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Somayeh Fallah, Farshid Mehrdoust

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On the existence and uniqueness of the solution to the double Heston model equation and valuing Lookback option

Somayeh Fallah and Farshid Mehrdoust
Department of Applied Mathematics, Faculty of Mathematical Sciences
University of Guilan, P. O. Box: 41938-1914, Rasht, Iran

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Abstract

In this work, we study the existence and uniqueness of the solution to the stochastic differential equation of the double Heston model which is defined by two independent variance processes with non-Lipschitz diffusions. Besides, we present a Monte Carlo algorithm based on the Euler discretization method to price the Lookback options under this model.

Keywords: double Heston model; strong solution; Lookback option; Monte Carlo simulation

MSC: 35K99; 91B28; 35Q80; 35Cxx; 65M06

1 Introduction

The Black-Scholes option pricing model based on Brownian motion and normal distribution is one of the most popular option pricing model in the financial theory. Although, this model due to its specific assumptions can be used as an effective approximation, researches have widely shown the limitations associated with this model. In particular, assuming constant volatility across different option strike prices is one of the most significant ones. In fact, the volatility smile observed in the real life does not allow to deal with a constant volatility. To model the smile effectively, one remedy is to use stochastic volatility models.

The Heston model [11] is a stochastic volatility model assumes that market's volatility follows a mean reverting Cox-Ingersoll-Ross (CIR) process. It is motivated by the two following features: First, the market's volatility is stochastic and second, the distribution of risky asset returns has heavier tails than those of the normal distribution. But along with all the advantages of the original Heston model, empirical studies demonstrate that this model is not always able to fit the implied volatility smile very well for options with short maturities [8]. To deal with this issue some alternative models such as a time-dependent Heston model [3, 7, 16, 22] have been proposed. In [13] and [15], respectively, it is presented a fractional and mixed fractional Heston model. Another suggested model is to add additional parameters to the Heston model which makes the model more flexible. Representative generalizations include the Bates model [2] in which the Heston model is enhanced with a jump process, the double Heston model [5] which defines a two-factor structure for the volatility, and the Wishart model [6, 10], where the Cox-Ingersoll-Ross variance process is replaced by a Wishart process.

There are strong reasons to confirm that the double Heston model in comparison to the original Heston model is more compatible with the real market [19]. The asset price dynamic under the double Heston model is defined as follows:

$$\begin{aligned}
dS_t &= rS_t\, dt + \sqrt{V_t^1} S_t\, dW_t^1 + \sqrt{V_t^2} S_t\, dW_t^2\\
dV_t^1 &= \kappa_1(\theta_1 - V_t^1)\, dt + \sigma_1 \sqrt{V_t^1}\, dB_t^1, \quad V_0^1 > 0,\\
dV_t^2 &= \kappa_2(\theta_2 - V_t^2)\, dt + \sigma_2 \sqrt{V_t^2}\, dB_t^2, \quad V_0^2 > 0,\\
dW_t^1\, dB_t^1 &= \rho_1,\\
dW_t^2\, dB_t^2 &= \rho_2,\\
dW_t^1\, dB_t^2 &= 0,\\
dW_t^2\, dB_t^1 &= 0,
\end{aligned} \tag{1.1}$$

where S_t and V_t^i, $i = 1, 2$, represent the price and volatilities of the underlying asset and also, W_t^1, B_t^1 and W_t^2, B_t^2 are the Brownian motions with correlations ρ_1 and ρ_2, respectively. Besides, the parameters κ_i, θ_i, and σ_i, $i = 1, 2$, are the mean reversion speed of the volatility, the long-run mean, and the volatility of volatility, respectively. r is the interest rate. It is widely acknowledged that, if the parameters of the volatility processes obey the condition $2\kappa_i\theta_i > \sigma_i^2$, $i = 1, 2$, known as the Feller condition, then V_t^i, $i = 1, 2$, are strictly positive [1].

The main goal of this work is to investigate the uniqueness and existence of solutions to the double Heston model equations. We follow this aim in Section 2. Besides, we numerically study the price of the Lookback options under this model. Despite the fact that there are many different algorithms, such as [4, 12, 17, 21, 23, 24], to further this aim, we perform the Monte Carlo algorithm based on the different discretization schemes, i.e. Euler, Milstein, Transform Volatility (TV) schemes [20].

Before the birth of the Lookback options, as a type of the exotic options, there was no way that could help investors to cope with a problem called "regret", which all investors caught up with. Regret selling too early and regret holding on for too long to be suddenly swept by a correction. Since Lookback call options would allow investors to buy at the lowest price during the life of the options while Lookback put options would allow investors to sell at the highest price, these options act as an insurance against regret. With Lookback options, investors would never again face the punishing decision of timing an entry or exit. There are two kinds of Lookback options: Lookback option with floating and fixed strike price. In Section 3, we study the price of these kinds of the option under double Heston model. Besides, we provide some numerical results in Section 4.

2 Existence and Uniqueness

In this section, we show the existence and uniqueness of the solution to the double Heston model.

Let us state the following conditions that guarantee the existence and uniqueness of the solution to the stock price equation of the double Heston model.

Assumption 2.1. ***Locally Lipschitz condition.*** *For every integer $n \geq 1$, there exists a positive constant C_n such that for all $t \in [0,T]$ and all $x, \bar{x}, y, \bar{y}, z, \bar{z} \in \mathbb{R}^d$ with $|y| \vee |\bar{y}| \vee |z| \vee |\bar{z}| \leq n$, we have*

$$
\begin{aligned}
|f(t,x,y,z) - f(t,\bar{x},\bar{y},\bar{z})| + |g_1(t,x,y,z) - g_1(t,\bar{x},\bar{y},\bar{z})| \\
+ |g_2(t,x,y,z) - g_2(t,\bar{x},\bar{y},\bar{z})| \\
\leq C_n\big(|x-\bar{x}| + |y-\bar{y}| + |z-\bar{z}|\big),
\end{aligned} \tag{2.1}
$$

Linear growth condition. *There exists a positive constant C such that*

$$
|f(t,x,y,z)| + |g_1(t,x,y,z)| + |g_2(t,x,y,z)| \leq C\big(1 + |x| + |y| + |z|\big), \tag{2.2}
$$

for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$.

