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Delay-dependent stability of Runge–Kutta methods for linear delay differential–algebraic equations[☆]

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ABSTRACT

This paper is concerned with the stability analysis of numerical solutions for linear delay differential–algebraic equations. By means of the argument principle, the delay-dependent stability of Runge–Kutta methods is investigated. Several numerical examples are given to illustrate the effectiveness of the proposed results.

1. Introduction

In this paper, we consider the following linear delay differential–algebraic equations (DDAEs)

$$\begin{cases} E\dot{x}(t) = Lx(t) + Mx(t-\tau), \quad t \geq 0, \\ x(t) = \phi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1}$$

where $x(t) \in C^d$ is the state. The delay τ is given and $\tau > 0$. $E, M, L \in C^{d\times d}$. The parameter matrix E is singular and $rankE = q < d$. In contrast to standard linear systems for which E is invertible, DDAEs (1) may not possess a solution for certain initial conditions. Therefore, the vector valued initial function $\phi(t)$ is required to be consistent with DDAEs (1) which ensures that the associated initial value problem has at least one solution, see e.g. [1,2].

Over the past few decades, a considerable amount of research has been done in the field of stability analysis of numerical methods for (delay) differential–algebraic equations (see e.g. [2–6]). Compared with classical delay system, delay differential–algebraic equations endow many special features such as impulse terms and input derivatives in the state response, non-causality between input and state (or output), consistent initial conditions, etc., which make the study of delay differential–algebraic equations more sophisticated than classical linear delay system.

Stability on numerical methods for classical delay differential equations has been investigated by numerous authors, see e.g. [7–13]. As we know, the stability of numerical solutions for delay differential equations can be fallen into two categories according to its dependence upon the size of time-delays. The first one is called delay-independent stability, which does not depend on time-delays; Otherwise, it is referred to as delay-dependent stability. Each case can naturally

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be extended to delay differential–algebraic equations. Delay-independent stability analysis of numerical methods for the linear neutral differential–algebraic equations with a constant delay

$$E\dot{x}(t) = Lx(t) + Mx(t-\tau) + N\dot{x}(t-\tau), \quad t > 0 \tag{2}$$

or with multiple delays has been investigated in [5,6,14,15], etc. But, to our knowledge, there is almost no result on the delay-dependent stability analysis of Runge–Kutta (RK) methods for delay differential–algebraic equations.

It is well known that the analysis of delay-dependent stability is much more difficult than the delay-independent case. One of the reasons is that the delay-dependent stability region is larger and more complicated to describe. Recently, to study the delay-dependent stability of numerical methods for the delay differential system of neutral type

$$\dot{x}(t) = Lx(t) + Mx(t-\tau) + N\dot{x}(t-\tau), \quad t > 0, \tag{3}$$

Hu and Mitsui in [16] gave a new definition which is known as weak delay-dependent stability. Compared to the definition of $D-$stability, the proposed weak delay-dependent stability is more relaxed. After that, sufficient conditions for delay-dependent stability of RK methods for linear neutral system with multiple delays are presented in [17]. Following the same line as in [16], delay-dependent stability of linear multistep (LM) methods for linear differential–algebraic equations with multiple delays was analyzed in [18].

In comparison with our previous work [18], the present paper aims to assess the stability of RK methods for linear DDAEs (1). In particular, weak delay-dependent stability criteria of semi-implicit and fully implicit RK methods are proposed by means of the argument principle. As pointed out in [19], RK methods have interesting computational and theoretical properties. Compared to LM methods, RK methods combine higher order with better stability and are self-starting. We also notice that delay-dependent stability analysis of RK methods is more difficult than LM methods. These give the principle motivation of the present paper.

The paper is organized as follows. In Section 2, some preliminary results are introduced. Section 3 addresses delay-dependent stability of RK methods. Finally, we present three numerical examples to demonstrate the effectiveness of the obtained results in Section 4 and conclude the paper in Section 5.

2. Preliminaries

In this section, we present some definitions and lemmas, including the stability criteria of the analytical solutions which are essential for the main results in this paper.

Definition 2.1 (*e.g. [20]*)**.** (i) DDAEs (1) is called regular if the characteristic polynomial $\det(sE - L)$ is not identically zero. (ii) DDAEs (1) is called causal if $deg(\det(sE - L)) = rankE = q$.

