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An optimal and efficient general eighth-order derivative free scheme for simple roots

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Abstract

The main motivation of this study is to present an optimal scheme in a general way that can be applied to any existing optimal multipoint fourth-order iterative scheme whose first substep employs Steffensen's method or Steffensen like method to further produce optimal eighth-order iterative schemes. A rational function approximation approach is used in the construction of proposed scheme. In addition, we also discussed the theoretical and computational properties of our scheme. Each member of the presented scheme satisfies the optimality conjecture for multipoint iterative methods without memory which was given by Kung and Traub in 1970. Finally, we also concluded on the basis of obtained numerical results that our methods have faster convergence in contrast to the existing methods of same order because they have minimum residual errors, minimum error difference between two consecutive iterations and minimum asymptotic error constants corresponding to the considered test function.

Keywords: Nonlinear equations, Simple roots, Computational order of convergence, Steffensen's type method.

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1 Introduction

The construction of new solution techniques have always been a paramount importance in the field of numerical analysis in order to find the approximate solutions of nonlinear equations. Newton's method is

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one of the most important and classic iterative methods, which converges quadratically for a simple root in the neighborhood of the required zero.

However, it is also well known that there are some major difficulties in the application of Newton's method. Computation of the first-order derivative at each step is one of such difficulties. Generally, there are many practical situations where the calculations of derivatives are expensive and/or it requires a great deal of time to be given or calculated. In order to overcome this problem, Steffensen's proposed a quadratically convergent method [9], which is defined as follows:

$$x_{n+1} = x_n - \frac{f(x)^2}{f(x_n + f(x_n)) - f(x_n)}, \quad n \geq 0, \tag{1.1}$$

where $x_n = e_n + \alpha$, e_n is the error at nth step and α is required zero of the involved function.

In the past and recent years, many higher order derivation of classical Steffensen's method or Steffensen like method have been proposed [[1–5, 8–11, 13–17]. All of them need additional functional evaluations. All these modifications are targeted at increasing the local order of convergence with a view of increasing efficiency index [14]. Some of them belong to the optimal class of fourth-order methods [1, 8, 11, 17] and some other are in the optimal class of eighth-order methods [3–5,13,15,16]. Very recently, Sharma et al. [18] proposed an optimal scheme which is applicable to every optimal fourth-order method whose first substep should be Newton to further extend eighth-order convergence. But, there are two main problems with this scheme: first one is that the scheme involve first-order derivative; second one is that they randomly consider the third substep without any justification of this substep. According to our knowledge, there is no an optimal derivative free method to produce an eight-order scheme from a fourth-order one.

Nowadays, a constructive development of optimal derivative free schemes of eighth-order which are applicable to every optimal fourth-order derivative free iterative method/family of iterative methods with Steffensen's method or Steffensen type method applied to the first substep iteration rather than the usual development dependent on particular fourth-order methods becomes a more interesting and challenging task in the filed of numerical analysis. This is the main purpose of this study.

In order to develop new iterative schemes it is necessary to use some methods to approximate the value of a function at a given point. Geometric derivation [14, 19–23] from the different quadratic curves (e.g. parabola, hyperbola, circle or ellipse), functional approach [24–26], power mean approach [27, 28], quadrature approach [29–34], Adomian decomposition approach [35–39], sampling approach [14, 40–43], inverse interpolation approach [14], weight function approach [4, 13, 44, 45], compositional approach [46–48], are some well–known approximations techniques available in the literature. It is worthy to note that every approach has some advantages and disadvantages because it's dependent on the problem under consideration. The choice of suitable approximation approach can save considerable amount of computation. Rational function approximation is an important case of nonlinear approximation that has been widely used in the literature for many different problems (for the details, see Jarratt and Nudds [3], Sharma et al. [12]).

In general, the free parameters associated with the chosen rational approximant are determined by imposing some conditions, for example, interpolation at some specific points. An improved method with higher-order convergence will be obtained as the number of undetermined constants in the rational function increases (for the details, see Jarratt and Nudds [3]). In recent years, Sharma et al. [12] also used this type of approach in order to obtain improved King's methods [6] with optimal order of convergence. However, their scheme required the computation of first-order derivative at each step.

The main contribution in this paper consists of a new general optimal eighth-order derivative free scheme based on the use of rational interpolation. It is able to produce new families of root finding numerical schemes from every existing optimal fourth-order method/family of methods, provided the first substep of the chosen scheme should be Steffensen's method or Steffensen type method. No doubts, there is a good variety of optimal eighth-order derivative free methods available in the literature [3–5, 13, 15, 16], but the advantage of our scheme is that it is not the extension of any particular well-known or existing method. The efficiency and accuracy of our methods is tested on a concrete variety of standard test problems. The numerical experiments demonstrate that our proposed methods are more efficient as compared to existing optimal methods of same order in term of minimum residual error, minimum error between the two consecutive iterations and minimum asymptotic error constant to the corresponding test function.

2 Derivation of an optimal eighth-order scheme

Here, we consider a general optimal fourth-order derivative free scheme in the following way

$$\begin{cases} u_n = x_n + \beta f(x_n), \\ w_n = x_n - \dfrac{f(x_n)}{f[u_n,\ x_n]}, \\ z_n = \phi_4(u_n,\ x_n,\ w_n), \end{cases} \tag{2.1}$$

where f is a nonlinear operator defined on a non–empty open convex subset of a Banach space X with values in a Banach space Y, $n \geq 0,\ 0 \neq \beta \in \mathbb{R}$, $f[u_n,\ x_n] = \dfrac{f(u_n) - f(x_n)}{u_n - x_n}$ is a divided difference of order one and ϕ_4 is any optimal fourth-order derivative free method [1, 2, 5, 8, 11, 17].