Theorem 2.2. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \geq 0\}$ satisfying the usual assumptions, that is, right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $W_t = (W_t^1, W_t^2)$, for all $t \geq 0$, be a vector of two-dimension independent Brownian motion on this probability space. Let Z be a random variable which is independent of the σ-algebra $\mathbb{F}$ generated by W_t and such that $\mathbb{E}[|Z|^2] < \infty$. Let coefficients $f(.,.,.,.)$, $g_1(.,.,.,.)$, and $g_2(.,.,.,.)$ apply to Assumption 2.1 . Then, for every $t \in [0,T]$, the stochastic differential equation*

$$
dS_t = f(t, S_t, V_t^1, V_t^2)\, dt + g_1(t, S_t, V_t^1, V_t^2)\, dW_t^1 + g_2(t, S_t, V_t^1, V_t^2)\, dW_t^2, \tag{2.3}
$$

where $S_0 = Z$ and V_t^1, $V_t^2 > 0$, has a unique solution S adopted to the filtration $\mathbb{F}^Z$ generated by Z and $W_s = \{(W_s^1, W_s^2);\ s \leq t\}$. Moreover, $\mathbb{E}[\int_0^T |S_t|^2\, dt] < \infty$.

Proof. First, we show that the solution of (2.3) is unique. To do so, let $\hat{S}_t$ and $\bar{S}_t$ are two solutions of equation (2.3) with initial values $\hat{Z}$ and $\bar{Z}$ respectively, i.e. $\hat{S}(0,\omega) = \hat{Z}(\omega)$ and $\bar{S}(0,\omega) = \bar{Z}(\omega)$, $\omega \in \Omega$. For every integer $n \geq 1$, define

$$
\hat{\tau}_n := \inf\{t \in [0,T]\ : |V_t^1| \geq n\},
$$

and also,

$$
\bar{\tau}_n := \inf\{t \in [0,T]\ : |V_t^2| \geq n\}.
$$

Set $\tau_n = \hat{\tau}_n \wedge \bar{\tau}_n$. Then

$$\begin{aligned}
&\mathbb{E}[|\hat{S}_{t\wedge\tau_n} - \bar{S}_{t\wedge\tau_n}|^2] \\
&\quad = \mathbb{E}\big[\big|\hat{Z} - \bar{Z} + \int_0^{t\wedge\tau_n} \big(f(u,\hat{S}_u,V_u^1,V_u^2) - f(u,\bar{S}_u,V_u^1,V_u^2)\big)\,du \\
&\qquad + \int_0^{t\wedge\tau_n} \big(g_1(u,\hat{S}_u,V_u^1,V_u^2) - g_1(u,\bar{S}_u,V_u^1,V_u^2)\big)\,dW_u^1 \\
&\qquad + \int_0^{t\wedge\tau_n} \big(g_2(u,\hat{S}_u,V_u^1,V_u^2) - g_2(u,\bar{S}_u,V_u^1,V_u^2)\big)\,dW_u^2\big|^2\big] \\
&\quad \le \mathbb{E}\big[4|\hat{Z} - \bar{Z}|^2 + 4\big|\int_0^t \big(f(u,\hat{S}_u,V_u^1,V_u^2) - f(u,\bar{S}_u,V_u^1,V_u^2)\big)\,du\big|^2 \\
&\qquad + 4\big|\int_0^t \big(g_1(u,\hat{S}_u,V_u^1,V_u^2) - g_1(u,\bar{S}_u,V_u^1,V_u^2)\big)\,dW_u^1\big|^2 \\
&\qquad + 4\big|\int_0^t \big(g_2(u,\hat{S}_u,V_u^1,V_u^2) - g_2(u,\bar{S}_u,V_u^1,V_u^2)\big)\,dW_u^2\big|^2\big].
\end{aligned} \tag{2.4}$$

Using Jensen inequality, we get

$$\begin{aligned}
\mathbb{E}[|\hat{S}_{t\wedge\tau_n} - \bar{S}_{t\wedge\tau_n}|^2] &\le 4\mathbb{E}\big[|\hat{Z} - \bar{Z}|^2\big] \\
&\quad + 4\mathbb{E}\big[\big|\int_0^t f(u,\hat{S}_u,V_u^1,V_u^2) - f(u,\bar{S}_u,V_u^1,V_u^2)\,du\big|^2\big] \\
&\quad + 4\mathbb{E}\big[\big|\int_0^{t\wedge\tau_n} g_1(u,\hat{S}_u,V_u^1,V_u^2) - g_1(u,\bar{S}_u,V_u^1,V_u^2)\,dW_u^1\big|^2\big] \\
&\quad + 4\mathbb{E}\big[\big|\int_0^{t\wedge\tau_n} g_2(u,\hat{S}_u,V_u^1,V_u^2) - g_2(u,\bar{S}_u,V_u^1,V_u^2)\,dW_u^2\big|^2\big].
\end{aligned} \tag{2.5}$$

By the Ito isometry,

$$\begin{aligned}
\mathbb{E}[|\hat{S}_{t\wedge\tau_n} - \bar{S}_{t\wedge\tau_n}|^2] &\le 4\mathbb{E}\big[|\hat{Z} - \bar{Z}|^2\big] \\
&\quad + 4t\ \mathbb{E}\big[\int_0^t \big|f(u,\hat{S}_u,V_u^1,V_u^2) - f(u,\bar{S}_u,V_u^1,V_u^2)\big|^2\,du\big] \\
&\quad + 4\mathbb{E}\big[\int_0^{t\wedge\tau_n} \big|g_1(u,\hat{S}_u,V_u^1,V_u^2) - g_1(u,\bar{S}_u,V_u^1,V_u^2)\big|^2\,du\big] \\
&\quad + 4\mathbb{E}\big[\int_0^{t\wedge\tau_n} \big|g_2(u,\hat{S}_u,V_u^1,V_u^2) - g_2(u,\bar{S}_u,V_u^1,V_u^2)\big|^2\,du\big] \\
&\le 4\mathbb{E}\big[|\hat{Z} - \bar{Z}|^2\big] + 4t\ C_n^2 \int_0^{t\wedge\tau_n} \mathbb{E}[|\hat{S}_u - \bar{S}_u|^2]\,du \\
&\qquad + 4C_n^2 \int_0^{t\wedge\tau_n} \mathbb{E}[|\hat{S}_u - \bar{S}_u|^2]\,du + 4C_n^2 \int_0^{t\wedge\tau_n} \mathbb{E}[|\hat{S}_u - \bar{S}_u|^2]\,du \\
&= 4\mathbb{E}\big[|\hat{Z} - \bar{Z}|^2\big] + 4(t+2)C_n^2 \int_0^t \mathbb{E}[|\hat{S}_{u\wedge\tau_n} - \bar{S}_{u\wedge\tau_n}|^2]\,du.
\end{aligned} \tag{2.6}$$

