta]^\alpha \in C_{\text{rd}}^1([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$. Our attention is restricted to those solutions of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution x of Eq. (1.1) is said to be oscillatory on $[T_x, \infty)_{\mathbb{T}}$ if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

If $\alpha = 1$, $\tau(t) = t$, then (1.1) simplifies to the third-order nonlinear dynamic equation

$$(r_2(t) [(r_1(t)x^\Delta(t))^\Delta]^\Delta + q(t)f(x(t))) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \quad (1.2)$$

If, furthermore, $r_1(t) = r_2(t) = 1, f(x) = x, \tau(t) = t$, then (1.1) reduces to the third-order linear dynamic equation

$$x^{\Delta\Delta\Delta}(t) + q(t)x(t) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \quad (1.3)$$

If, in addition, $\alpha = 1$, then (1.1) reduces to the nonlinear delay dynamic equation

$$(r_2(t) [(r_1(t)x^\Delta(t))^\Delta]^\Delta + q(t)f(x[\tau(t)])) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \quad (1.4)$$

In 2005, Erbe et al. [16] considered the general third-order nonlinear dynamic equation (1.2). By employing generalized Riccati transformation techniques, they established some sufficient conditions which ensure that every solution of Eq. (1.2) is oscillatory or converges to zero. In 2007, Erbe et al. [17] studied the third-order linear dynamic equation (1.3), and they obtained Hille and Nehari type oscillation criteria for it. In 2011, Han, Li, Sun, and Zhang [18] extended and improved the results of [16,17], meanwhile obtaining some oscillatory criteria for Eq. (1.4). On this basis, we discuss the oscillation of solutions of Eq. (1.1). By using the generalized Riccati transformation and the inequality technique, we obtain some sufficient conditions which guarantee that every solution of Eq. (1.1) is oscillatory or converges to zero.

The paper is organized as follows. In Section 2, we present some basic definitions and useful results from the theory of calculus on time scales. In Section 3, we give several lemmas. In Section 4, we use the generalized Riccati transformation and the inequality technique to obtain some sufficient conditions which guarantee that every solution of Eq. (1.1) is either oscillatory or converges to zero.

2. Some preliminaries

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g of two differentiable functions f and g :

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)), \quad (2.1)$$

$$\left(\frac{f}{g} \right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))} \quad \text{if } gg^\sigma \neq 0. \quad (2.2)$$

For $b, c \in \mathbb{T}$ and a differentiable function f , the Cauchy integral of f^Δ is defined by

$$\int_b^c f^\Delta(t) \Delta t = f(c) - f(b).$$

The integration by parts formula reads

$$\int_b^c f^\Delta(t)g(t) \Delta t = f(c)g(c) - f(b)g(b) - \int_b^c f^\sigma(t)g^\Delta(t) \Delta t,$$

and infinite integrals are defined by

$$\int_b^\infty f(s) \Delta s = \lim_{t \rightarrow \infty} \int_b^t f(s) \Delta s.$$

For more details, see [4,5].

3. Several lemmas

In this section, we present several lemmas that will be needed in the proofs of our results in Section 4.

Lemma 3.1. Assume that $x(t)$ is an eventually positive solution of (1.1). Then there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that either

- (I) $x(t) > 0$, $x^\Delta(t) > 0$, $(r_1(t)x^\Delta(t))^\Delta > 0$, $t \in [T, \infty)_{\mathbb{T}}$;
 or
 (II) $x(t) > 0$, $x^\Delta(t) < 0$, $(r_1(t)x^\Delta(t))^\Delta > 0$, $t \in [T, \infty)_{\mathbb{T}}$.

The proof is similar to that of [16, Lemma 1].

Lemma 3.2 ([4, Theorem 1.90]). If x is differentiable, then

$$(x^\gamma)^\Delta = \gamma x^\Delta \int_0^1 [hx^\sigma + (1-h)x]^{\gamma-1} dh. \quad (3.1)$$

Lemma 3.3 ([19, Theorem 41]). Assume that X and Y are nonnegative real numbers. Then

$$\lambda XY^{\lambda-1} - X^\lambda \leq (\lambda - 1)Y^\lambda \quad \text{for all } \lambda > 1, \quad (3.2)$$

where the equality holds if and only if $X = Y$.

Throughout this paper, for sufficiently large T , we denote

$$R_1(t, T) = \int_T^t \left(\frac{1}{r_2(s)} \right)^{\frac{1}{\alpha}} \Delta s,$$

$$R_2(t, T) = \int_T^t \frac{R_1(s, T)}{r_1(s)} \Delta s.$$

Lemma 3.4. Assume that $x(t)$ is an eventually positive solution of (1.1) which satisfies case (I) in Lemma 3.1. Then there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that

$$x^\Delta(t) \geq \frac{R_1(t, T)}{r_1(t)} r_2^{\frac{1}{\alpha}}(t) (r_1(t)x^\Delta(t))^\Delta, \quad t \in [T, \infty)_{\mathbb{T}} \quad (3.3)$$

$$x(t) > R_2(t, T) r_2^{\frac{1}{\alpha}}(t) (r_1(t)x^\Delta(t))^\Delta, \quad t \in [T, \infty)_{\mathbb{T}}. \quad (3.4)$$

Proof. Pick $T \in [t_0, \infty)_{\mathbb{T}}$ so that $x[\tau(t)] > 0$ on $[T, \infty)_{\mathbb{T}}$. Using (1.1), we obtain

$$\left(r_2(t) \left[(r_1(t)x^\Delta(t))^\Delta \right]^\alpha \right)^\Delta = -q(t)f(x[\tau(t)]) \leq -Kq(t)x^\alpha[\tau(t)] < 0, \quad T \in [t_0, \infty)_{\mathbb{T}}.$$