We choose the rational function

$$\tau(x) = \frac{(x - x_n) + \lambda_1}{\lambda_2(x - x_n)^2 + \lambda_3(x - x_n) + \lambda_4}, \tag{2.2}$$

where the parameters $\lambda_i, 1 \leq i \leq 4$ are determined by imposing the interpolation conditions

$$\tau(x_n) = f(x_n),\ \ \tau(u_n) = f(u_n),\ \ \tau(w_n) = f(w_n),\ \ \tau(z_n) = f(z_n). \tag{2.3}$$

From the first tangency condition, we obtain

$$\lambda_1 = \lambda_4 f(x_n). \tag{2.4}$$

Again, with help of last three interpolation conditions, we obtain the following three independent relations

$$\begin{aligned} \lambda_2(u_n - x_n) + \lambda_3 &= \frac{1 - \lambda_4 f[u_n,\ x_n]}{f(u_n)}, \\ \lambda_2(w_n - x_n) + \lambda_3 &= \frac{1 - \lambda_4 f[w_n,\ x_n]}{f(w_n)}, \\ \lambda_2(z_n - x_n) + \lambda_3 &= \frac{1 - \lambda_4 f[z_n,\ x_n]}{f(z_n)}, \end{aligned} \tag{2.5}$$

which further yield

$$\lambda_2 = \frac{f[u_n, x_n]\,[\lambda_4(f(u_n)f[w_n, x_n] - f(w_n)f[u_n, x_n]) - f(u_n) + f(w_n)]}{f(u_n)f(w_n)(f(x_n) + f[u_n, x_n](u_n - x_n))},$$

$$\lambda_3 = \frac{\lambda_2 f(u_n)(x_n - u_n) - \lambda_4 f[u_n, x_n] + 1}{f(u_n)},$$

$$\lambda_4 = \frac{\theta - f(u_n)f(w_n)f(x_n)(\beta f[u_n, x_n] + 1) + f(w_n)f(z_n)[f(x_n) + f[u_n, x_n](z_n - x_n)]}{\theta f[w_n, x_n] - f(u_n)f(w_n)f(x_n)f[z_n, x_n](\beta f[u_n, x_n] + 1) + f(w_n)f[u_n, x_n]f(z_n)[f(x_n) + f[u_n, x_n](z_n - x_n)]}, \tag{2.6}$$

where $\theta = f(u_n)f[u_n, x_n]f(z_n)(\beta f(x_n) + x_n - z_n)$.

Now, we assume that the above rational function (2.2) meets the x–axis at the point $x = x_{n+1}$, in order to find the final iteration. Then, we obtain

$$\tau(x_{n+1}) = 0, \tag{2.7}$$

which further yields

$$x_{n+1} = x_n - \lambda_4 f(x_n). \tag{2.8}$$

Finally, with the help of expressions (2.1) and (2.8), we have

$$\begin{cases} w_n = x_n - \dfrac{f(x_n)}{f[u_n,\ x_n]}, \\ z_n = \phi_4(u_n,\ x_n,\ w_n), \\ x_{n+1} = x_n - \lambda_4 f(x_n), \end{cases} \tag{2.9}$$

where $n \geq 0$, λ_4 and ϕ_4 are defined earlier in this section. In the next theorem 3.1, we demonstrate that the order of convergence of the proposed scheme will reach at the optimal eighth-order without using any additional functional evaluations. It is interesting to observe that only a single coefficient η_0 of $\phi_4(u_n,\ w_n,\ z_n)$, the asymptomatic error constant of ϕ_4, contributes its role in the construction of the desired eighth-order convergence(see Theorem 3.1 for details).

3 Convergence analysis

Theorem 3.1 *Let $x = \alpha$ be a simple zero of an analytic function $f : \mathbb{C} \to \mathbb{C}$ in the region enclosing the required zero. Assume that $\phi_4(u_n,\ x_n,\ w_n)$ is any optimal fourth-order scheme and that the initial guess $x = x_0$ is sufficiently close to the required zero to guarantee the convergence of the method. Then, the proposed iterative scheme* (2.9) *is eighth-order of convergence.*

Proof The Taylor's series expansion of the function $f(x_n)$ around the point $x = \alpha$ leads us to

$$f(x_n) = \sum_{j=1}^{8} c_j e_n^j + O(e_n^9), \tag{3.1}$$

where $e_n = x_n - \alpha$ and $c_j = \dfrac{f^{(j)}(\alpha)}{j!}$ for $j = 1, 2,\ \ldots,\ 8$.

Similarly, we obtain the expansion of $f(u_n)$ around the point $x = \alpha$, which is given as follows:

$$f(u_n) = c_1 \left(\beta c_1 + 1\right) e_n + c_2 \left(\beta^2 c_1^2 + 3\beta c_1 + 1\right) e_n^2 + \sum_{m=1}^{6} G_m e_n^{k+2} + O(e_n^9), \tag{3.2}$$

where every constant G_m depends on $\beta, c_1, \ldots, c_8$. Namely,

$$G_1 := 2\beta(1+\beta c_1)c_2^2 + \beta c_1 c_3(1+\beta c_1)^3 c_3,$$

$$G_2 := 3\beta(1+\beta c_1)^2 c_2 c_3 + \beta c_2\{\beta c_2^2 + 2(1+\beta c_1)c_3\} + \beta c_1 c_4 + (1+\beta c_1)^4 c_4,$$

$$G_3 := c_5(\beta c_1+1)^5 + 4\beta c_2 c_4(\beta c_1+1)^3 + 3\beta c_3\Big(\beta c_2^2 + c_3(\beta c_1+1)\Big)(\beta c_1+1) + 2\beta c_2\Big(\beta c_2 c_3 + c_4(\beta c_1+1)\Big) + \beta c_1 c_5,$$