Thus the function

$$G(t) = \mathbb{E}\big[|\hat{S}_{t\wedge\tau_n} - \bar{S}_{t\wedge\tau_n}|^2\big], \qquad 0 \le t \le T,$$

satisfies

$$G(t) \le F + A\int_0^t G(u)\ du,$$

where $F = 4\mathbb{E}\big[|\hat{Z}-\bar{Z}|^2\big]$ and $A = 4(t+2)C_n^2$. Now assume that $\hat{Z} = \bar{Z}$. By the Gronwall inequality [18] and since $\lim_{n\to\infty}\tau_n = \infty$, it follows that

$$\mathbb{E}\big[\big|\hat{S}_t - \bar{S}_t\big|^2\big] = \mathbb{E}\big[\liminf_{n\to\infty}\big|\hat{S}_t - \bar{S}_t\big|^2\big] \le \liminf_{n\to\infty}\mathbb{E}\big[\big|\hat{S}_t - \bar{S}_t\big|^2\big] = 0.$$

Consequently, by continuity of $t \to \big|\hat{S}_t - \bar{S}_t\big|$ we get

$$\mathbb{P}\big[\big|\hat{S}_t(\omega) - \bar{S}_t(\omega)\big| = 0, \quad \forall\ t \in [0,T]\big] = 1,$$

and the uniqueness is proved.

To prove the existence of the solution, we define $S_t^{(0)} = S_0$ and $S_t^{(k)} = S_t^{(k)}(\omega)$ inductively as follows

$$\begin{aligned} S_t^{(k+1)} = S_0 &+ \int_0^t f(u, S_u^{(k)}, V_u^1, V_u^2)\ du \\ &+ \int_0^t g_1(u, S_u^{(k)}, V_u^1, V_u^2)\ dW_u^1 + \int_0^t g_2(u, S_u^{(k)}, V_u^1, V_u^2)\ dW_u^2. \end{aligned} \tag{2.7}$$

Then, similar computation as the uniqueness, we have

$$\mathbb{E}\big[|S^{(k+1)}_{t\wedge\tau_n} - S^{(k)}_{t\wedge\tau_n}|^2\big] \le 4(t+2)C_n^2\int_0^t \mathbb{E}\big[|S^{(k)}_{u\wedge\tau_n} - S^{(k-1)}_{u\wedge\tau_n}|^2\big]\ du, \tag{2.8}$$

for $k \ge 1$, $t \le T$ and

$$\begin{aligned} &\mathbb{E}\big[|S_t^{(1)} - S_t^{(0)}|^2\big] \\ &\quad \le 3t^2C^2\big(1 + \mathbb{E}\big[|S_0|^2 + |V_0^1|^2 + |V_0^2|^2\big]\big) + 2\times 3tC^2\big(1 + \mathbb{E}\big[|S_0|^2 + |V_0^1|^2 + |V_0^2|^2\big]\big) \\ &\quad \le A_1 t, \end{aligned} \tag{2.9}$$

where the constant A_1 only depends on C, T and $\mathbb{E}\big[|S_0|^2 + |V_0^1|^2 + |V_0^2|^2\big]$. By induction on k we have

$$\mathbb{E}\big[|S_t^{(k+1)} - S_t^{(k)}|^2\big] \le \frac{A_2^{k+1}t^{k+1}}{(k+1)!}, \qquad k \ge 0,\ t \in [0,T], \tag{2.10}$$

for some suitable constant A_2 depending only on C, C_n, T and $\mathbb{E}\big[|S_0|^2 + |V_0^1|^2 + |V_0^2|^2\big]$. The required assertion now follows from the Doob martingale inequality and the Fatou lemma. □

In what follows, we use the *local Lipschitz condition* and the *linear growth condition* [14] to verify the existence and uniqueness of solutions to the volatility equations satisfying the CIR dynamics

$$dV_t^i = \kappa_i(\theta_i - V_t^i)\ dt + \sigma_i\sqrt{V_t^i}\ dB_t^i, \quad i = 1, 2.$$

Theorem 2.3. *Assuming $\kappa_i\theta_i > \frac{\sigma_i^2}{2}$, $i=1,2$, the volatility equations of the double Heston model have unique positive solutions V_t^1, V_t^2 where $t\in[0,T]$.*

Proof. *Theorem 2.3.4* of [14] guarantees the existence of solutions to the volatility equations.

To prove the uniqueness of the solutions, let $\hat{V}^i(t,\omega)$ and $\bar{V}^i(t,\omega)$, $i=1,2$, be solutions of the volatility equations with initial values γ. For every integer $n\geq 1$, define