Then $r_2(t) \left[(r_1(t)x^\Delta(t))^\Delta \right]^\alpha$ is strictly decreasing on $[T, \infty)_{\mathbb{T}}$. We get

$$\begin{aligned} r_1(t)x^\Delta(t) &\geq r_1(t)x^\Delta(t) - r_1(T)x^\Delta(T) = \int_T^t \frac{\left(r_2(s) \left[(r_1(s)x^\Delta(s))^\Delta \right]^\alpha \right)^{\frac{1}{\alpha}}}{r_2^{\frac{1}{\alpha}}(s)} \Delta s \\ &\geq \left(r_2(t) \left[(r_1(t)x^\Delta(t))^\Delta \right]^\alpha \right)^{\frac{1}{\alpha}} \int_T^t \left(\frac{1}{r_2(s)} \right)^{\frac{1}{\alpha}} \Delta s. \end{aligned}$$

Hence, we obtain

$$x^\Delta(t) \geq \frac{R_1(t, T)}{r_1(t)} r_2^{\frac{1}{\alpha}}(t) (r_1(t)x^\Delta(t))^\Delta, \quad T \in [t_0, \infty)_{\mathbb{T}}.$$

Integrating both sides of the above inequality yields

$$x(t) > R_2(t, T) r_2^{\frac{1}{\alpha}}(t) (r_1(t)x^\Delta(t))^\Delta, \quad T \in [t_0, \infty)_{\mathbb{T}}.$$

This completes the proof.

Lemma 3.5. Assume that $x(t)$ is an eventually positive solution of (1.1) which satisfies case (I) in Lemma 3.1. Furthermore, assume that $r_1^\Delta(t) \leq 0$ and

$$\int_{t_0}^{\infty} r_1(s) \tau^\alpha(s) \Delta s = \infty. \quad (3.5)$$

Then there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > tx^\Delta(t)$, and $\frac{x(t)}{t}$ is strictly decreasing on $[T, \infty)_{\mathbb{T}}$.

The proof is similar to that of Han et al. [18, Lemma 2.3].

Lemma 3.6. Assume that $x(t)$ is an eventually positive solution of (1.1) which satisfies case (II) in Lemma 3.1. Furthermore,

$$\int_{t_0}^{\infty} \frac{1}{r_1(t)} \int_t^{\infty} \left[\frac{1}{r_2(s)} \int_s^{\infty} q(u) \Delta u \right]^{\frac{1}{\alpha}} \Delta s \Delta t = \infty. \quad (3.6)$$

Then $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Assume that $x(t)$ is an eventually positive solution of (1.1) which satisfies case (II) in Lemma 3.1. Then $x(t)$ is decreasing and $\lim_{t \rightarrow \infty} x(t) = l \geq 0$. We assert that $l = 0$. If not, then $x[\tau(t)] \geq x(t) \geq l > 0$ for $t \in [T, \infty)_{\mathbb{T}}$. Integrating (1.1) from t to ∞ , we get

$$-r_2(t) \left[(r_1(t)x^\Delta(t))^\Delta \right]^\alpha \leq -K \int_t^{\infty} q(s) x^\alpha[\tau(s)] \Delta s \leq -K l^\alpha \int_t^{\infty} q(s) \Delta s, \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

Hence, we obtain

$$-(r_1(t)x^\Delta(t))^\Delta \leq -l \left[\frac{1}{r_2(t)} \int_t^{\infty} Kq(s) \Delta s \right]^{\frac{1}{\alpha}}.$$

Integrating the above inequality from t to ∞ , we obtain

$$r_1(t)x^\Delta(t) \leq -K l^{\frac{1}{\alpha}} \int_t^{\infty} \left[\frac{1}{r_2(s)} \int_s^{\infty} q(u) \Delta u \right]^{\frac{1}{\alpha}} \Delta s.$$

Integrating the last inequality again from T to t , we obtain

$$x(t) - x(T) \leq -K l^{\frac{1}{\alpha}} \int_T^t \frac{1}{r_1(s)} \int_s^{\infty} \left[\frac{1}{r_2(u)} \int_u^{\infty} q(v) \Delta v \right]^{\frac{1}{\alpha}} \Delta u \Delta s.$$

Since condition (3.6) holds, we obtain $\lim_{t \rightarrow \infty} x(t) = -\infty$, which contradicts $x(t) > 0$. Hence $l = 0$. This completes the proof.

4. Main results

The following is the main result of this paper.

Theorem 4.1. Let (3.5), (3.6), and $r_1^\Delta(t) \leq 0$ hold. Assume that there exists a positive function $\delta \in C_{\text{rd}}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, and, for all sufficiently large T , that there exists $T_0 > T$,

$$\limsup_{t \rightarrow \infty} \int_{T_0}^t \left[Q(s) - \frac{r_1(s) \sigma^\alpha(s) (\delta^\Delta(s))^2}{4\alpha s^\alpha \delta(\sigma(s)) R(s, T)} \right] \Delta s = \infty, \quad (4.1)$$

where $R(t, T) = R_1(t, T) R_2^{\alpha-1}(t, T)$, $Q(t) = Kq(t) \delta(\sigma(t)) \left(\frac{\tau(t)}{\sigma(t)} \right)^\alpha$. Then every solution $x(t)$ of Eq. (1.1) is either oscillatory or converges to zero.

Proof. Assume that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists sufficiently large $T \geq t_0$ such that $x(t) > 0$ and $x[\tau(t)] > 0$ for all $t \in [T, \infty)_{\mathbb{T}}$. By Lemma 3.1, we see that $x(t)$ satisfies either case (I) or case (II).