It is worth noting that the definition of causality of differential–algebraic system imposes a stronger requirement than causality in classical system theoretic meaning. As shown in [20], the causality of differential–algebraic system implies that the index is one. Throughout this paper, we first make an assumption that DDAEs (1) is regular and causal. Hence there are two invertible matrices S and $T \in C^{d\times d}$ such that

$$\widetilde{E} = SET = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}, \qquad \widetilde{L} = SLT = \begin{bmatrix} L_1 & 0 \\ 0 & I_{d-q} \end{bmatrix},$$

$$\widetilde{M} = SMT = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}. \tag{4}$$

Setting

$$x(t) = Ty(t), \tag{5}$$

where $y(t) = [\eta_1(t), \eta_2(t)]^T$ with $\eta_1(t) \in C^q$, $\eta_2(t) \in C^{d-q}$. Then DDAEs (1) reduces to the following form

$$\begin{cases} \dot{\eta}_1(t) = L_1\eta_1(t) + M_1\eta_1(t-\tau) + M_2\eta_2(t-\tau), \\ \eta_2(t) = -M_3\eta_1(t-\tau) - M_4\eta_2(t-\tau). \end{cases} \tag{6}$$

In this case, the initial condition can be written as

$$\eta_1(t) = \phi_1(t), \qquad \eta_2(t) = \phi_2(t), \quad t \in [-\tau, 0]. \tag{7}$$

Hence, the study of stability for DDAEs (1) is equal to study this of the canonical form (6)–(7).

The stability analysis is usually based on the roots of the characteristic equation

$$P(s) = \det \Delta(s) = 0 \tag{8}$$

associated with the Laplace transform of (6), where

$$\Delta(s) = \begin{bmatrix} sI_q - L_1 - M_1e^{-s\tau} & -M_2e^{-s\tau} \\ -M_3e^{-s\tau} & -I_{d-q} - M_4e^{-s\tau} \end{bmatrix}.$$

Lemma 2.2 (*e.g.* [21]). *Assume that DDAEs* (1) *is regular and causal. Then DDAEs* (1) *is asymptotically stable iff* $\sigma < 0$, *where*

$$\sigma = \sup\{Res : det\,\Delta(s) = 0\},$$

and Res stands for the real part of a complex number s.

The following lemmas are useful to check the existence of inverse matrix.

Lemma 2.3 (*e.g.* [22]). *For any matrix* $F \in C^{d\times d}$,

$$\rho(F) \le \|F\|,$$

where $\rho(F)$, $\|F\|$ *stand for the spectral radius and compatible matrix norm of matrix F, respectively.*

Lemma 2.4 (*e.g.* [22]). *Let* $F \in C^{d\times d}$, *if* $\rho(F) < 1$, *then* $(I-F)^{-1}$ *exists, and*

$$\frac{1}{1+\|F\|} \le \|(I-F)^{-1}\| \le \frac{1}{1-\|F\|}.$$

The argument principle is presented as follows.

Lemma 2.5 (*e.g.* [23]). *Suppose that*

(A1) a function $H(s)$ *is meromorphic in the domain interior to a positively oriented simple closed counter* γ*;*

(A2) $H(s)$ *is analytic and nonzero on* γ*;*

(A3) counting multiplicities, Z is the number of zeros and Y is the number of poles of $H(s)$ *inside* γ.

Then

$$\frac{1}{2\pi}\Delta_\gamma \arg H(s) = Z - Y,$$

where $\Delta_\gamma \arg H(s)$ *represents the change of the argument of* $H(s)$ *along* γ.

By means of the argument principle, the delay-dependent stability criteria of differential–algebraic equations with multiple delays are obtained in [18]. To proceed, we assume that the condition

$$\|M_4\| < 1 \tag{9}$$

holds throughout this paper. The following lemma presents the location of all unstable characteristic roots of DDAEs (1).

Lemma 2.6 ([18]). *Suppose that condition* (9) *holds. Let s be a characteristic root of DDAEs* (1) *with* $Res \ge 0$, *then*

$$|s| \le \|L_1\| + \|M_1\| + \frac{\|M_2\|\cdot\|M_3\|}{1-\|M_4\|} \doteq r. \tag{10}$$

Define the semi-circle region $D_r = \{s : |s| \le r \text{ and } Res \ge 0\}$. Let Γ_r be the boundary of the bounded region D_r. See Fig. 1 in detail, in which $d_1 = (r, \frac{\pi}{2})$, $d_2 = (r, -\frac{\pi}{2})$ are in polar coordinates. Then the following lemma is a special case of stability criteria of analytical solutions obtained in [18].