$$\begin{aligned}G_4 := {}&\beta^2 c_2 c_3^2 + 2\beta^2 c_2^2 c_4 + \beta c_3\Big(\beta^2 c_2^3 + 6\beta c_3 c_2(\beta c_1+1) + 3c_4(\beta c_1+1)^2\Big) + 2\beta^2 c_1 c_2 c_5 + c_6(\beta c_1+1)^6 + 5\beta c_2 c_5(\beta c_1+1)^4\\ &+ 2\beta c_4\Big(3\beta c_2^2 + 2c_3(\beta c_1+1)\Big)(\beta c_1+1)^2 + 2\beta c_2 c_5 + \beta c_1 c_6,\end{aligned}$$

$$\begin{aligned}G_5 := {}&4\beta c_4\Big(\beta^2 c_2^3 + 3\beta c_3 c_2(\beta c_1+1) + c_4(\beta c_1+1)^2\Big)(\beta c_1+1) + 3\beta c_3\Big(\beta^2 c_3 c_2^2 + (\beta c_1+1)(2\beta c_4 c_2 + \beta c_3^2 + c_5(\beta c_1+1))\Big)\\ &+ c_7(\beta c_1+1)^7 + 6\beta c_2 c_6(\beta c_1+1)^5 + 5\beta c_5\left(2\beta c_2^2 + c_3(\beta c_1+1)\right)(\beta c_1+1)^3 + 2\beta c_2\left(\beta c_3 c_4 + \beta c_2 c_5 + c_6(\beta c_1+1)\right) + \beta c_1 c_7,\end{aligned}$$

$$\begin{aligned}G_6 := {}&5\beta c_5\Big(2\beta^2 c_2^3 + 4\beta c_3 c_2(\beta c_1+1) + c_4(\beta c_1+1)^2\Big)(\beta c_1+1)^2 + \beta^2 c_2 c_4^2 + 2\beta^2 c_2 c_3 c_5 + 2\beta^2 c_2^2 c_6 + 3\beta c_3\Big[\beta^2 c_4 c_2^2\\ &+ \beta c_2\{\beta c_3^2 + 2c_5(\beta c_1+1)\} + (\beta c_1+1)\{2\beta c_3 c_4 + c_6(\beta c_1+1)\}\Big] + 2\beta^2 c_1 c_2 c_7 + \beta c_4\Big[\beta^3 c_2^4 + 12\beta^2 c_3 c_2^2(\beta c_1+1)\\ &+ 2(\beta c_1+1)^2\Big(6\beta c_4 c_2 + 3\beta c_3^2 + 2c_5(\beta c_1+1)\Big)\Big] + 2\beta c_2 c_7 + \beta c_1 c_8 + c_8(\beta c_1+1)^8 + 7\beta c_2 c_7(\beta c_1+1)^6\\ &+ 3\beta c_6\left(5\beta c_2^2 + 2c_3(\beta c_1+1)\right)(\beta c_1+1)^4.\end{aligned}$$

From (3.1) and (3.2), it is straightforward to prove that

$$w_n - \alpha = c_2\left(\beta + \frac{1}{c_1}\right)e_n^2 + \frac{\left(c_1 c_3\left(\beta^2 c_1^2 + 3\beta c_1 + 2\right) - c_2^2\left(\beta^2 c_1^2 + 2\beta c_1 + 2\right)\right)}{c_1^2}e_n^3 + \sum_{m=1}^{5}\bar{G}_m e_n^{k+3} + O(e_n^9). \tag{3.3}$$

It is easy to derive the equality

$$f(w_n) = (1+\beta c_1)c_2 e_n^2 + \frac{c_1(2+3\beta c_1+\beta^2 c_1^2)c_3 - (2+2\beta c_1+\beta^2 c_1^2)c_2^2}{c_1}e_n^3 + H_1 e_n^4 + H_2 e_n^5 + O(e_n^6), \tag{3.4}$$

where

$$H_1 := \frac{1}{c_1^2}\big[(5+7\beta c_1+4\beta^2 c_1^2+\beta^3 c_1^3)c_2^3 - c_1(7+10\beta c_1+7\beta^2 c_1^2+2\beta^3 c_1^3)c_2 c_3 + c_1^2(3+6\beta c_1+4\beta^2 c_1^2+\beta^3 c_1^3)c_4\big],$$

$$\begin{aligned}H_2 := {}&\frac{1}{c_1^3}\big[c_1(24+43\beta c_1+35\beta^2 c_1^2+15\beta^3 c_1^3+3\beta^4 c_1^4)c_2^2 c_3 - (2+\beta c_1)^2(3+2\beta c_1+\beta^2 c_1^2)c_2^4 - 2c_1^2(5+9\beta c_1+8\beta^2 c_1^2\\ &+ 4\beta^3 c_1^3+\beta^4 c_1^4)c_2 c_4 - c_1^2\{(6+12\beta c_1+12\beta^2 c_1^2+6\beta^3 c_1^3+\beta^4 c_1^4)c_3^2 - c_1(4+10\beta c_1+10\beta^2 c_1^2+5\beta^3 c_1^3+\beta^4 c_1^4)c_5\}\big].\end{aligned}$$

Since $\phi_4(u_n, x_n,\ w_n)$ is of fourth order, it follows that

$$z_n - \alpha = \eta_0 e_n^4 + \eta_1 e_n^5 + \eta_2 e_n^6 + \eta_3 e_n^7 + \eta_4 e_n^8 + O(e_n^9), \tag{3.5}$$

where $\eta_0 \neq 0$ and η_i $(0 \leq i \leq 4)$ are asymptotic error constants.