$$\hat{\tau}_n := T\wedge \inf\{t\in[0,T] \ : |\hat{V}_t^i| \geq n, \ i=1,2\},$$

and also,

$$\bar{\tau}_n := T\wedge \inf\{t\in[0,T] \ : |\bar{V}_t^i| \geq n, \ i=1,2\}.$$

Set $\tau_n = \hat{\tau}_n \wedge \bar{\tau}_n$. Then, for $i=1,2$

$$\begin{aligned}
&\mathbb{E}\big[|\hat{V}^i_{t\wedge\tau_n} - \bar{V}^i_{t\wedge\tau_n}|^2\big] \\
&\quad = \mathbb{E}\big[\big|\int_0^{t\wedge\tau_n} \big(-\kappa_i(\hat{V}_u^i - \bar{V}_u^i)\big)\ du + \int_0^{t\wedge\tau_n} \sigma_i\big(\sqrt{\hat{V}_u^i} - \sqrt{\bar{V}_u^i}\big)\ dB_u^i\big|^2\big] \\
&\quad \leq 2\ \mathbb{E}\big[\big|\int_0^{t\wedge\tau_n} \big(-\kappa_i(\hat{V}_u^i - \bar{V}_u^i)\big)\ du\big|^2\big] + 2\ \mathbb{E}\big[\big|\int_0^{t\wedge\tau_n} \sigma_i\big(\sqrt{\hat{V}_u^i} - \sqrt{\bar{V}_u^i}\big)\ dB_u^i\big|^2\big] \\
&\quad \leq 2t\kappa_i^2\ \mathbb{E}\big[\int_0^{t\wedge\tau_n} |\hat{V}_u^i - \bar{V}_u^i|^2\ du\big] + 2\sigma_i^2\ \mathbb{E}\big[\int_0^{t\wedge\tau_n} \big|\sqrt{\hat{V}_u^i} - \sqrt{\bar{V}_u^i}\big|^2\ du\big]. \qquad (2.11)
\end{aligned}$$

Given local condition, there exists $\epsilon > 0$, so that

$$\mathbb{E}\big[|\hat{V}^i_{t\wedge\tau_n} - \bar{V}^i_{t\wedge\tau_n}|^2\big] \leq 2T\kappa_i^2\ \mathbb{E}\big[\int_0^{t\wedge\tau_n} |\hat{V}_u^i - \bar{V}_u^i|^2\ du\big] + \frac{2\sigma_i^2}{\epsilon^2}\ \mathbb{E}\big[\int_0^{t\wedge\tau_n} |\hat{V}_u^i - \bar{V}_u^i|^2\ du\big], \qquad (2.12)$$

and from the local Lipschitz condition [14] we have

$$\mathbb{E}\big[|\hat{V}^i_{t\wedge\tau_n} - \bar{V}^i_{t\wedge\tau_n}|^2\big] \leq 2\big(T\kappa_i^2 + \frac{\sigma_i^2}{\epsilon^2}\big)C_n^2\ \int_0^t \mathbb{E}\big[|\hat{V}^i_{u\wedge\tau_n} - \bar{V}^i_{u\wedge\tau_n}|^2\ du\big]. \qquad (2.13)$$

We now apply the Gronwall inequality to conclude that $\{\hat{V}^i_{t\wedge\tau_n};\ 0\leq t\leq T\}$ and $\{\bar{V}^i_{t\wedge\tau_n};\ 0\leq t\leq T\}$ are modification of each another and thus are indiscernible. Letting $n\to\infty$ we see that the same is true for $\{\hat{V}_t^i;\ 0\leq t\leq T\}$ and $\{\bar{V}_t^i;\ 0\leq t\leq T\}$. □

3 Lookback option pricing

Assuming $T>0$ as the time to maturity, we simulate the equations of the double Heston model over the time interval $[0,T]$, given a discretization as $0=t_1<t_2<\cdots<t_n=T$, where the time increments are equally spaced with width dt and $t_k = k\ dt$, for all $k=1,2,\cdots,n$. The volatility processes in (1.1), V_t^i, $i=1,2$, are written in integral form as

$$V_{t+dt}^i = V_t^i + \int_t^{t+dt} \kappa_i(\theta_i - V_u^i)\ du + \int_t^{t+dt} \sigma_i\sqrt{V_u^i}\ dB_u^i, \quad i=1,2.$$

The Euler discretization approximates the integrals using the left-point rule

$$\int_t^{t+dt} \kappa_i(\theta_i - V_u^i)\ du \approx \kappa_i(\theta_i - V_t^i)\ dt$$

$$\begin{aligned}\int_t^{t+dt} \sigma_i \sqrt{V_u^i}\ dB_u^i &\approx \sigma_i \sqrt{V_t^i}\ (B_{t+dt}^i - B_t^i) \\ &= \sigma_i \sqrt{V_t^i\ dt} Z_v^i,\end{aligned}$$

where Z_v^i, $i = 1, 2$, are two standard normal random variables. The right hand side involves $(\theta_i - V_t^i)$ rather than $(\theta_i - V_{t+dt}^i)$ since it is indistinct the value of V_{t+dt}^i at time t. Thus we have

$$V_{t+dt}^i = V_t^i + \kappa_i(\theta_i - V_t^i)\ dt + \sigma_i \sqrt{V_t^i\ dt} Z_v^i.$$

Here, the volatility processes stay positive provided $\sigma_i^2 < 2\kappa_i\theta_i$. Unfortunately, this is rarely satisfied and for this reason, it is applied, as common approaches, the full truncation scheme where V_t^i is replaced with $V_t^{i+} = \max(0, V_t^i)$ or the reflection scheme where V_t^i is replaced with $|V_t^i|$. In this work, we exert the full truncation scheme.

To simulate the asset price process in (1.1), we write this process in exponential form. By the Ito lemma, we obtain

$$d \ln S_t = (r - \frac{V_t^1 + V_t^2}{2})\ dt + \sqrt{V_t^1}\ dW_t^1 + \sqrt{V_t^2}\ dW_t^2,$$

or in integral form

$$\ln S_{t+dt} = \ln S_t + \int_0^t (r - \frac{V_u^1 + V_u^2}{2})\ du + \int_0^t \sqrt{V_u^1}\ dW_u^1 + \int_0^t \sqrt{V_u^2}\ dW_u^2.$$

Consequently, the Euler discretization of the $\ln S_t$ is as follows

$$\begin{aligned}\ln S_{t+dt} &= \ln S_t + (r - \frac{V_t^1 + V_t^2}{2})\ dt + \sqrt{V_t^1}\ (W_{t+dt}^1 - W_t^1) + \sqrt{V_t^2}\ (W_{t+dt}^2 - W_t^2) \\ &= \ln S_t + (r - \frac{V_t^1 + V_t^2}{2})\ dt + \sqrt{V_t^1\ dt}\ Z_s^1 + \sqrt{V_t^2\ dt}\ Z_s^2.\end{aligned}$$

To discrete the asset price process S_t we set

$$S_{t+dt} = S_t\ \exp\left((r - \frac{V_t^1 + V_t^2}{2})\ dt + \sqrt{V_t^1\ dt}\ Z_s^1 + \sqrt{V_t^2\ dt}\ Z_s^2\right),$$

where Z_s^i, $i = 1, 2$ are, as usual, two standard normal random variables. Again, to handle the volatilities when they become negative, we must apply the full truncation or reflection schemes by replacing V_t^i, $i = 1, 2$, everywhere with V_t^{i+} or with $\left|V_t^i\right|$.