Lemma 2.7 ([18]). *Suppose that condition* (9) *holds. Then DDAEs* (1) *is asymptotically stable iff*

$$P(s) \ne 0 \quad \text{for } s \in \Gamma_r \tag{11}$$

and

$$\Delta_{\Gamma_r} \arg P(s) = 0 \tag{12}$$

hold, where $\Delta_{\Gamma_r} \arg P(s)$ *represents the change of the argument of* $P(s)$ *along the closed semi-circumference* Γ_r.

Remark 2.8. According to Lemma 2.7, an algorithm is presented in [18] for checking the asymptotic stability of DDAEs (1).

Remark 2.9. It should be pointed out that condition (9) is a restriction to our main results. For a given delay differential–algebraic equation, $\|M_4\|$ is a constant since it is determined by the coefficients of DDAEs (1). But when $\|M_4\|$ approaches 1, the radius r of the semi-circle region becomes large and it needs a lot of computational efforts.

3. Main results

In this section, we investigate the delay-dependent stability of RK methods for DDAEs (1). In what follows, we adopt the uniform step-size $h = \tau/m$, where $m \ge 1$ is an integer. We first present the definition of weak delay-dependent stability for DDAEs (1), which is the adaptation for the case of neutral delay differential equations proposed in [16] by Hu and Mitsui.

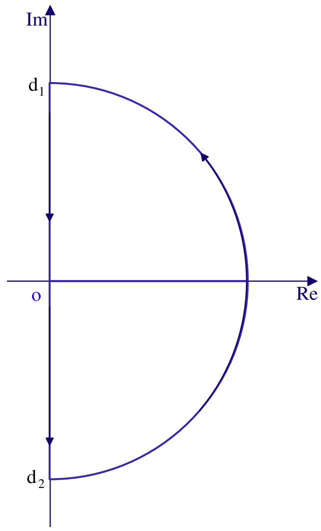


Fig. 1. Semi-circle region D_r.

Definition 3.1. Assume that DDAEs (1) is asymptotically stable for given matrices E, L, M and time-delay τ. A numerical method is said to be weakly delay-dependently stable for DDAEs (1) if there exists a positive integer m such that the step-size $h = \tau/m$ and the numerical solution x_n with h satisfies

$$x_n \to 0 \ \ as \ \ n \to \infty$$

for any consistent initial condition.

Before proceeding, we simply sketch RK schemes for the initial value problem of ordinary differential equations (ODEs)

$$\begin{cases} \dot{u}(t) = f(t, u(t)), \quad t \geq 0, \\ u(0) = u_0, \end{cases} \tag{13}$$

where $u(t) \in R^d$. It is well known that a $v-$stage RK method for ODEs (13) (see e.g. [24]) is defined by

$$K_i = hf(t_n + c_i h, u_n + h\sum_{j=1}^{v} a_{ij}K_j), \quad i = 1, 2, \ldots, v, \tag{14}$$

and

$$u_{n+1} = u_n + \sum_{i=1}^{v} b_i K_i, \quad n = 0, 1, 2, \ldots, \tag{15}$$

where $h > 0$ denotes a given uniform step-size, $t_n = nh$ $(n = 0, 1, 2, \ldots)$, u_n is the approximation of $u(t_n)$. The RK method (14)–(15) can be represented as the Butcher tableau

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \ldots & a_{1v} \\ c_2 & a_{21} & a_{22} & \ldots & a_{2v} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_v & a_{v1} & a_{v2} & \ldots & a_{vv} \\ \hline & b_1 & b_2 & \ldots & b_v \end{array} .$$

An RK method is said to be explicit if $a_{ij} = 0$ for $i \leq j$; otherwise it is of implicit type; particularly, it is semi-implicit if $a_{ij} = 0$ for $i < j$.

We are now in a position to describe the RK methods for DDAEs (1). In view of the algebraic restrictions, in particular to those that are not explicitly given in the system, there are many difficulties in applying standard RK schemes for ODEs directly to DDAEs. As pointed out in [25], explicit RK methods cannot be used directly to differential–algebraic equations. Hence, to obtain a reasonable RK scheme for DDAEs (1), it is necessary that the matrix A in the Butcher tableau is nonsingular, which implies that we are confined to implicit RK methods.

Implemented with the uniform step-size $h = \tau/m$, where m is a positive integer, and applying the $v-$stage RK scheme (14)–(15) to DDAEs (1), we have

$$EK_{n,i} = hL(x_n + \sum_{j=1}^{v} a_{ij}K_{n,j}) + hM(x_{n-m} + \sum_{j=1}^{v} a_{ij}K_{n-m,j}), \quad i = 1, 2, \dots, v, \tag{16}$$

and

$$x_{n+1} = x_n + \sum_{i=1}^{v} b_i K_{n,i}, \quad n = 0, 1, 2, \dots, \tag{17}$$

in which x_n is a sequence of approximate values of $x(t_n)$, $t_n = nh$ $(n = 1, 2, \dots)$ are equidistant step-values with the step-size $h = \tau/m$, $m \geq 1$ integer. The symbol $K_{n,i}$ means the $i-$stage value of the RK method at the $n-$th step-point.