Now, the expansion of $f(z_n)$ around α provides the equality

$$f(z_n) = c_1\eta_0 e_n^4 + c_1\eta_1 e_n^5 + c_1\eta_2 e_n^6 + c_1\eta_3 e_n^7 + \left(c_2\eta_0^2 + c_1\eta_4\right) e_n^8 + O(e_n^9). \tag{3.6}$$

With the help of expression (3.1) – (3.6), we have

$$\lambda_4 f(x_n) = e_n - \frac{(1+\beta c_1)^2 c_2 \left(c_2^3 - 2c_1c_2c_3 + c_1^2 c_4\right)\eta_0}{c_1^4} e_n^8 + O(e_n^9). \tag{3.7}$$

Finally, by inserting the expression (3.7) in the last substep of the proposed scheme (2.9), we will obtain the following error equation

$$e_{n+1} = \frac{\eta_0(1+\beta c_1)^2 c_2(c_2^3 - 2c_1c_2c_3 + c_1^2 c_4)}{c_1^4} e_n^8 + O(e_n^9). \tag{3.8}$$

This reveals that the proposed scheme (2.9) reaches an eighth-order convergence by using only four functional evaluations (viz. $f(u_n)$, $f(x_n)$, $f(w_n)$ and $f(z_n)$) per iteration. So, it satisfies the optimality Kung–Traub conjecture for multipoint iterative methods without memory. The expression (3.8) also shows that only a single coefficient η_0 from $\phi_4(u_n,\ w_n,\ z_n)$ contributes its role in the asymptotic error constant term of proposed scheme. This completes the proof. □

4 Numerical analysis

Here, we will check the effectiveness and validity of our theoretical results on a concrete variety of nonlinear equations which are mentioned in Table 1. In addition, we mentioned the number of iteration indexes (n), approximated zeros (x_n), absolute residual error of the corresponding function $(|f(x_n)|)$, error in the consecutive iterations $|x_{n+1} - x_n|$, $\left|\frac{x_{n+1} - x_n}{(x_n - x_{n-1})^8}\right|$, the asymptotic error constant $\eta = \lim_{n\to\infty}\left|\frac{x_{n+1} - x_n}{(x_n - x_{n-1})^8}\right|$, and ρ in the Tables 2–7. In order to calculate the computational order of convergence (ρ), we use the following formula

$$\rho = \left| \frac{(x_{n+1} - x_n)/\eta}{(x_n - x_{n-1})} \right|, \; n = 2, 3.$$

We calculate the computational order of convergence, asymptotic error constant and other constants up to several number of significant digits (around 1000 significant digits) to minimize the round off error.

As we mentioned in the above paragraph that we calculate the values of all the constants and functional residuals up to several number of significant digits but due to the limited paper space, we display the value of x_n and ρ up to 15 and 6 significant digits, respectively. In addition, we also display $\left| \frac{x_{n+1} - x_n}{(x_n - x_{n-1})^8} \right|$ and η up to 10 significant digits. Moreover, absolute residual error in the function $|f(x_n)|$ and error in the consecutive iterations $|x_{n+1} - x_n|$ are display up to 2 significant digits with exponent power in the Tables 2–7. Furthermore, the approximated zeros up to 30 significant digits and initial guesses are also displayed in the Table 1 although minimum 1000 significant digits are available with us.

Now, we will consider some special cases of the proposed scheme (2.9) in order to compare our methods with other existing methods of the same order. Therefore, we consider:

(i) Firstly, we deal with the optimal family of fourth-order derivative free methods proposed by Liu et al. Liu et al. [8]. Then, we obtain the following optimal family of eighth-order methods with the help of our proposed scheme (2.9):

$$\begin{cases} w_n = x_n - \dfrac{f(x_n)}{f[u_n,\ x_n]}, \\ z_n = w_n - \dfrac{f(w_n)(f[u_n,\ x_n] - f[u_n,\ w_n] + f[w_n,\ x_n])}{f[w_n,\ x_n]^2}, \\ x_{n+1} = x_n - \lambda_4 f(x_n), \end{cases} \tag{4.1}$$

Let us chose $\beta = 1$ for the computational work in the above scheme, denoted by $(OM1)$.

(ii) Now, we shall choose another optimal family of fourth-order methods from Zheng et al. [17], which further yields

$$\begin{cases} w_n = x_n - \dfrac{f(x_n)}{f[u_n,\ x_n]}, \\ z_n = w_n - \dfrac{f(w_n)\left[(p-1)(f[u_n,\ w_n] - f[u_n,\ x_n]) + f[w_n,\ x_n] - b(w_n - u_n)(w_n - x_n)\right]}{f[w_n,\ x_n]\left(pf[u_n,\ w_n] - pf[u_n,\ x_n] + f[w_n,\ x_n] - \alpha_1(w_n - u_n)(w_n - x_n)\right)}, \\ x_{n+1} = x_n - \lambda_4 f(x_n). \end{cases} \tag{4.2}$$

Let us consider $p = 2,\ b = \alpha_1 = 0$ and $\beta = 1$ in the above scheme, known as $(OM2)$.

(iii) Again, we choose another optimal family of fourth-order methods from Cordero and Torregrosa [1] to further extend eighth-order derivative free methods, which is defined as follows:

$$\begin{cases} w_n = x_n - \dfrac{f(x_n)}{f[u_n,\ x_n]}, \\ z_n = w_n - \dfrac{f(w_n)}{\dfrac{af(w_n) - bf(u_n)}{w_n - u_n} + \dfrac{cf(w_n) - df(x_n)}{w_n - x_n}}, \\ x_{n+1} = x_n - \lambda_4 f(x_n). \end{cases} \tag{4.3}$$

Let us pick up $a = b = c = 1,\ d = 0$ and $\beta = \dfrac{1}{2}$ in the above scheme, called by $(OM3)$.