If case (I) holds, then $x^\Delta(t) > 0$, $t \in [T, \infty)_{\mathbb{T}}$. Define the function $W(t)$ by

$$W(t) = \delta(t) r_2(t) \left(\frac{(r_1(t)x^\Delta(t))^\Delta}{x(t)} \right)^\alpha, \quad t \in [T, \infty)_{\mathbb{T}}.$$

Then $W(t) > 0$. By the product (2.1) and then the quotient rule (2.2),

$$W^\Delta(t) = \frac{\delta^\Delta(t)}{\delta(t)} W(t) - \delta(\sigma(t)) \frac{q(t) f(x[\tau(t)])}{x^\alpha(\sigma(t))} - \delta(\sigma(t)) r_2(t) \left[\frac{(r_1(t)x^\Delta(t))^\Delta}{x(t)} \right]^\alpha \frac{(x^\alpha(t))^\Delta}{x^\alpha(\sigma(t))}.$$

By Lemma 3.2, we get

$$W^\Delta(t) \leq \frac{(\delta^\Delta(t))_+}{\delta(t)} W(t) - \delta(\sigma(t)) \frac{q(t)f(x[\tau(t)])}{x^\alpha(\sigma(t))} - \alpha \delta(\sigma(t)) r_2(t) \left[\frac{(r_1(t)x^\Delta(t))^\Delta}{x(t)} \right]^\alpha \frac{x^\alpha(t)x^\Delta(t)}{x^\alpha(\sigma(t))x(t)},$$

where $(\delta^\Delta(t))_+ = \max\{0, \delta^\Delta(t)\}$.

Using (3.3) and (3.4), there exists $T_0 \in (T, \infty)_\mathbb{T}$ such that

$$\begin{aligned} \frac{x^\Delta(t)}{x(t)} &> \frac{R_1(t, T)}{r_1(t)} r_2(t) \left[\frac{(r_1(t)x^\Delta(t))^\Delta}{x(t)} \right]^\alpha \left[\frac{x(t)}{r_2^\frac{1}{\alpha}(t)(r_1(t)x^\Delta(t))^\Delta} \right]^{\alpha-1} \\ &> \frac{R_1(t, T)R_2^{\alpha-1}(t, T)}{r_1(t)} r_2(t) \left[\frac{(r_1(t)x^\Delta(t))^\Delta}{x(t)} \right]^\alpha \\ &= \frac{R(t, T)}{r_1(t)} r_2(t) \left[\frac{(r_1(t)x^\Delta(t))^\Delta}{x(t)} \right]^\alpha, \quad t \geq T_0. \end{aligned}$$

Also, from Lemma 3.5, we obtain $\frac{x(\tau(t))}{x(\sigma(t))} \geq \frac{\tau(t)}{\sigma(t)}, \frac{x(t)}{x(\sigma(t))} \geq \frac{t}{\sigma(t)}$, so we obtain

$$W^\Delta(t) \leq -K\delta(\sigma(t))q(t) \left(\frac{\tau(t)}{\sigma(t)} \right)^\alpha + \frac{(\delta^\Delta(t))_+}{\delta(t)} W(t) - \alpha \delta(\sigma(t)) \left(\frac{t}{\sigma(t)} \right)^\alpha \frac{R(t, T)}{r_1(t)} \left[r_2(t) \left(\frac{(r_1(t)x^\Delta(t))^\Delta}{x(t)} \right)^\alpha \right]^2.$$

By the definition of $W(t)$, we obtain

$$W^\Delta(t) \leq -K\delta(\sigma(t))q(t) \left(\frac{\tau(t)}{\sigma(t)} \right)^\alpha + \frac{(\delta^\Delta(t))_+}{\delta(t)} W(t) - \alpha \frac{\delta(\sigma(t))}{\delta^2(t)} \left(\frac{t}{\sigma(t)} \right)^\alpha \frac{R(t, T)}{r_1(t)} W^2(t). \quad (4.2)$$

From (4.2), noting the definition of $Q(t)$, we obtain

$$Q(t) \leq -W^\Delta(t) + \frac{(\delta^\Delta(t))_+}{\delta(t)} W(t) - \alpha \frac{\delta(\sigma(t))}{\delta^2(t)} \left(\frac{t}{\sigma(t)} \right)^\alpha \frac{R(t, T)}{r_1(t)} W^2(t). \quad (4.3)$$

By the averaging technique, we obtain

$$\frac{(\delta^\Delta(t))_+}{\delta(t)} W(t) - \alpha \frac{\delta(\sigma(t))}{\delta^2(t)} \left(\frac{t}{\sigma(t)} \right)^\alpha \frac{R(t, T)}{r_1(t)} W^2(t) \leq \frac{r_1(t)\sigma^\alpha(t)(\delta^\Delta(t))^2}{4\alpha t^\alpha \delta(\sigma(t))R(t, T)},$$

and hence we obtain

$$W^\Delta(t) \leq - \left[Q(t) - \frac{r_1(t)\sigma^\alpha(t)(\delta^\Delta(t))^2}{4\alpha t^\alpha \delta(\sigma(t))R(t, T)} \right]. \quad (4.4)$$

Integrating (4.4) from T_0 to t , we get

$$-W(T_0) \leq W(t) - W(T_0) \leq - \int_{T_0}^t \left[Q(s) - \frac{r_1(s)\sigma^\alpha(s)(\delta^\Delta(s))^2}{4\alpha s^\alpha \delta(\sigma(s))R(s, T)} \right] \Delta s.$$

This yields

$$\int_{T_0}^t \left[Q(s) - \frac{r_1(s)\sigma^\alpha(s)(\delta^\Delta(s))^2}{4\alpha s^\alpha \delta(\sigma(s))R(s, T)} \right] \Delta s \leq W(T_0).$$

This is contrary to (4.1).

If case (II) holds, from (3.6), by Lemma 3.6, $\lim_{t \rightarrow \infty} x(t) = 0$. This completes the proof.