With the non-singular linear transformation (5), the stability analysis of RK scheme (16)–(17) for DDAEs (1) can be reduced to those of the canonical form (6) with the initial condition (7), whose RK method can be described by

$$\widetilde{E}K_{n,i} = h\widetilde{L}(y_n + \sum_{j=1}^{v} a_{ij}K_{n,j}) + h\widetilde{M}(y_{n-m} + \sum_{j=1}^{v} a_{ij}K_{n-m,j}), \quad i = 1, 2, \dots, v, \tag{18}$$

and

$$y_{n+1} = y_n + \sum_{i=1}^{v} b_i K_{n,i}, \quad n = 0, 1, 2, \dots, \tag{19}$$

where y_n is a sequence of approximate value of $y(t_n)$, and the matrices $\widetilde{E}, \widetilde{L}, \widetilde{M}$ are given in (4).

Lemma 3.2. *The characteristic polynomial of the resulting difference system (18)–(19) is given by*

$$P_{RK}(z) = \det \widetilde{\Delta}(z), \tag{20}$$

where

$$\widetilde{\Delta}(z) = \begin{bmatrix} I_v \otimes \widetilde{E} - hA \otimes \widetilde{L} & 0 \\ -b^T \otimes I_d & I_d \end{bmatrix} z^{m+1} - \begin{bmatrix} 0 & he \otimes \widetilde{L} \\ 0 & I_d \end{bmatrix} z^m - \begin{bmatrix} hA \otimes \widetilde{M} & 0 \\ 0 & 0 \end{bmatrix} z - \begin{bmatrix} 0 & he \otimes \widetilde{M} \\ 0 & 0 \end{bmatrix}.$$

Proof. Denote

$$K_n = [K_{n,1}^T, K_{n,2}^T, \dots, K_{n,v}^T]^T \in C^{vd}, \qquad e = [1, 1, \dots, 1]^T \in C^v.$$

By means of the Kronecker product, we can rewrite the difference system (18)–(19) in a compact form

$$\begin{bmatrix} I_v \otimes \widetilde{E} - hA \otimes \widetilde{L} & 0 \\ -b^T \otimes I_d & I_d \end{bmatrix} \begin{bmatrix} K_n \\ y_{n+1} \end{bmatrix} - \begin{bmatrix} 0 & he \otimes \widetilde{L} \\ 0 & I_d \end{bmatrix} \begin{bmatrix} K_{n-1} \\ y_n \end{bmatrix}$$

$$- \begin{bmatrix} hA \otimes \widetilde{M} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K_{n-m} \\ y_{n-m+1} \end{bmatrix} - \begin{bmatrix} 0 & he \otimes \widetilde{M} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K_{n-m-1} \\ y_{n-m} \end{bmatrix} = 0, \tag{21}$$

where the vector $\begin{bmatrix} K_n \\ y_{n+1} \end{bmatrix} \in C^{(v+1)d}$, I_v denotes $v-$order identity matrix.

Taking z-transform to system (21) and introducing $Z\{\begin{bmatrix} K_{n-m+1} \\ y_{n-m} \end{bmatrix}\} = Y(z)$, we get

$$\{\begin{bmatrix} I_v \otimes \widetilde{E} - hA \otimes \widetilde{L} & 0 \\ -b^T \otimes I_d & I_d \end{bmatrix} z^{m+1} - \begin{bmatrix} 0 & he \otimes \widetilde{L} \\ 0 & I_d \end{bmatrix} z^m - \begin{bmatrix} hA \otimes \widetilde{M} & 0 \\ 0 & 0 \end{bmatrix} z - \begin{bmatrix} 0 & he \otimes \widetilde{M} \\ 0 & 0 \end{bmatrix}\} Y(z) = 0.$$

Hence, the characteristic polynomial of the difference system (18)–(19) is described as (20) and the proof is completed.

The computational effort in implementing the semi-implicit RK methods is substantially less than for a fully implicit method. For this reason, we first formulate a delay-dependent stability criterion for the semi-implicit RK methods for DDAEs (1).