The payoff of the lookback option is dependent on the maximum or minimum of the asset price achieved during the life of the option. This allows the holder to look back over time to determine the payoff. Lookback option is categorized as a path dependent securities since its payoff depends on the path followed by the price of the underlying asset and not just on its final value. The payoff of the lookback option with a fixed strike price K is the maximum difference between the optimal underlying asset price and the strike. For the call option, the holder prefer to exercise the option at the point when the underlying asset price is at its highest level and for the put option, the holder prefer to exercise at the lowest price of the underlying asset. Finally, by defining $\tau = T - t$, the value of the lookback call and put options with fixed strike price K are described as

$$\begin{aligned}
\textit{Lookback Call} &= e^{-r\tau}\ \max(G_t - K, 0),\\
\textit{Lookback Put} &= e^{-r\tau}\ \max(K - H_t, 0),
\end{aligned}$$

where G_t and H_t are, respectively, the maximum and the minimum of the price process over the time interval $[0, T]$. Moreover, the payoff of the lookback option with a float strike price is the maximum difference between the market asset's price at maturity and the floating strike. For the call, the strike price is fixed at the asset's lowest price during the option's life, and, for the put, it is fixed at the asset's highest price. Note that these options are not really options, as they will be always exercised by their holder. In fact, the option is never out-of-the-money, which makes it more expensive than a standard option. Again, by defining $\tau = T - t$, the value of the lookback call and put options with floating strike price are described as

$$\begin{aligned}
\textit{Lookback Call} &= e^{-r\tau}\ \max(S_T - H_t, 0),\\
\textit{Lookback Put} &= e^{-r\tau}\ \max(G_t - S_T, 0),
\end{aligned}$$

where S_T is the underlying asset's price at maturity time T. These equations are used to produce Algorithm 1 for the Monte Carlo simulation of a Lookback call option with fixed and floating strike price under double Heston model.

Algorithm 1 Monte Carlo simulation of a Lookback call option with fixed and floating strike.

$m \leftarrow$ number of path
$n \leftarrow$ number of steps
$sum_{fix} \leftarrow 0$
$sum_{float} \leftarrow 0$
for $j = 1, \cdots, m$ **do**
  $S \leftarrow$ initial price of asset
  $V^1 \leftarrow$ initial value of the first volatility process
  $V^2 \leftarrow$ initial value of the second volatility process
  for $i = 1, \cdots, n$ **do**
    $Z_s^1 \leftarrow$ randn;
    $u_1 \leftarrow$ randn;
    $Z_v^1 = Z_s^1 * \rho_1 + u_1 * \sqrt{1-\rho_1^2}$;
    $Z_s^2 \leftarrow$ randn;
    $u_2 \leftarrow$ randn;
    $Z_v^2 = Z_s^2 * \rho_2 + u_2 * \sqrt{1-\rho_2^2}$;

$$S(i+1) = S(i) * \exp\Bigg(\Big(r - \frac{\max(V^1(i),0) + \max(V^2(i),0)}{2} \Big) * dt + \sqrt{\max(V^1(i),0) * dt} * Z_s^1 + \sqrt{\max(V^2(i),0) * dt} * Z_s^2 \Bigg);$$

$$V^1(i+1) = V^1(i) + \kappa_1 * (\theta_1 - \max(V^1(i),0)) * dt + \sigma_1 * \sqrt{\max(V^1(i),0) * dt} * Z_v^1;$$

$$V^2(i+1) = V^2(i) + \kappa_2 * (\theta_2 - \max(V^2(i),0)) * dt + \sigma_2 * \sqrt{\max(V^2(i),0) * dt} * Z_v^2;$$

  end for
  $G \leftarrow$ the maximum value of the asset path
  $H \leftarrow$ the minimum value of the asset path
  $sum_{fix} = sum_{fix} + e^{-r\tau} * \max(G-K,0)$;
  $sum_{float} = sum_{float} + e^{-r\tau} * \max(S(n+1)-H,0)$;
end for
$value_{fix} = sum_{fix}/m$;
$value_{float} = sum_{float}/m$;
return $value_{fix}$
return $value_{float}$

4 Numerical result

In what follows, we present some simulation test to examine how the parameters of the double Heston model affect it. All our numerical results have been performed with the parameters in the Table 1 which are selected from [9].

Table 1: Parameters used for simulation

initial price of asset	100
interest rate	0.04
initial volatility 1	0.05
initial volatility 2	0.03
volatility of volatiliy 1	0.3
volatility of volatiliy 2	0.1
mean reversion 1	4
mean reversion 2	2
long-run mean 1	0.05
long-run mean 2	0.03
correlation 1	-0.5
correlation 2	0.5

In Table 2, it is presented the price of the Lookback call option with fixed strike under the double Heston model with various values of the time maturity T and strike price K. Results show that a higher value is obtained for the price of the option when the maturity time increases and the strike price decreases (see Figure 1).

In Table 3, it is presented the price of the Lookback call option with fixed strike under the double Heston model with various values of the time maturity T and strike price K. Results show that a higher value is obtained for the price of the option when the maturity time increases and the strike price decreases (see Figure 2).

In the Tables 4, 5, and 6, we point to the effect of the parameters time to maturity T, interest rate r, and strike price K on the value of the Lookback option with fixed strike price by considering the Euler, Milstein, and TV discretization schemes.

In the following, we investigate the impact of the parameters of the volatility processes to the double Heston model on the implied volatility surface (See Figure 3).