Theorem 4.2. Let (3.5), (3.6), and $r_1^\Delta(t) \leq 0$ hold. Assume that there exist functions $H, m \in C_{rd}(\mathbb{D}, \mathbb{R})$, where $\mathbb{D} \equiv \{(t, s) : t \geq s \geq T\}$ such that

$$H(t, t) = 0, \quad t \geq T; \quad H(t, s) > 0, \quad t > s \geq T, \quad (4.5)$$

and H has a nonpositive continuous Δ -partial derivative $H^{\Delta_s}(t, s)$ with respect to the second variable and satisfies

$$-H^{\Delta_s}(\sigma(t), \sigma(s)) - \frac{(\delta^\Delta(s))_+}{\delta(s)} H(\sigma(t), \sigma(s)) = \frac{m(t, s)}{\delta(s)} \sqrt{H(\sigma(t), \sigma(s))}, \quad (4.6)$$

and, for all sufficiently large T , that there exists $T_0 > T$,

$$\limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \left[H(\sigma(t), s)Q(s) - \frac{m_-^2(t, s)\sigma^\alpha(s)r_1(s)}{4\alpha\delta(\sigma(s))s^\alpha R(s, T)} \right] \Delta s = \infty, \quad (4.7)$$

where $\delta(t)$ is a positive Δ -differentiable function and $R(t, T)$, $Q(t)$ are defined in Theorem 4.1.

$$m_-(t, s) = \max\{0, -m(t, s)\}, \quad m_+(t, s) = \max\{0, m(t, s)\}.$$

Then every solution $x(t)$ of Eq. (1.1) is either oscillatory or converges to zero.

Proof. Assume that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists sufficiently large $T \geq t_0$ such that $x(t) > 0$ and $x[\tau(t)] > 0$ for all $t \in [T, \infty)_{\mathbb{T}}$. By Lemma 3.1, we see that $x(t)$ satisfies either case (I) or case (II).

If case (I) holds, we proceed as in the proof of Theorem 4.1 and get (4.3). Multiplying both sides of (4.3), with t replaced by s , by $H(\sigma(t), \sigma(s))$, and integrating with respect to s from T_0 to $\sigma(t)$, $t \geq T_0$, we get

$$\begin{aligned} \int_{T_0}^{\sigma(t)} H(\sigma(t), \sigma(s))Q(s)\Delta s &\leq - \int_{T_0}^{\sigma(t)} H(\sigma(t), \sigma(s))W^{\Delta}(s)\Delta s + \int_{T_0}^{\sigma(t)} \frac{H(\sigma(t), \sigma(s))(\delta^{\Delta}(s))_+}{\delta(s)} W(s)\Delta s \\ &\quad - \int_{T_0}^{\sigma(t)} \frac{\alpha H(\sigma(t), \sigma(s))\delta(\sigma(s))s^\alpha R(s, T)}{\delta^2(s)\sigma^\alpha(s)r_1(s)} W^2(s)\Delta s. \end{aligned}$$

Integrating by parts and using (4.5), we obtain

$$\begin{aligned} \int_{T_0}^{\sigma(t)} H(\sigma(t), \sigma(s))Q(s)\Delta s &\leq H(\sigma(t), T_0)W(T_0) + \int_{T_0}^{\sigma(t)} H^{\Delta s}(\sigma(t), s)W(s)\Delta s \\ &\quad + \int_{T_0}^{\sigma(t)} \frac{H(\sigma(t), \sigma(s))(\delta^{\Delta}(s))_+}{\delta(s)} W(s)\Delta s \\ &\quad - \int_{T_0}^{\sigma(t)} \frac{\alpha H(\sigma(t), \sigma(s))\delta(\sigma(s))s^\alpha R(s, T)}{\delta^2(s)\sigma^\alpha(s)r_1(s)} W^2(s)\Delta s \\ &\leq H(\sigma(t), T_0)W(T_0) + \int_{T_0}^{\sigma(t)} \left[-\frac{m(t, s)\sqrt{H(\sigma(t), \sigma(s))}}{\delta(s)} W(s) \right. \\ &\quad \left. - \frac{\alpha H(\sigma(t), \sigma(s))\delta(\sigma(s))s^\alpha R(s, T)}{\delta^2(s)\sigma^\alpha(s)r_1(s)} W^2(s) \right] \Delta s \\ &\leq H(\sigma(t), T_0)W(T_0) + \int_{T_0}^{\sigma(t)} \left[\frac{m_-(t, s)\sqrt{H(\sigma(t), \sigma(s))}}{\delta(s)} W(s) \right. \\ &\quad \left. - \frac{\alpha H(\sigma(t), \sigma(s))\delta(\sigma(s))s^\alpha R(s, T)}{\delta^2(s)\sigma^\alpha(s)r_1(s)} W^2(s) \right] \Delta s. \end{aligned} \quad (4.8)$$

Using the averaging technique, we obtain

$$\frac{m_-(t, s)\sqrt{H(\sigma(t), \sigma(s))}}{\delta(s)} W(s) - \frac{\alpha H(\sigma(t), \sigma(s))\delta(\sigma(s))s^\alpha R(s, T)}{\delta^2(s)\sigma^\alpha(s)r_1(s)} W^2(s) \leq \frac{m_-^2(t, s)\sigma^\alpha(s)r_1(s)}{4\alpha\delta(\sigma(s))s^\alpha R(s, T)}. \quad (4.9)$$

Combining (4.8) and (4.9), we get

$$\frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \left[H(\sigma(t), s)Q(s) - \frac{m_-^2(t, s)\sigma^\alpha(s)r_1(s)}{4\alpha\delta(\sigma(s))s^\alpha R(s, T)} \right] \Delta s \leq W(T_0),$$

which contradicts (4.7).