Theorem 3.3. *Assume that DDAEs (1) is asymptotically stable for given matrices E, L, M and delay τ and the $v-$stage RK method (16)–(17) is semi-implicit with the constant step-size $h = \tau/m$ for a certain integer $m \geq 1$. If*

(H1) $a_{ii} \neq 0$ and $\rho(a_{ii}hL_1) < 1$ for $i = 1, 2, \ldots, v$;

(H2) the characteristic polynomial $P_{RK}(z)$ satisfies

$$P_{RK}(z) \neq 0 \quad \text{for} \quad z \in \Gamma,$$

and

$$\frac{1}{2\pi}\Delta_\Gamma \arg P_{RK}(z) = d(v+1)(m+1),$$

where $\Gamma = \{z : |z| = 1\}$ stands for the boundary of the unit circle region. Then the RK method (16)–(17) for DDAEs (1) is weakly delay-dependently stable.

Proof. The difference system (16)–(17) is asymptotically stable iff all the characteristic roots of $P_{RK}(z) = 0$ lie in the inner of the unit circle. Noting that the underlying RK method is semi-implicit, we have that $a_{ij} = 0$ for $i < j$. Then, from the form of matrices $\widetilde{E}, \widetilde{L}$, we can find that

$$I_v \otimes \widetilde{E} - hA \otimes \widetilde{L} = \begin{bmatrix} I_q - a_{11}hL_1 & & & & \\ & -a_{11}hI_{d-q} & & & 0 \\ & & \ddots & & \\ & * & & I_q - a_{vv}hL_1 & \\ & & & & -a_{vv}hI_{d-q} \end{bmatrix},$$

which is a lower triangular. By condition (H1) and according to Lemmas 2.3 and 2.4, the matrices $I_q - a_{ii}hL_1$ $(i = 1, 2, \ldots, v)$ are nonsingular. Then the matrix $I_v \otimes \widetilde{E} - hA \otimes \widetilde{L}$ is nonsingular, whence the $(v+1)d$ matrix

$$\begin{bmatrix} I_v \otimes \widetilde{E} - hA \otimes \widetilde{L} & 0 \\ -b^T \otimes I_d & I_d \end{bmatrix}$$

in the characteristic polynomial $P_{RK}(z)$ is also nonsingular. It means that the degree of the polynomial $P_{RK}(z)$ is $d(v+1)(m+1)$. Thus, counting multiplicities, the polynomial $P_{RK}(z)$ has $d(v+1)(m+1)$ zeros. By the argument principle, condition (H2) implies that the condition $|z| < 1$ holds for all the $d(v+1)(m+1)$ roots of $P_{RK}(z) = 0$. The proof is finished.

For a fully implicit RK method applied to DDAEs (1), we derive the following result.

Theorem 3.4. *Suppose that DDAEs (1) is asymptotically stable for given matrices E, L, M and delay τ and the $v-$stage RK method (16)–(17) is fully implicit with the constant step-size $h = \tau/m$ for a certain integer $m \geq 1$. If*

(H3) the matrix $I_v \otimes \widetilde{E} - hA \otimes \widetilde{L}$ is invertible;

(H4) the characteristic polynomial $P_{RK}(z)$ satisfies

$$P_{RK}(z) \neq 0 \quad \text{for} \quad z \in \Gamma,$$

and

$$\frac{1}{2\pi}\Delta_\Gamma \arg P_{RK}(z) = d(v+1)(m+1),$$

where $\Gamma = \{z : |z| = 1\}$ stands for the boundary of the unit circle region. Then the RK method (16)–(17) is weakly delay-dependently stable.

Proof. The proof can be carried out similarly to that of Theorem 3.3. By condition (H3), we know that the matrix

$$\begin{bmatrix} I_v \otimes \widetilde{E} - hA \otimes \widetilde{L} & 0 \\ -b^T \otimes I_d & I_d \end{bmatrix}$$

is nonsingular. So the degree of the polynomial $P_{RK}(z)$ becomes $d(v+1)(m+1)$. Thus, by condition (H4) and the argument principle, the characteristic equation $P_{RK}(z) = 0$ has no roots on or outside the region $\{z : |z| < 1\}$. The proof is completed.

Now we describe an algorithm to check the weak delay-dependent stability of semi-implicit RK method for DDAEs (1) due to Theorem 3.3.

Algorithm 3.5. Suppose that DDAEs (1) is asymptotically stable and the RK method is semi-implicit and condition (H1) holds, then we implement the following step to check the stability of the underlying RK method.