The parameters ρ_1 and ρ_2 will produce asymmetric volatility smiles that look more like skews (It is obtained a symmetric smile with $\rho = 0$). This effect is illustrated in Figure 4.

The parameters volatility of volatility σ_1 and σ_2 and parameters mean reversion κ_1 and κ_2 affect the smile effect but results show that the effect of them opposes each other. Figure 5 demonstrates the duty of the parameters σ_1 and σ_2 where it can be identified that increasing these parameters will magnify the smile effect. Besides, Figure 6 illustrates that increasing the value of the parameters κ_1 and κ_2 will result in lower effect.

The initial volatilities V_0^1 and V_0^2 and their long-run mean volatilities θ_1 and θ_2, in addition to the smile effect, will impact the level of the smile. This effect is illustrated in Figure 7.

Table 2: Lookback call option price with fixed strike under the double Heston model.

T(years)	0.5	1	1.5	2
$K = 90$	26.3363	34.2185	40.9113	[illegible].8507
$K = 95$	21.3909	29.2172	35.9271	42.261[illegible]
$K = 100$	16.3910	24.3322	30.9304	37.355[illegible]
$K = 105$	12.1534	20.2324	26.88[illegible]	33.1380
$K = 110$	8.8112	16.4168	22.8[illegible]18	29.2039

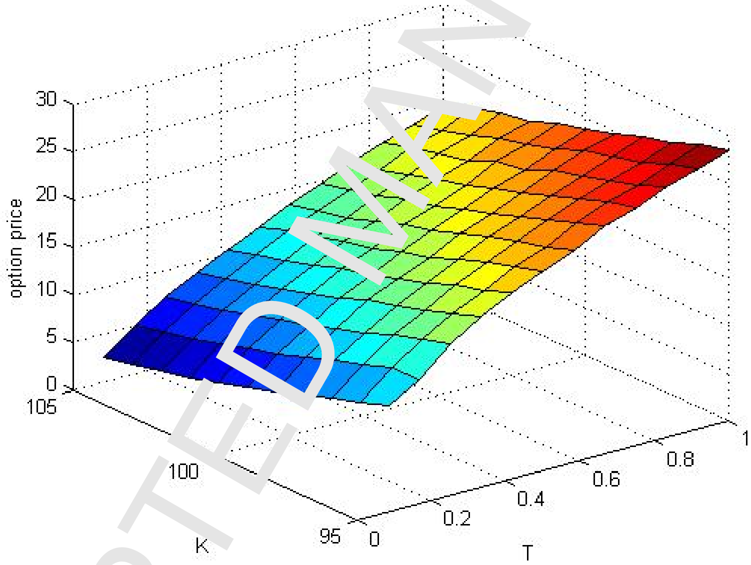

Figure 1: Lookback call option price surface with fixed strike under the double Heston model.

Table 3: Lookback put option price with fixed strike under the double Heston model.

T(years)	0.5	1	1.5	2
$K = 90$	5.6707	9.4989	12.0791	14.1583
$K = 95$	8.8162	12.9955	15.8293	17.6736
$K = 100$	13.0097	17.2249	20.0535	21.9678
$K = 105$	17.8711	21.9838	24.6868	26.5135
$K = 110$	22.8230	26.8653	29.3836	31.0167

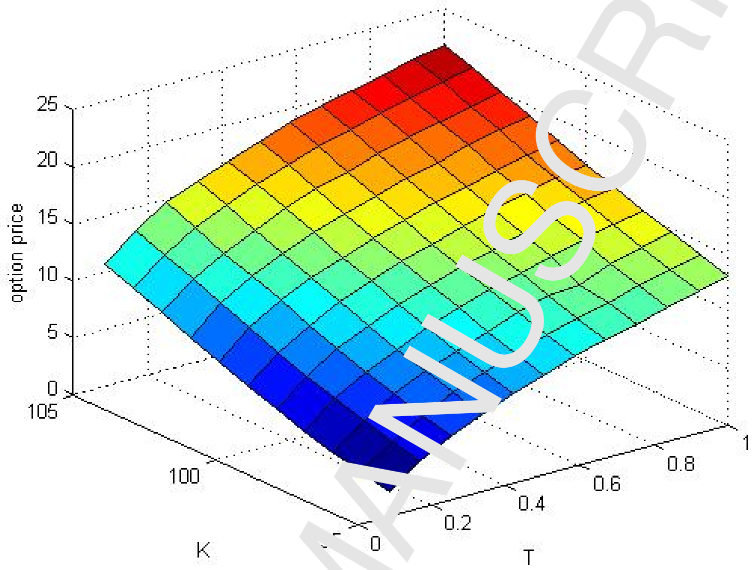

Figure 2: Lookback put option price surface with fixed strike under the double Heston model.

Table 4: The value of the Lookback call option with fixed strike price under the double Heston model by different discretization schemes.

Method	$T = 1/12$	$T = 4/12$	$T = 8/12$	$T = 1$
Euler	6.3[illegible]97	13.1465	19.2434	24.4866
Milstein	6.2944	13.0964	19.3189	24.4010
TV	6.2333	13.2351	19.3439	24.4894

Table 5: The value of the Lookback call option with fixed strike price under the double Heston model by different discretization schemes.

Methods	$r = 0.01$	$r = 0.03$	$r = 0.05$	$r = 0.07$
Euler	22.6792	23.9604	25.0815	26.3310
Milstein	22.5938	23.9330	25.1265	26.3142
TV	22.4587	23.8900	25.6935	26.2591

Table 6: The value of the Lookback call option with fixed strike price under the double Heston model by different discretization schemes.