If case (II) holds, from (3.6), by Lemma 3.6, $\lim_{t \rightarrow \infty} x(t) = 0$. This completes the proof.

Theorem 4.3. Let (3.5), (3.6), and $r_1^{\Delta}(t) \leq 0$ hold. Assume that there exist functions $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$ such that

$$H^{\Delta s}(\sigma(t), \sigma(s)) + \frac{(\delta^{\Delta}(s))_+}{\delta(s)} H(\sigma(t), \sigma(s)) = -\frac{h(t, s)}{\delta(s)} H^{\frac{\alpha}{\alpha+1}}(\sigma(t), \sigma(s)), \quad (4.10)$$

and, for all sufficiently large T , that there exists $T_0 > T$,

$$\limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \left[H(\sigma(t), s)Q(s) - \frac{h_-^{\alpha+1}(t, s)[\sigma^\alpha(s)r_1(s)]^\alpha}{(\alpha+1)^{\alpha+1}[\delta(\sigma(s))s^\alpha R_1(s, T)]^\alpha} \right] \Delta s = \infty, \quad (4.11)$$

where $\delta(t)$ is a positive Δ -differentiable function, H is defined in Theorem 4.2, and $Q(t)$ is defined in Theorem 4.1.

$$h_-(t, s) = \max\{0, -h(t, s)\}, \quad h_+(t, s) = \max\{0, h(t, s)\}.$$

Then every solution $x(t)$ of Eq. (1.1) is either oscillatory or converges to zero.

Proof. Assume that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists sufficiently large $T \geq t_0$ such that $x(t) > 0$ and $x[\tau(t)] > 0$ for all $t \in [T, \infty)_{\mathbb{T}}$. By Lemma 3.1, we see that $x(t)$ satisfies either case (I) or case (II).

If case (I) holds, we proceed as in the proof of Theorem 4.1, and get

$$W^\Delta(t) \leq \frac{(\delta^\Delta(t))_+}{\delta(t)} W(t) - \delta(\sigma(t)) \frac{q(t)f(x[\tau(t)])}{x^\alpha(\sigma(t))} - \alpha \delta(\sigma(t)) r_2(t) \left[\frac{(r_1(t)x^\Delta(t))^\Delta}{x(t)} \right]^\alpha \frac{x^\alpha(t)x^\Delta(t)}{x^\alpha(\sigma(t))x(t)}.$$

From Lemma 3.5, we obtain $\frac{x(\tau(t))}{x(\sigma(t))} \geq \frac{\tau(t)}{\sigma(t)}$, $\frac{x(t)}{x(\sigma(t))} \geq \frac{t}{\sigma(t)}$, and, using (3.3), we obtain

$$\begin{aligned} W^\Delta(t) &\leq -K\delta(\sigma(t))q(t) \left(\frac{\tau(t)}{\sigma(t)} \right)^\alpha + \frac{(\delta^\Delta(t))_+}{\delta(t)} W(t) - \frac{\alpha \delta(\sigma(t))R_1(t, T)}{r_1(t)} \\ &\quad \times \left(\frac{t}{\sigma(t)} \right)^\alpha r_2^{1+\frac{1}{\alpha}}(t) \left[\frac{(r_1(t)x^\Delta(t))^\Delta}{x(t)} \right]^{\alpha+1}. \end{aligned}$$

Hence, by the definition of $W(t)$, $Q(t)$, we obtain

$$W^\Delta(t) \leq -Q(t) + \frac{(\delta^\Delta(t))_+}{\delta(t)} W(t) - \frac{\alpha \delta(\sigma(t))R_1(t, T)}{\delta^{1+\frac{1}{\alpha}}(t)r_1(t)} \left(\frac{t}{\sigma(t)} \right)^\alpha W^{1+\frac{1}{\alpha}}(t). \quad (4.12)$$

Multiplying both sides of (4.12), with t replaced by s , by $H(\sigma(t), \sigma(s))$, and integrating with respect to s from T_0 to $\sigma(t)$, $t \geq T_0$, we get

$$\begin{aligned} \int_{T_0}^{\sigma(t)} H(\sigma(t), \sigma(s))Q(s)\Delta s &\leq - \int_{T_0}^{\sigma(t)} H(\sigma(t), \sigma(s))W^\Delta(s)\Delta s + \int_{T_0}^{\sigma(t)} \frac{H(\sigma(t), \sigma(s))(\delta^\Delta(s))_+}{\delta(s)} W(s)\Delta s \\ &\quad - \int_{T_0}^{\sigma(t)} \frac{\alpha H(\sigma(t), \sigma(s))\delta(\sigma(s))s^\alpha R_1(s, T)}{\delta^{1+\frac{1}{\alpha}}(s)\sigma^\alpha(s)r_1(s)} W^{1+\frac{1}{\alpha}}(s)\Delta s. \end{aligned}$$