Step 0. Take a sufficiently large integer N and distribute N node points z_j $(j = 1, 2, \ldots, N)$ on the unit circle Γ uniformly so as to $\arg z_j = \frac{2\pi j}{N}$. For each z_j, we evaluate $P_{RK}(z_j)$ by computing the determinant as

$$P_{RK}(z_j) = \det \widetilde{\Delta}(z_j), \quad j = 1, 2, \ldots, N.$$

Also we decompose $P_{RK}(z_j)$ into its real and imaginary parts.

Step 1. For each z_j $(j = 1, 2, \ldots, N)$, we examine whether $P_{RK}(z_j) = 0$ by evaluating its modulus satisfies $|P_{RK}(z_j)| \le \eta_1$ with the preassigned tolerance $\eta_1 > 0$. If it holds, the RK method is unstable; then we end the algorithm. Otherwise, we turn to the next step.

Step 2. Check whether $\frac{1}{2\pi}\Delta_\Gamma \arg P_{RK}(z) = d(v+1)(m+1)$ along the sequence $P_{RK}(z_j)$ by evaluating $|\frac{1}{2\pi}\arg P_{RK}(z_j) - d(v+1)(m+1)| \le \eta_2$ with the preassigned tolerance $\eta_2 > 0$. If it holds, it means the change of the argument is $d(v+1)(m+1)$ along Γ; then the semi-implicit RK method is asymptotically stable, otherwise not stable.

Remark 3.6. To obtain the given accuracy, the number of the node points N must be enough large. We can distribute the node points z_j $(j = 1, 2, \ldots, N)$ on the boundary Γ uniformly, i.e., we divide the interval $[0, 2\pi)$ with a sufficiently small constant step-size h_1. Therefore, the number of node points $N = [\frac{2\pi}{h_1}]$, where the symbol $[x]$ denotes the greatest integer less than or equal to the real number x.

Remark 3.7. For the full implicit RK method, a similar algorithm can be described to check Theorem 3.4 by minor modifications in Algorithm 3.5.

Remark 3.8. Associated with Lagrange interpolations, delay-dependent stability analysis of RK methods for the linear DDAEs with multiple delays is similar to that of the present paper.

4. Numerical examples

This section provides three numerical examples to shed light on the effectiveness of our main results. Without loss in generality, we may adopt the spectral matrix norm (see e.g. [22]) given by

$$\|F\| = \sqrt{\lambda_{max}(F^H F)}.$$

In what follows, the 3-stage diagonally implicit Runge–Kutta (DIRK) method of order 4, which is described by the Butcher array

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|ccc} \frac{1+w}{2} & \frac{1+w}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{w}{2} & \frac{1+w}{2} & 0 \\ \frac{1-w}{2} & 1+w & -1-2w & \frac{1+w}{2} \\ \hline & \frac{1}{6w^2} & 1-\frac{1}{3w^2} & \frac{1}{6w^2} \end{array},$$

is our underlying scheme of the semi-implicit RK method for linear DDAEs, in which $w = \frac{2}{\sqrt{3}}\cos\frac{\pi}{18}$, one of the roots of $3w^3 - 3w = 1$, see e.g. [24]. Note that $a_{11} = a_{22} = a_{33} = \frac{1+w}{2} \approx 1.0686 \neq 0$.

Example 4.1. Consider the 2-dimensional DDAEs with the parameter matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} -0.3 & 0 \\ 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} -0.5 & -1 \\ 0.8 & 0.7 \end{bmatrix} \tag{22}$$

and with the initial condition

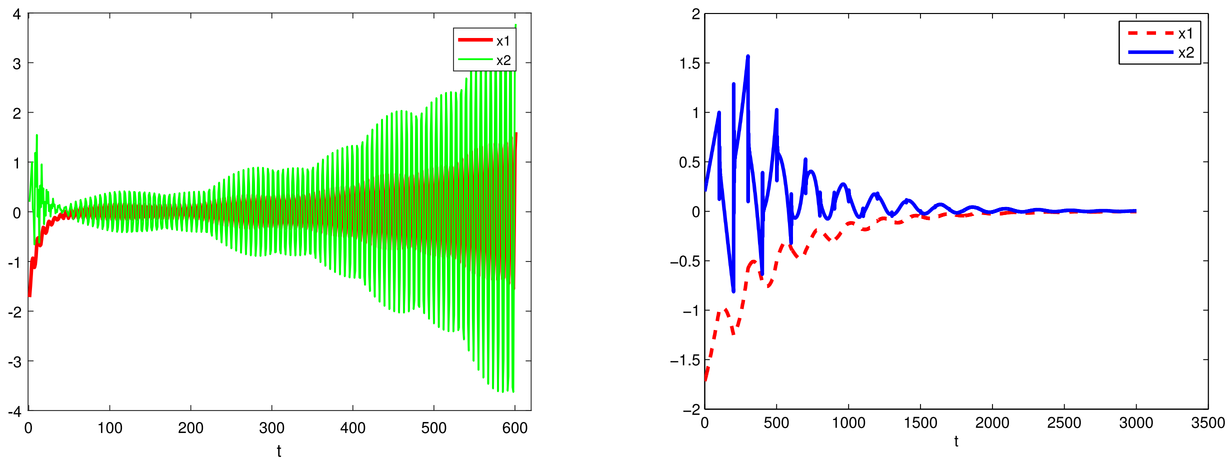
$$x(t) = \begin{bmatrix} \sin t - 1 \\ t + 1 \end{bmatrix}, \quad t \in [-\tau, 0]. \tag{23}$$