Methods	$K = 90$	$K = 95$	$K = 100$	$K = 105$	$K = 110$
Euler	34.3228	29.3635	24.4110	19.9880	16.4348
Milstein	34.1877	29.4644	24.3092	20.0471	16.3936
TV	34.1535	29.2037	24.4062	20.0142	16.3394

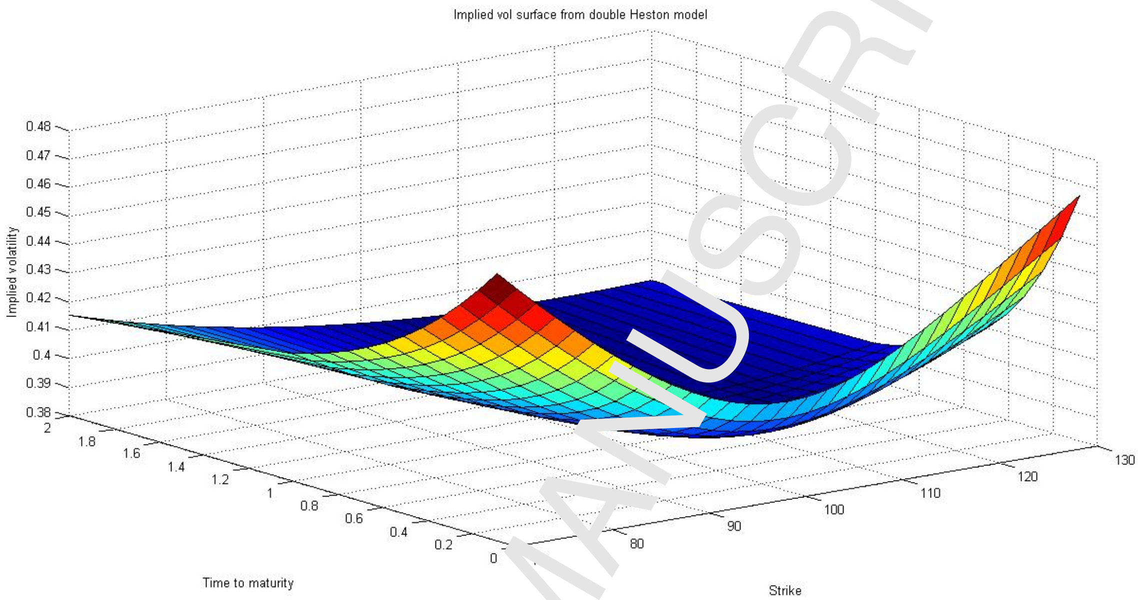

Figure 3: Implied volatility surface from double Heston model.

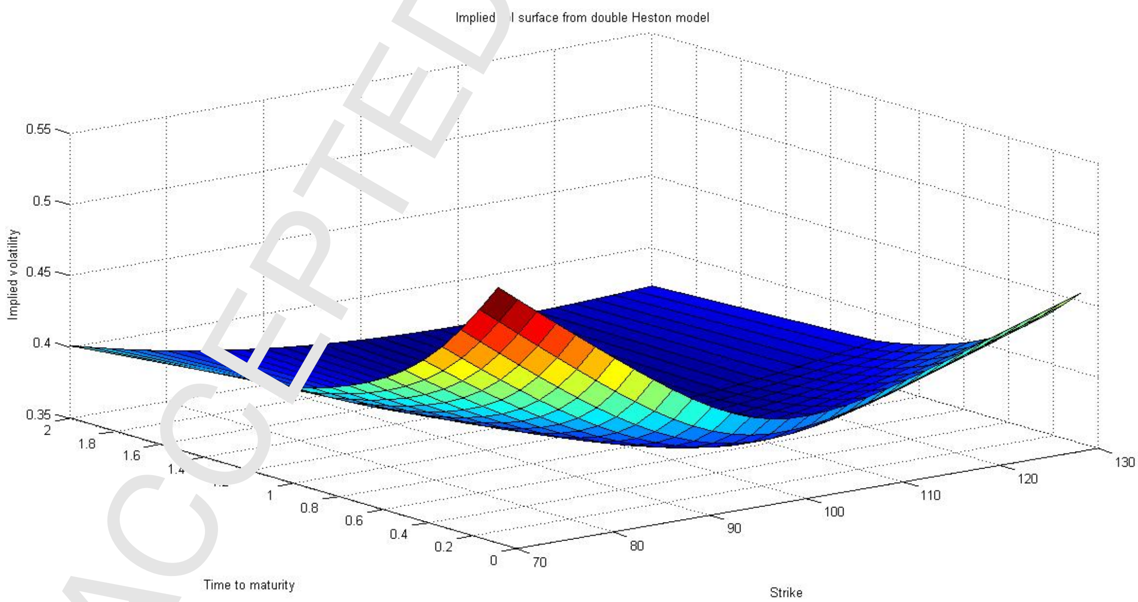

Figure 4: Implied volatility surface from double Heston model whit changed correlations.

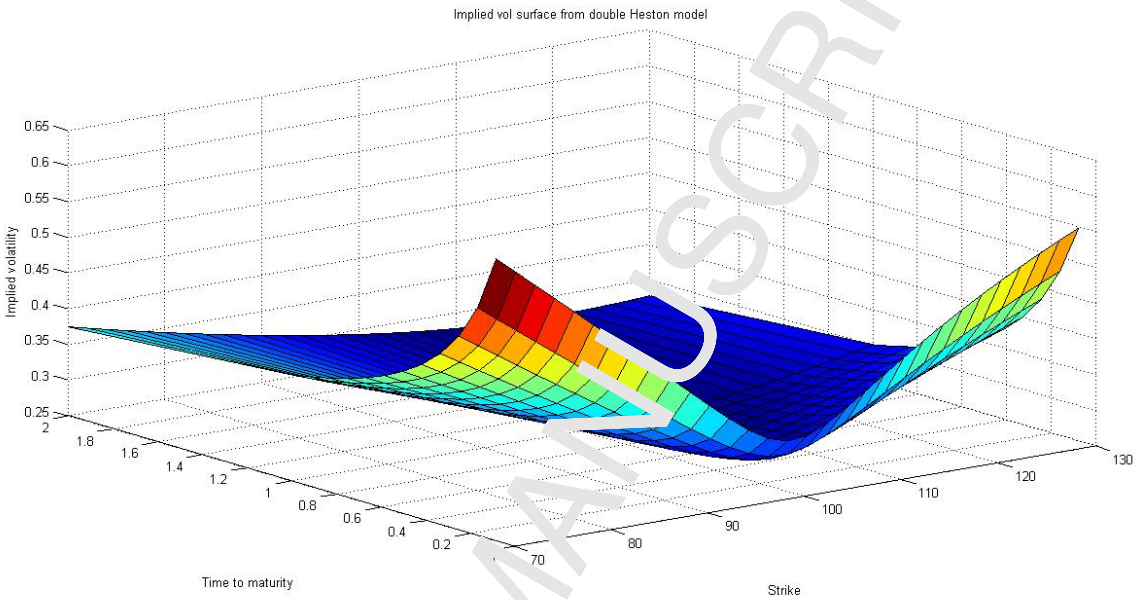

Figure 5: Implied volatility surface from double Heston model whit changed volatility of volatilities.