Integrating by parts and using (4.5), we obtain

$$\begin{aligned} \int_{T_0}^{\sigma(t)} H(\sigma(t), \sigma(s))Q(s)\Delta s &\leq H(\sigma(t), T_0)W(T_0) + \int_{T_0}^{\sigma(t)} H^\Delta(s)(\sigma(t), s)W(s)\Delta s \\ &\quad + \int_{T_0}^{\sigma(t)} \frac{H(\sigma(t), \sigma(s))(\delta^\Delta(s))_+}{\delta(s)} W(s)\Delta s \\ &\quad - \int_{T_0}^{\sigma(t)} \frac{\alpha H(\sigma(t), \sigma(s))\delta(\sigma(s))s^\alpha R_1(s, T)}{\delta^{1+\frac{1}{\alpha}}(s)\sigma^\alpha(s)r_1(s)} W^{1+\frac{1}{\alpha}}(s)\Delta s \\ &\leq H(\sigma(t), T_0)W(T_0) + \int_{T_0}^{\sigma(t)} \left[-\frac{h(t, s)H^{\frac{\alpha}{1+\alpha}}(\sigma(t), \sigma(s))}{\delta(s)} W(s) \right. \\ &\quad \left. - \frac{\alpha H(\sigma(t), \sigma(s))\delta(\sigma(s))s^\alpha R_1(s, T)}{\delta^{1+\frac{1}{\alpha}}(s)\sigma^\alpha(s)r_1(s)} W^{1+\frac{1}{\alpha}}(s) \right] \Delta s \\ &\leq H(\sigma(t), T_0)W(T_0) + \int_{T_0}^{\sigma(t)} \left[\frac{h_-(t, s)H^{\frac{\alpha}{1+\alpha}}(\sigma(t), \sigma(s))}{\delta(s)} W(s) \right. \\ &\quad \left. - \frac{\alpha H(\sigma(t), \sigma(s))\delta(\sigma(s))s^\alpha R_1(s, T)}{\delta^{1+\frac{1}{\alpha}}(s)\sigma^\alpha(s)r_1(s)} W^{1+\frac{1}{\alpha}}(s) \right] \Delta s. \end{aligned} \quad (4.13)$$

Now, set

$$X^\lambda = \frac{\alpha H(\sigma(t), \sigma(s)) \delta(\sigma(s)) s^\alpha R_1(s, T)}{\delta^{1+\frac{1}{\alpha}}(s) \sigma^\alpha(s) r_1(s)} W^\lambda(s),$$

$$Y^{\lambda-1} = \frac{h_-(t, s) [\sigma^\alpha(s) r_1(s)]^{\frac{1}{\lambda}}}{\lambda [\alpha \delta(s) s^\alpha R_1(s, T)]^{\frac{1}{\lambda}}},$$

where $\lambda = \frac{\alpha+1}{\alpha} > 1$, $X \geq 0$ and $Y \geq 0$. Using inequality (3.2), we obtain

$$\frac{h_-(t, s) H^{\frac{1}{\lambda}}(\sigma(t), \sigma(s))}{\delta(s)} W(s) - \frac{\alpha H(\sigma(t), \sigma(s)) \delta(\sigma(s)) s^\alpha R_1(s, T)}{\delta^{\lambda-1}(s) \sigma^\alpha(s) r_1(s)} W^\lambda(s) \leq \frac{h_-^{\alpha+1}(t, s) [\sigma^\alpha(s) r_1(s)]^\alpha}{(\alpha+1)^{\alpha+1} [\delta(\sigma(s)) s^\alpha R_1(s, T)]^\alpha}. \quad (4.14)$$

Combining (4.13) and (4.14), we get

$$\frac{1}{H(\sigma(t), T_0)} \int_{T_0}^t \left[H(\sigma(t), s) Q(s) - \frac{h_-^{\alpha+1}(t, s) [\sigma^\alpha(s) r_1(s)]^\alpha}{(\alpha+1)^{\alpha+1} [\delta(\sigma(s)) s^\alpha R_1(s, T)]^\alpha} \right] \Delta s \leq W(T_0),$$

which contradicts (4.11).

If case (II) holds, from (3.6), by Lemma 3.6, $\lim_{t \rightarrow \infty} x(t) = 0$. This completes the proof.

Remark 4.1. If we let $\alpha = 1$ in Theorems 4.1 and 4.2, conditions (4.1) and (4.7) are (2.7) and (2.14) in [18]. Therefore, the result that $\alpha = 1$ in [18] is improved to be that α is a ratio of positive odd integers. When $\tau(t) = t$, $\alpha = 1$, and $H(t, s) = (\frac{t-s}{t})^m$ in Theorem 4.2, condition (4.7) is converted to (3.10) in [16], which can be considered as an extension of the oscillatory criterion for third-order nonlinear dynamic equations (see [16]).

Example 4.1. Consider the third-order nonlinear delay dynamic equation

$$\left(t^{\frac{2}{3}} (x^{\Delta\Delta}(t))^{\frac{5}{3}} \right)^\Delta + \frac{1}{t^2} \left(x \left(\frac{t}{2} \right) \right)^{\frac{5}{3}} \left(1 + x^2 \left(\frac{t}{2} \right) \right) = 0, \quad t \in \overline{2\mathbb{Z}}, \quad t \geq t_0 := 2. \quad (4.15)$$

Here, $\alpha = \frac{5}{3}$, $r_1(t) = 1$, $r_2(t) = t^{\frac{2}{3}}$, $q(t) = \frac{1}{t^2}$, $f(x) = x^{\frac{5}{3}}(1+x^2)$, and $\tau(t) = \frac{t}{2} < t$.

Conditions (H₁)–(H₃) are clearly satisfied, and (H₄) holds with $K = 1$. $r_1^\Delta(t) = 0$, and

$$\int_2^\infty r_1(s) \tau^\alpha(s) \Delta s = \int_2^\infty \left(\frac{s}{2} \right)^{\frac{5}{3}} \Delta s = 2^{-\frac{5}{3}} \frac{s^{\frac{8}{3}}}{2^{\frac{8}{3}-1}} \Big|_2^\infty = \infty,$$

$$\int_2^\infty \frac{1}{r_1(t)} \int_t^\infty \left[\frac{1}{r_2(s)} \int_s^\infty q(u) \Delta u \right]^{\frac{1}{\alpha}} \Delta s \Delta t = \int_2^\infty 1 \int_t^\infty \left[\frac{1}{s^{\frac{2}{3}}} \int_s^\infty \frac{1}{u^2} \Delta u \right]^{\frac{3}{5}} \Delta s \Delta t = \int_2^\infty \int_t^\infty \frac{2^{\frac{3}{5}}}{s} \Delta s \Delta t = \infty,$$