First, we consider the stability of analytic solutions for the initial value problem (22)–(23). It is not difficult to calculate that

$$\|M_4\| = 0.7 < 1, \; r = \|L_1\| + \|M_1\| + \frac{\|M_2\|\|M_3\|}{1 - \|M_4\|} \approx 3.4667.$$

Then, by Lemma 2.7, we can evaluate that system (22) is asymptotically stable iff $\tau \le 1.42$.

Now, we employ Algorithm 3.5 to check the stability of the numerical solutions derived by the DIRK method with the uniform step-size $h = \tau/m$ in the case of $\tau = 0.8$. First, observe that the condition $\rho(a_{ii}hL_1) < 1$ means that $h < 1/0.3a_{ii} \approx 3.1193$ for $i = 1, 2, 3$. According to Remark 3.6, to distribute the node points z_j $(j = 1, 2, \ldots, N)$ on the



(a) Numerical solutions with $m = 3$

(b) Numerical solutions with $m = 100$

Fig. 2. Numerical solutions are not asymptotically stable in Example 4.1.

unit circle Γ, we choose the constant step-size $h_1 = 0.0001$. Then, taking $m = 3$, $h = \tau/m = \frac{1}{3} < 3.1193$, implementing Algorithm 3.5 we can obtain that

$$\frac{1}{2\pi}\Delta_\Gamma \arg P_{RK}(z) = 25.9996 \neq 32 = d(v+1)(m+1).$$

It follows, from Theorem 3.3, that the numerical solutions of system (22) are divergent, which are shown in Fig. 2(a).

However, if we take $m = 100$, $h = \tau/m = 0.008 < 3.1193$ and utilizing Algorithm 3.5, we can check that

$$\frac{1}{2\pi}\Delta_\Gamma \arg P_{RK}(z) = 807.9907 \approx 808 = d(v+1)(m+1),$$

which shows the conditions in Theorem 3.3 are fulfilled. So the numerical solutions of DDAEs (1) with the given parameter matrices are converging to 0, which are depicted in Fig. 2(b). Hence, the numerical solutions of the system under consideration are weakly delay-dependently stable.

Example 4.2. Consider the 3-dimensional DDAEs

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{x}(t) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -0.3 & 0 \\ 0 & 0 & 1 \end{bmatrix}x(t) + \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ -0.5 & 1.2 & 0.8 \end{bmatrix}x(t-\tau) \tag{24}$$

with the initial condition

$$x(t) = \begin{bmatrix} \sin t \\ t+1 \\ e^t \end{bmatrix}, \quad -\tau \le t \le 0. \tag{25}$$

By direct computation, we have that $\|M_4\| = 0.8 < 1$, and

$$r = \|L_1\| + \|M_1\| + \frac{\|M_2\|\|M_3\|}{1-\|M_4\|} = 2 + 1.618 + \frac{1 \times 1.3}{1-0.8} \approx 10.118.$$

Also by Lemma 2.7, we can conclude that the DDAEs (24) is asymptotically stable iff $\tau \le 1.28$.