(iv) Similarly, we choose another optimal family of fourth-order methods from Ren et al. in [11], which further yield

$$\begin{cases} w_n = x_n - \dfrac{f(x_n)}{f[u_n,\ x_n]}, \\ z_n = w_n - \dfrac{f(w_n)}{a(w_n - u_n)(w_n - x_n) + f[u_n,\ w_n] - f[u_n,\ x_n] + f[w_n,\ x_n]}, \\ x_{n+1} = x_n - \lambda_4 f(x_n). \end{cases} \tag{4.4}$$

Let us assume $a = 1$ and $\beta = \dfrac{1}{2}$ in the above scheme, denoted by $(OM4)$.

Now, we will compare them with the optimal families of eighth-order methods which were proposed by Zheng et al. in [16] and Soleymani and Vanani [13], out of these families we shall choose the expression (8) (for $\gamma = 1$) and expression (21), called by ZM and SV, respectively. In addition, we will also compare them with a general class of multipoint iterative methods without memory given by Kung and Traub [7], out their general class we choose an optimal eight-order derivative free method, denoted by KT. Moreover, we will compare with another optimal family of eighth-order derivative free methods presented by Khattri and Steihaug [5], out them we pick up expression (17) (for $\alpha = 1$), known as KS. Finally, we will also compare our methods with an optimal eighth-order method (2.8) (for $\beta = 1$), proposed by Thukral [15], called by TM.

For the computer programming, all computations have been performed using the programming package *Mathematica* 9 with multiple precision arithmetic. Further, the meaning of $a(\pm b)$ is $a \times 10^{(\pm b)}$ in the following Tables 2–7.

Table 1: Test problems.

$f(x)$	x_0	$Zeros(\alpha)$
$f_1(x) = 10xe^{x^2} - 1$; [11]	1.7	1.67963061042844994067492033884
$f_2(x) = \exp(-x) - \tan^{-1}(x) - 1$; [16]	0.5	0
$f_3(x) = \sin^2(x) + x$; [13]	0.2	0
$f_4(x) = x^3 + 4x^2 - 10$; [5]	1.3	1.36523001341409684576080682898
$f_5(x) = \log(x^2 + x + 2) - x + 1$; [11]	3.6	4.15259073675715827499698900477
$f_6(x) = x^4 + (5 + 2i)x + \sqrt{5}i + 1$; [10]	$0.7 + 16i$	$0.76743794129744696507885721826 + 1.71313115253563442344370391956i$

Table 2: Convergence behavior of different iterative methods for $f_1(x)$.

Cases	n	x_n	$\|f(x_n)\|$	$\|x_{n+1} - x_n\|$	$\frac{x_{n+1} - x_n}{(x_n - x_{n-1})^8}$	η	ρ
	1	1.67963061042841	1.2(−13)	4.2(−14)			
ZM	2	1.67963061042845	4.3(−107)	1.6(−107)	1.703121240	1.703121240	8.00000
	3	1.67963061042845	1.7(−854)	6.2(−855)	1.703121240		8.00000
	1	1.67963061042766	2.2(−12)	7.9(−13)			
SV	2	1.67963061042845	1.3(−95)	4.7(−96)	30.91187283	30.91187283	8.00000
	3	1.67963061042845	2.1(−761)	7.7(−762)	30.91187283		8.00000
	1	1.67963061042712	3.7(−12)	1.3(−12)			
KT	2	1.67963061042845	1.4(−93)	5.2(−94)	53.69988442	53.69988442	8.00000
	3	1.67963061042845	7.8(−745)	2.8(−745)	53.69988442		8.00000
	1	1.67963061042841	1.2(−13)	4.2(−14)			
KS	2	1.67963061042845	4.3(−107)	1.6(−107)	1.703121240	1.703121240	8.00000
	3	1.67963061042845	1.7(−854)	6.2(−855)	1.703121240		8.00000
	1	1.67963061042851	1.6(−13)	5.7(−14)			
TM	2	1.67963061042845	4.8(−106)	1.7(−106)	1.564440184	1.564440184	8.00000
	3	1.67963061042845	3.5(−846)	1.3(−846)	1.564440184		8.00000
	1	1.67963061042844	3.2(−14)	1.2(−14)			
$OM1$	2	1.67963061042845	4.3(−112)	1.6(−112)	0.4628860562	0.4628860562	8.00000
	3	1.67963061042845	4.3(−895)	1.5(−895)	0.4628860562		8.00000
	1	1.67963061042840	1.4(−13)	4.9(−14)			
$OM2$	2	1.67963061042845	1.7(−106)	6.3(−107)	1.937439498	1.937439498	8.00000
	3	1.67963061042845	1.3(−849)	4.7(−850)	1.937439498		8.00000
	1	1.67963061042845	1.7(−16)	6.0(−17)			
$OM3$	2	1.67963061042845	1.2(−132)	4.3(−133)	0.002638519681	0.002638519681	8.00000
	3	1.67963061042845	8.1(−1062)	2.9(−1062)	0.002638519681		8.00000
	1	1.67963061042845	9.4(−17)	3.4(−17)			
$OM4$	2	1.67963061042845	7.2(−135)	2.6(−135)	0.001506709240	0.001506709240	8.00000
	3	1.67963061042845	8.5(−1080)	3.1(−1080)	0.001506709240		8.00000

Table 3: Convergence behavior of different iterative methods for $f_2(x)$.