so (3.5) and (3.6) hold. Noting that

$$R_1(t, T) = \int_T^t \left(\frac{1}{r_2(s)} \right)^{\frac{1}{\alpha}} \Delta s = \int_T^t s^{-\frac{2}{5}} \Delta s = \frac{s^{\frac{3}{5}}}{2^{\frac{3}{5}} - 1} \Big|_T^t = \frac{t^{\frac{3}{5}} - T^{\frac{3}{5}}}{2^{\frac{3}{5}} - 1},$$

$$R_2(t, T) = \int_T^t \frac{R_1(s, T)}{r_1(s)} \Delta s = \int_T^t \frac{s^{\frac{3}{5}} - T^{\frac{3}{5}}}{2^{\frac{3}{5}} - 1} \Delta s = \frac{t^{\frac{8}{5}}}{(2^{\frac{3}{5}} - 1)(2^{\frac{8}{5}} - 1)} - \frac{T^{\frac{3}{5}} t}{(2^{\frac{3}{5}} - 1)} + \frac{2T^{\frac{8}{5}}}{(2^{\frac{8}{5}} - 1)},$$

$$R(t, T) = R_1(t, T) R_2^{\alpha-1}(t, T) = \frac{t^{\frac{3}{5}} - T^{\frac{3}{5}}}{2^{\frac{3}{5}} - 1} \left[\frac{t^{\frac{8}{5}}}{(2^{\frac{3}{5}} - 1)(2^{\frac{8}{5}} - 1)} - \frac{T^{\frac{3}{5}} t}{(2^{\frac{3}{5}} - 1)} + \frac{2T^{\frac{8}{5}}}{(2^{\frac{8}{5}} - 1)} \right]^{\frac{2}{3}}.$$

Let $\delta(t) = \frac{t}{2}$. Since $\sigma(t) = 2t$, we obtain $Q(t) = Kq(t)\delta(\sigma(t))(\frac{\tau(t)}{\sigma(t)})^\alpha = \frac{1}{4^{\frac{3}{5}}t}$, and it follows that

$$\limsup_{t \rightarrow \infty} \int_{T_0}^t \left[Q(s) - \frac{r_1(s) \sigma^\alpha(s) (\delta^\Delta(s))^2}{4\alpha s^\alpha \delta(\sigma(s)) R(s, T)} \right] \Delta s = \limsup_{t \rightarrow \infty} \int_{T_0}^t \left[\frac{1}{4^{\frac{3}{5}}s} - \frac{3}{5 \cdot 2^{\frac{7}{5}} s R(s, T)} \right] \Delta s = \infty.$$

Then, by Theorem 4.1, every solution $x(t)$ of Eq. (4.15) is either oscillatory or converges to zero.

Example 4.2. Consider the third-order nonlinear delay dynamic equation

$$\left(t^{\frac{2}{3}} \left(\left(\frac{1}{t} x^{\Delta}(t) \right)^{\Delta} \right)^{\frac{5}{3}} \right)^{\Delta} + \frac{1}{t^2} \left(x \left(\frac{t}{2} \right) \right)^{\frac{5}{3}} \left(1 + \ln \left(1 + x^2 \left(\frac{t}{2} \right) \right) \right) = 0, \quad t \in \overline{2\mathbb{Z}}, \quad t \geq t_0 := 2. \quad (4.16)$$

Here, $\alpha = \frac{5}{3}$, $r_1(t) = \frac{1}{t}$, $r_2(t) = t^{\frac{2}{3}}$, $q(t) = \frac{1}{t^2}$, $f(x) = x^{\frac{5}{3}}(1 + \ln(1 + x^2))$, and $\tau(t) = \frac{t}{2} < t$.

Conditions (H₁)–(H₃) are clearly satisfied, and (H₄) holds with $K = 1$. $r_1^{\Delta}(t) = -\frac{1}{2t^2} < 0$, and

$$\int_{t_0}^{\infty} r_1(s) \tau^{\alpha}(s) \Delta s = 2^{-\frac{5}{3}} \int_2^{\infty} s^{\frac{2}{3}} \Delta s = \infty, \\ \int_{t_0}^{\infty} \frac{1}{r_1(t)} \int_t^{\infty} \left[\frac{1}{r_2(s)} \int_s^{\infty} q(u) \Delta u \right]^{\frac{1}{\alpha}} \Delta s \Delta t = \int_2^{\infty} t \int_t^{\infty} \left[\frac{1}{s^{\frac{2}{3}}} \int_s^{\infty} \frac{1}{u^2} \Delta u \right]^{\frac{3}{5}} \Delta s \Delta t = \int_2^{\infty} t \int_t^{\infty} \frac{2^{\frac{3}{5}}}{s} \Delta s \Delta t = \infty,$$

so (3.5) and (3.6) hold. Noting that

$$R_1(t, T) = \int_T^t \left(\frac{1}{r_2(s)} \right)^{\frac{1}{\alpha}} \Delta s = \int_T^t s^{-\frac{2}{5}} \Delta s = \frac{t^{\frac{3}{5}} - T^{\frac{3}{5}}}{2^{\frac{3}{5}} - 1}, \\ R_2(t, T) = \int_T^t \frac{R_1(s, T)}{r_1(s)} \Delta s = \int_T^t \frac{s^{\frac{8}{5}} - T^{\frac{3}{5}} s}{2^{\frac{3}{5}} - 1} \Delta s = \frac{t^{\frac{13}{5}} - T^{\frac{13}{5}}}{(2^{\frac{3}{5}} - 1)(2^{\frac{13}{5}} - 1)} - \frac{T^{\frac{3}{5}}(t^2 - T^2)}{3(2^{\frac{3}{5}} - 1)}, \\ R(t, T) = R_1(t, T) R_2^{\alpha-1}(t, T) = \frac{t^{\frac{3}{5}} - T^{\frac{3}{5}}}{2^{\frac{3}{5}} - 1} \left[\frac{t^{\frac{13}{5}} - T^{\frac{13}{5}}}{(2^{\frac{3}{5}} - 1)(2^{\frac{13}{5}} - 1)} - \frac{T^{\frac{3}{5}}(t^2 - T^2)}{3(2^{\frac{3}{5}} - 1)} \right]^{\frac{2}{3}}.$$