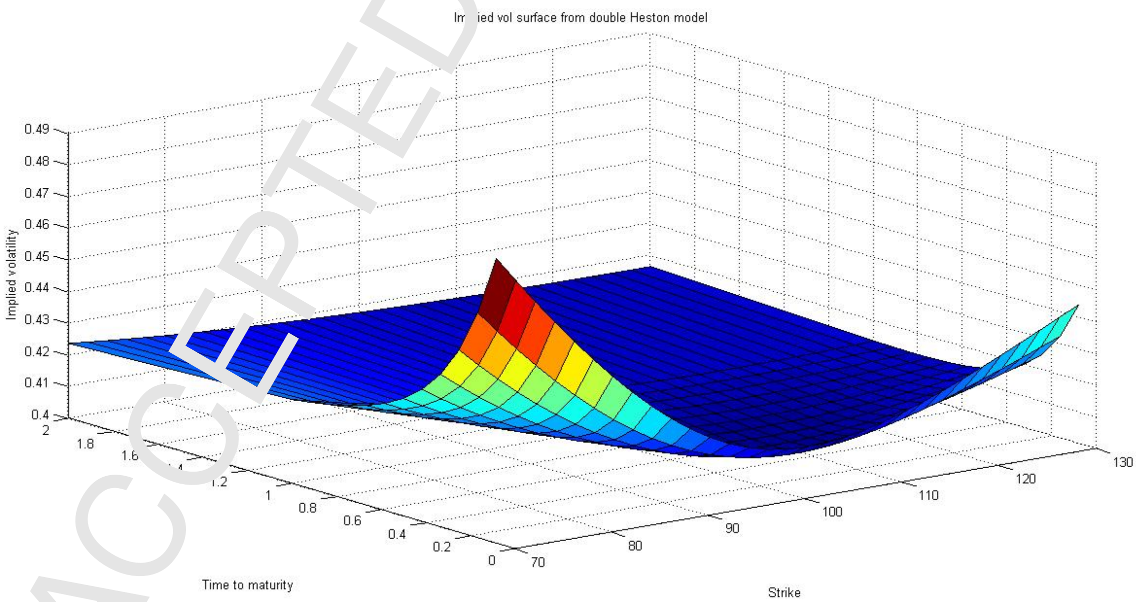

Figure 6: Implied volatility surface from double Heston model whit changed mean reversions.

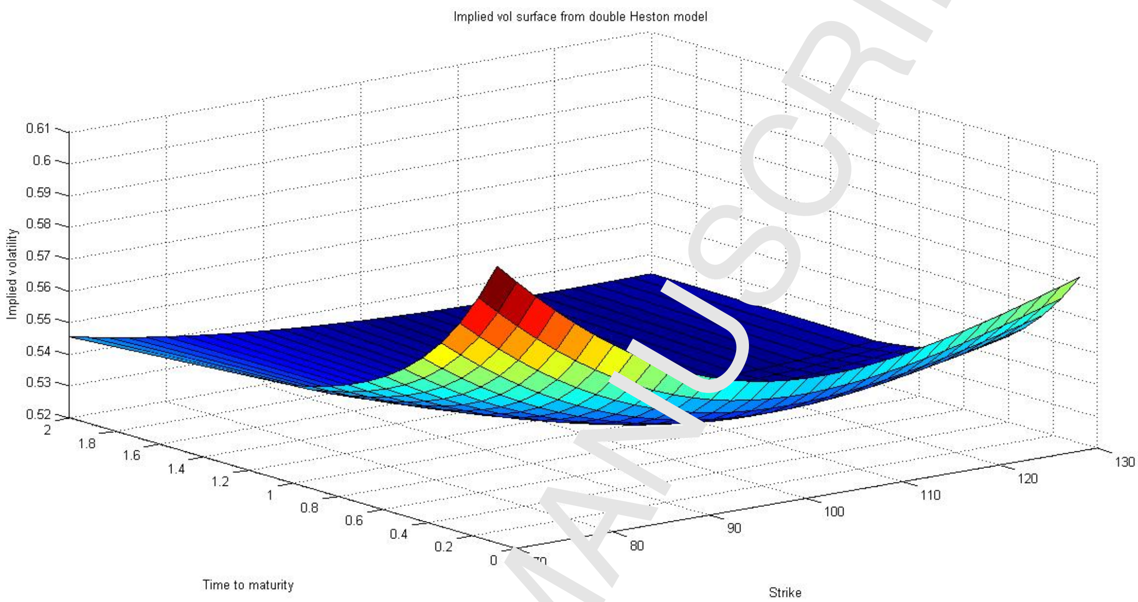

Figure 7: Implied volatility surface from double Heston model whit changed long-run volatilities.

5 Conclusion

The main goal of this paper is to examine the existence and uniqueness of solutions of the double Heston model. Generally, Lipschitz and linear growth are conditions which guarantee the existence and uniqueness of the solution. However, our under consideration model is defined based on two stochastic volatility processes which follow the CIR processes. Thus, it is replaced the Lipschitz condition by the local Lipschitz condition. Besides, we numerically study the value of the Lookback options under the double Heston model by applying the Monte Carlo algorithm. We use different discretization schemes, i.e. Euler, Milstein, and transform volatility schemes, to evaluate the Lookback option price under this model and verify the effect of the some parameters on the results. Finally, since, among the stochastic volatility models, the double Heston model is one of the simple yet powerful model to present the smile volatility, we study the behaviour of the smile volatility generated by this model under the related parameters. But one of the limitations of this study is that, as there is not an analytic pricing formula for the Lookback options under the double Heston model, we present our results based on a numerical experiments, of course, providing such a formula is one of the our noticeable end in the future.

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