Cases	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	$\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}$	η	ρ
	1	$-4.71152315888097(-7)$	$9.4(-7)$	$4.7(-7)$			
ZM	2	$-1.26800021474312(-54)$	$2.5(-54)$	$1.3(-54)$	0.0005221901622	0.0005221896701	8.00000
	3	$-3.48965821455630(-435)$	$7.0(-435)$	$3.5(-435)$	0.0005221896701		8.00000
	1	$-2.57499475484165(-6)$	$5.1(-6)$	$2.6(-6)$			
SV	2	$-3.80144307772904(-48)$	$7.6(-48)$	$3.8(-48)$	0.001966698117	0.001966688368	8.00000
	3	$-8.57676446442167(-383)$	$1.7(-382)$	$8.6(-383)$	0.001966688368		8.00000
	1	$-3.66227202022631(-6)$	$7.3(-6)$	$3.7(-6)$			
KT	2	$-8.55874586608907(-47)$	$1.7(-46)$	$8.6(-47)$	0.002644876143	0.002644856771	8.00000
	3	$-7.61523319936559(-372)$	$1.5(-371)$	$7.6(-372)$	0.002644856771		8.00000
	1	$-4.71152315888098(-7)$	$9.4(-7)$	$4.7(-7)$			
KS	2	$-1.26800021474312(-54)$	$2.5(-54)$	$1.3(-54)$	0.0005221901622	0.0005221896701	8.00000
	3	$-3.48965821455630(-435)$	$7.0(-435)$	$7.0(-435)$	0.0005221896701		8.00000
	1	$-1.31238499881984(-6)$	$2.6(-6)$	$1.3(-6)$			
TM	2	$-1.07424020348132(-50)$	$2.1(-50)$	$1.1(-50)$	0.001220706737	0.001220703125	8.00000
	3	$-2.16481740525581(-403)$	$4.3(-403)$	$2.2(-403)$	0.001220703125		8.00000
	1	$-1.49115722562163(-7)$	$3.0(-7)$	$1.5(-7)$			
$OM1$	2	$-9.94664123838956(-59)$	$2.0(-58)$	$9.9(-59)$	0.0004069011858	0.0004069010417	8.00000
	3	$-3.89852597547628(-468)$	$7.8(-468)$	$3.9(-468)$	0.0004069010417		8.00000
	1	$-1.12071503148622(-6)$	$2.2(-6)$	$1.1(-6)$			
$OM2$	2	$-2.53157826882851(-51)$	$5.1(-51)$	$2.5(-51)$	0.001017254884	0.001017252604	8.00000
	3	$-1.71616753640922(-408)$	$3.4(-408)$	$1.7(-408)$	0.001017252604		8.00000
	1	$-1.25164089193913(-9)$	$2.5(-9)$	$1.3(-9)$			
$OM3$	2	$-4.11191193309798(-113)$	$8.2(-113)$	$4.1(-113)$	$6.826622022(-42)$	$7.951738754(-456)$	8.00000
	3	$-6.49849127234668(-1355)$	$1.3(-1355)$	$6.5(-1355)$	$7.951738754(-456)$		8.00000
	1	$3.23043587378188(-9)$	$6.5(-9)$	$3.2(-9)$			
$OM4$	2	$8.72511705555499(-108)$	$1.7(-107)$	$8.7(-108)$	$7.356686794(-40)$	$3.914919307(-434)$	8.00000
	3	$1.31490056504395(-1290)$	$2.6(-1290)$	$1.3(-1290)$	$3.914919307(-434)$		8.00000

Table 4: Convergence behavior of different iterative methods for $f_3(x)$.

Cases	n	x_n	$\|f(x_n)\|$	$\|x_{n+1}-x_n\|$	$\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}$	η	ρ
	1	4.32157148397814(−6)	4.3(−6)	4.3(−6)			
ZM	2	1.29761375720156(−42)	1.3(−42)	1.3(−42)	10.66625180	10.66666667	8.00000
	3	8.57417322938934(−335)	8.6(−335)	8.6(−335)	10.66666667		8.00000
	1	0.0000295613210046354	3.0(−5)	3.0(−5)			
SV	2	3.57684323794609(−35)	3.6(−35)	3.6(−35)	61.33511830	61.33333333	8.00000
	3	1.64322041730108(−274)	1.6(−274)	1.6(−274)	61.33333333		8.00000
	1	0.0000321268900777766(−)	3.2(−5)	3.2(−5)			
KT	2	1.69406028762845(−34)	1.7(−34)	1.7(−34)	149.2723476	149.3333333	8.00004
	3	1.01294917159670(−268)	1.0(−268)	1.0(−268)	149.3333333		8.00000
	1	4.32157148397814(−6)	4.3(−6)	4.3(−6)			
KS	2	1.29761375720156(−42)	1.3(−42)	1.3(−42)	10.66625180	10.66666667	8.00000
	3	8.57417322938934(−335)	8.6(−335)	8.6(−335)	10.66666667		8.00000
	1	5.08888537729417(−6)	5.1(−6)	5.1(−6)			
TM	2	−4.07731320091187(−41)	4.1(−41)	4.1(−41)	90.65520040	90.66666667	8.00001
	3	−6.92532621678838(−322)	6.9(−322)	6.9(−322)	90.66666667		8.00000
	1	6.75678598280505(−6)	6.8(−6)	6.8(−6)			
$OM1$	2	6.95048127583966(−41)	7.0(−41)	7.0(−41)	15.99902705	16.00000000	8.00001
	3	8.71443037415132(−321)	8.7(−321)	8.7(−321)	16.00000000		8.00000
	1	4.31635655493395(−6)	4.3(−6)	4.3(−6)			
$OM2$	2	6.42580964419040(−43)	6.4(−43)	6.4(−43)	5.333218231	5.333333333	8.00000
	3	1.55032080832808(−337)	1.6(−337)	1.6(−337)	5.333333333		8.00000
	1	2.25247826417532(−6)	2.3(−6)	2.3(−6)			
$OM3$	2	2.23641402859162(−45)	2.2(−45)	2.2(−45)	3.374949953	3.375000000	8.00000
	3	2.11198796895783(−357)	2.1(−357)	2.1(−357)	3.375000000		8.00000
	1	4.15559203402444(−6)	4.2(−6)	4.2(−6)			
$OM4$	2	6.00280699931643(−43)	6.0(−43)	6.0(−43)	6.749803651	6.750000000	
	3	1.13799096727799(−337)	1.1(−337)	1.1(−337)	6.750000000		8.00000