Let $\delta(t) = t$. Since $\sigma(t) = 2t$, we obtain

$$Q(t) = Kq(t)\delta(\sigma(t)) \left(\frac{\tau(t)}{\sigma(t)} \right)^{\alpha} = \frac{1}{2^{\frac{7}{3}}t}.$$

Let $H(t, s) = (t - s)^2$, that there exists a function $m(t, s) = -\frac{2t^2 - 6st + 5s^2}{t - s}$ such that

$$-H^{\Delta_s}(\sigma(t), \sigma(s)) - \frac{(\delta^{\Delta}(s))_+}{\delta(s)} H(\sigma(t), \sigma(s)) = \frac{m(t, s)}{\delta(s)} \sqrt{H(\sigma(t), \sigma(s))}.$$

It follows that

$$\limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \left[H(\sigma(t), s) Q(s) - \frac{m^2(t, s) \sigma^{\alpha}(s) r_1(s)}{4\alpha \delta(\sigma(s)) s^{\alpha} R(s, T)} \right] \Delta s \\ = \limsup_{t \rightarrow \infty} \frac{1}{(2t - T_0)^2} \int_{T_0}^{2t} \left[\frac{(2t - s)^2}{2^{\frac{7}{3}}s} - \frac{3(2t^2 - 6st + 5s^2)^2}{5 \cdot 2^{\frac{4}{3}}s^2(t - s)^2 R(s, T)} \right] \Delta s = \infty.$$

Then, by Theorem 4.2, every solution $x(t)$ of Eq. (4.16) is either oscillatory or converges to zero.

Example 4.3. Consider the third-order nonlinear delay dynamic equation

$$\left(t \left(\left(\frac{1}{t} x^{\Delta}(t) \right)^{\Delta} \right)^3 \right)^{\Delta} + \frac{1}{t^2} \left(x^3 \left(\frac{t}{2} \right) + x^5 \left(\frac{t}{2} \right) \right) = 0, \quad t \in \overline{2\mathbb{Z}}, \quad t \geq t_0 := 2. \quad (4.17)$$

Here, $\alpha = 3$, $r_1(t) = \frac{1}{t}$, $r_2(t) = t$, $q(t) = \frac{1}{t^2}$, $f(x) = x^3(1 + x^2)$, and $\tau(t) = \frac{t}{2} < t$.

Conditions (H₁)–(H₃) are clearly satisfied, and (H₄) holds with $K = 1$. $r_1^{\Delta}(t) < 0$, and

$$\int_{t_0}^{\infty} r_1(s) \tau^{\alpha}(s) \Delta s = \int_2^{\infty} \frac{s^2}{8} \Delta s = \infty, \\ \int_{t_0}^{\infty} \frac{1}{r_1(t)} \int_t^{\infty} \left[\frac{1}{r_2(s)} \int_s^{\infty} q(u) \Delta u \right]^{\frac{1}{\alpha}} \Delta s \Delta t = \int_2^{\infty} t \int_t^{\infty} \left[\frac{1}{s} \int_s^{\infty} \frac{1}{u^2} \Delta u \right]^{\frac{1}{3}} \Delta s \Delta t = \infty,$$

so (3.5) and (3.6) hold. Noting that

$$R_1(t, T) = \int_T^t \left(\frac{1}{r_2(s)} \right)^{\frac{1}{\alpha}} \Delta s = \int_T^t s^{-\frac{1}{3}} \Delta s = \frac{t^{\frac{2}{3}} - T^{\frac{2}{3}}}{\frac{2}{3} - 1}.$$

Let $\delta(t) = t$. Since $\sigma(t) = 2t$, we obtain

$$Q(t) = Kq(t)\delta(\sigma(t)) \left(\frac{\tau(t)}{\sigma(t)} \right)^{\alpha} = \frac{1}{32t}.$$

Let $H(t, s) = (t - s)^2$, that there exists a function $h(t, s) = -\frac{2t^2 - 6st + 5s^2}{\sqrt{2(t-s)^3}}$ such that

$$H^{\Delta_s}(\sigma(t), \sigma(s)) + \frac{(\delta^{\Delta}(s))_+}{\delta(s)} H(\sigma(t), \sigma(s)) = -\frac{h(t, s)}{\delta(s)} H^{\frac{\alpha}{\alpha+1}}(\sigma(t), \sigma(s)).$$

It follows that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \left[H(\sigma(t), s)Q(s) - \frac{h^{\alpha+1}(t, s)[\sigma^{\alpha}(s)r_1(s)]^{\alpha}}{(\alpha+1)^{\alpha+1}[\delta(\sigma(s))s^{\alpha}R_1(s, T)]^{\alpha}} \right] \Delta s \\ &= \limsup_{t \rightarrow \infty} \frac{1}{(2t - T_0)^2} \int_{T_0}^{2t} \left[\frac{(2t - s)^2}{32s} - \frac{\left(2^{\frac{2}{3}} - 1\right)^3 (2t^2 - 6st + 5s^2)^4}{16s^6(t - s)^6 \left(s^{\frac{2}{3}} - T^{\frac{2}{3}}\right)^3} \right] \Delta s = \infty. \end{aligned}$$

Then, by Theorem 4.3, every solution $x(t)$ of Eq. (4.17) is either oscillatory or converges to zero.

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