Table 5: Convergence behavior of different iterative methods for $f_4(x)$.

Cases	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	$\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}$	η	ρ
	1	1.36523072544147	1.2(−5)	7.1(−7)			
ZM	2	1.36523001341410	3.9(−46)	2.4(−47)	358.3007926	358.3068564	8.00000
	3	1.36523001341410	5.8(−370)	3.5(−371)	358.3068564		8.00000
	1	1.36521210605856	3.0(−4)	1.8(−5)			
SV	2	1.36523001341410	3.8(−34)	2.3(−35)	2162.814365	2162.315716	7.99998
	3	1.36523001341410	2.7(−273)	1.6(−274)	2162.315716		8.00000
	1	1.36525034008719	3.4(−4)	2.0(−5)			
KT	2	1.36523001341410	2.0(−33)	1.2(−34)	4183.557611	4186.500190	8.00007
	3	1.36523001341410	3.4(−267)	2.0(−268)	4186.500190		8.00000
	1	1.36523072544147	1.2(−4)	7.1(−7)			
KS	2	1.36523001341410	3.9(−46)	2.4(−47)	358.3007926	358.3068564	8.00000
	3	1.36523001341410	5.8(−370)	3.5(−371)	358.3068564		8.00000
	1	1.36517777072192	8.6(−4)	5.2(−5)			
TM	2	1.36523001341410	1.7(−30)	1.0(−31)	1881.386068	1875.105210	7.99966
	3	1.36523001341410	4.4(−244)	2.6(−245)	1875.105210		8.00000
	1	1.36523053538902	8.6(−6)	5.2(−7)			
$OM1$	2	1.36523001341410	2.3(−47)	1.4(−48)	255.7552647	255.7584614	8.00000
	3	1.36523001341410	6.6(−380)	4.0(−381)	255.7584614		8.00000
	1	1.36523041486444	6.6(−6)	4.0(−7)			
$OM2$	2	1.36523001341410	2.4(−48)	1.5(−49)	219.4805922	219.4826007	8.00000
	3	1.36523001341410	8.4(−388)	5.1(−389)	219.4826007		8.00000
	1	1.36523002761610	2.3(−7)	1.4(−8)			
$OM3$	2	1.36523001341410	5.1(−61)	3.1(−62)	18.54499752	18.54500074	8.00000
	3	1.36523001341410	2.4(−490)	1.5(−491)	18.54500074		8.00000
	1	1.36523003293248	3.2(−7)	2.0(−8)			
$OM4$	2	1.36523001341410	8.6(−60)	5.2(−61)	24.79139475	24.79140087	8.00000
	3	1.36523001341410	2.3(−480)	1.4(−481)	24.79140087		8.00000

Table 6: Convergence behavior of different iterative methods for $f_5(x)$.

Cases	n	x_n	$\|f(x_n)\|$	$\|x_{n+1}-x_n\|$	$\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}$	η	ρ
	1	4.15259073677848	1.3(−11)	2.1(−11)			
ZM	2	4.15259073675716	3.1(−95)	5.2(−95)	1.207487804(−9)	1.207487804(−9)	8.00000
	3	4.15259073675716	3.6(−764)	6.0(−764)	1.207487804(−9)		8.00000
	1	4.15259073676591	5.3(−12)	8.8(−12)			
SV	2	4.15259073675716	2.6(−98)	4.3(−98)	1.246525864(−9)	1.246525864(−9)	8.00000
	3	4.15259073675716	8.8(−789)	1.5(−788)	1.246525864(−9)		8.00000
	1	4.15259073685407	5.8(−11)	9.7(−11)			
KT	2	4.15259073675716	2.2(−89)	3.7(−89)	4.761532982(−9)	4.761532983(−9)	8.00000
	3	4.15259073675716	1.0(−716)	1.7(−716)	4.761532983(−9)		8.00000
	1	4.15259073677848	1.3(−11)	2.1(−11)			
KS	2	4.15259073675716	3.1(−95)	5.2(−95)	1.207487804(−9)	1.207487804(−9)	8.00000
	3	4.15259073675716	3.6(−764)	6.0(−764)	1.207487804(−9)		8.00000
	1	4.15259073672872	1.7(−11)	2.8(−11)			
TM	2	4.15259073675716	1.3(−94)	2.2(−94)	5.120863156(−10)	5.120863155(−10)	8.00000
	3	4.15259073675716	1.6(−759)	2.7(−759)	5.120863155(−10)		8.00000
	1	4.15259073681408	3.4(−11)	5.7(−11)			
$OM1$	2	4.15259073675716	2.0(−91)	3.3(−91)	2.990945873(−9)	2.990945874(−9)	8.00000
	3	4.15259073675716	2.5(−733)	4.2(−733)	2.990945874(−9)		8.00000
	1	4.15259073675977	1.6(−12)	2.6(−12)			
$OM2$	2	4.15259073675716	2.2(−103)	3.6(−103)	1.662410892(−10)	1.662410892(−10)	8.00000
	3	4.15259073675716	2.7(−830)	4.4(−830)	1.662410892(−10)		8.00000
	1	4.15259073702082	1.6(−10)	2.6(−10)			
$OM3$	2	4.15259073675716	2.1(−85)	3.5(−85)	1.505000386(−8)	1.505000386(−8)	8.00000
	3	4.15259073675716	2.1(−684)	3.5(−684)	1.505000386(−8)		8.00000
	1	4.15259067408621	3.8(−8)	6.3(−8)			
$OM4$	2	4.15259073675716	3.5(−64)	5.8(−64)	2.426337637(−6)	2.426337479(−6)	8.00000
	3	4.15259073675716	1.8(−512)	3.0(−512)	2.426337479(−6)		8.00000

Table 7: Convergence behavior of different iterative methods for $f_6(x)$.

Cases	n	x_n	$\|f(x_n)\|$	$\|x_{n+1}-x_n\|$	$\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}$	η	ρ
ZM	1	$0.76751044638902 + 1.71325186317461i$	3.0(−3)	1.4(−4)			
	2	$0.76743794129745 + 1.71313115253563i$	2.5(−25)	1.2(−26)	75293.51358	74793.48502	7.99925
	3	$0.76743794129745 + 1.71313115253563i$	5.3(−202)	2.5(−203)	74793.48502		8.00000
SV	1	$0.76796181586919 + 1.71378291999293i$	1.8(−2)	8.4(−4)			
	2	$0.76743794129745 + 1.71313115253563i$	2.4(−18)	1.1(−19)	471149.9799	460962.1554	7.99692
	3	$0.76743794129745 + 1.71313115253563i$	2.5(−145)	1.2(−146)	460962.1554		8.00000
KT	1	$0.76775556228170 + 1.71350884641864i$	1.0(−2)	4.9(−4)			
	2	$0.76743794129745 + 1.71313115253563i$	6.9(−20)	3.3(−21)	933487.8048	901282.3882	7.99539
	3	$0.76743794129745 + 1.71313115253563i$	2.6(−157)	1.2(−158)	901282.3882		8.00000
KS	1	$0.76751044638902 + 1.71325186317461i$	3.0(−3)	1.4(−4)			
	2	$0.76743794129745 + 1.71313115253563i$	2.5(−25)	1.2(−26)	75293.51358	74793.48502	7.99925
	3	$0.76743794129745 + 1.71313115253563i$	5.3(−202)	2.5(−203)	74793.48502		8.00000
TM	1	$0.76752221942279 + 1.71325405150485i$	3.1(−3)	1.5(−4)			
	2	$0.76743794129745 + 1.71313115253563i$	2.2(−24)	1.0(−25)	425502.1429	418051.8567	7.99800
	3	$0.76743794129745 + 1.71313115253563i$	1.2(−193)	5.5(−195)	418051.8567		8.00000
$OM1$	1	$0.76747020061204 + 1.71318881174172i$	1.4(−3)	6.6(−4)			
	2	$0.76743794129745 + 1.71313115253563i$	2.6(−28)	1.2(−29)	34009.00220	33907.26306	7.99969
	3	$0.76743794129745 + 1.71313115253563i$	3.9(−226)	1.8(−227)	33907.26306		8.00000
$OM2$	1	$0.76747371017792 + 1.71319089448508i$	1.5(−3)	7.0(−4)			
	2	$0.76743794129745 + 1.71313115253563i$	4.6(−28)	2.2(−29)	39388.63601	39257.43557	7.99965
	3	$0.76743794129745 + 1.71313115253563i$	4.2(−224)	2.0(−225)	39257.43557		8.00000
$OM3$	1	$0.76744265012763 + 1.71314744730868i$	3.6(−4)	1.7(−5)			
	2	$0.76743794129745 + 1.71313115253563i$	2.7(−34)	1.3(−35)	1883.776475	1883.092772	7.99997
	3	$0.76743794129745 + 1.71313115253563i$	3.1(−275)	1.4(−276)	1883.092772		8.00000
$OM4$	1	$0.76744350876849 + 1.71314745329596i$	3.6(−4)	1.7(−5)			
	2	$0.76743794129745 + 1.71313115253563i$	3.2(−34)	1.5(−35)	1940.595596	1939.856566	7.99997
	3	$0.76743794129745 + 1.71313115253563i$	1.1(−274)	5.1(−276)	1939.856566		8.00000

5 Concluding remarks

In earlier studies, many scholars from worldwide presented a good number of optimal/non-optimal eighth-order derivative free methods. Most of them are the extensions or modifications of any particular fourth-order iterative method like Ostrowski's method or King's method, etc. The advantage of our proposed scheme over the existing schemes is that it is capable to produce several new interesting eighth-order schemes from every optimal fourth-order scheme whose first substep employs Steffensen's method or Steffensen type method. The derivation of the proposed scheme is based on rational functional approach. Each member of our scheme is optimal in the sense of Kung-Traub conjecture. We also present the convergence analysis of our scheme in the main Theorem 3.1 which demonstrate the theoretical eighth-order convergence. In addition, we also compare our methods with the existing robust methods of the same order on a concrete variety of standard test functions. The numerical results in the Tables 2–6 confirm that the minimum residual errors, minimum error in the consecutive iterations and smaller asymptotic error constants belongs to our methods. Such comparable or superior performance of our methods may be due to the inherent structure of our method with simple asymptotic error constants and rational functional approach. The future work based on the rational functional approach shall be devoted to a development of a new optimal higher-order scheme.

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