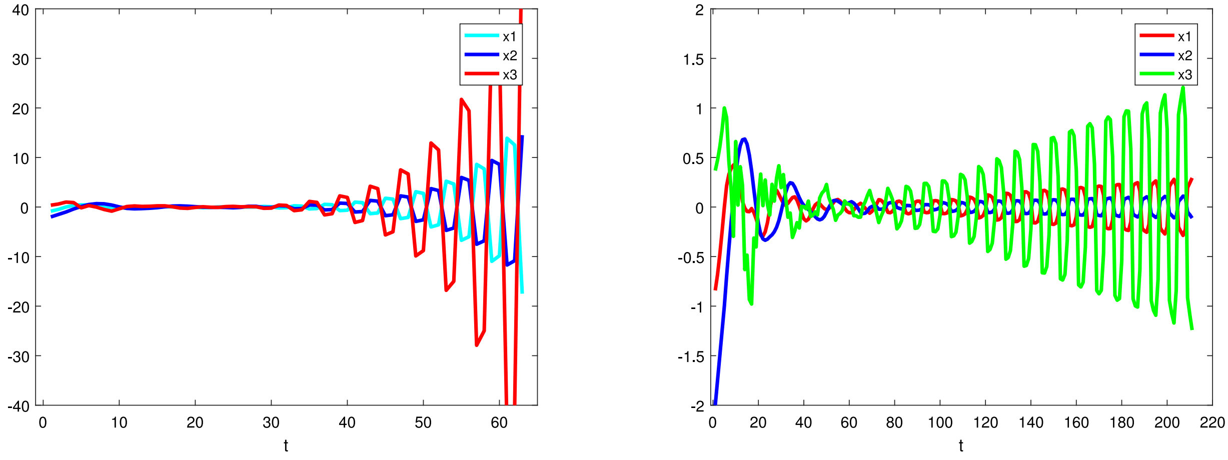
Next we aim at the weak delay-dependent stability of numerical solutions obtained by the DIRK scheme in the case of $\tau = 1$. Observe that the condition $\rho(a_{ii}hL_1) < 1$ $(i = 1, 2, 3)$ means that $2a_{ii}h < 1$, whence $h < 1/2a_{ii} \approx 0.4679$ for $i = 1, 2, 3$. To proceed, we choose the constant step-size $h_1 = 0.0001$ to divide the unit circle Γ.

Take $m = 3$, $h = \tau/m < 0.4679$. Then by Algorithm 3.5 we obtain that

$$\frac{1}{2\pi}\Delta_\Gamma \arg P_{RK}(z) = 41.9997 \neq 48 = d(v+1)(m+1).$$

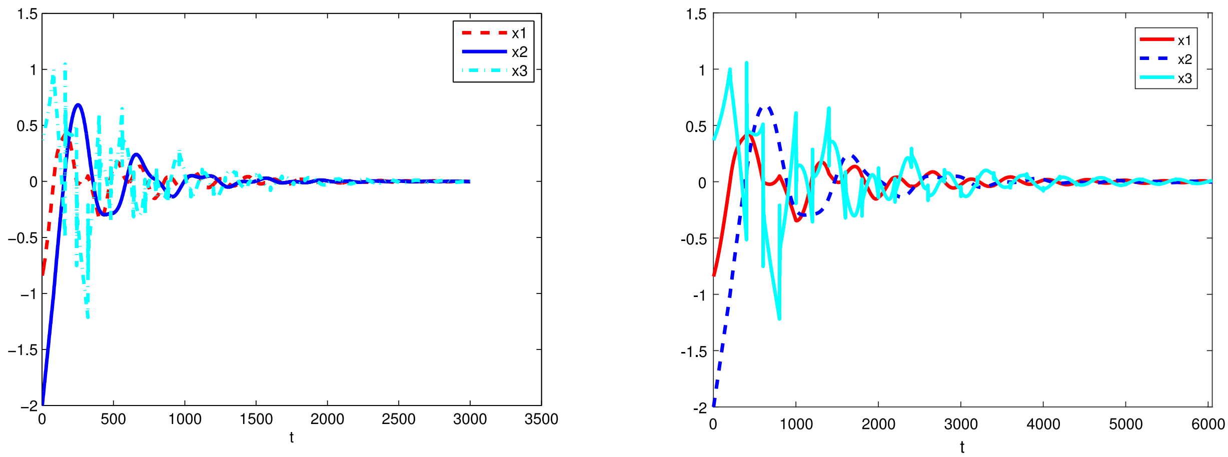
Taking $m = 4$, we obtain that

$$\frac{1}{2\pi}\Delta_\Gamma \arg P_{RK}(z) = 51.9993 \neq 60 = d(v+1)(m+1).$$



(a) Numerical solutions with $m = 3$ (b) Numerical solutions with $m = 4$

Fig. 3. Numerical solutions are not asymptotically stable in Example 4.2.



(a) Numerical solutions with $m = 80$ (b) Numerical solutions with $m = 200$

Fig. 4. Numerical solutions are not asymptotically stable in Example 4.2.

In such two cases, the assumptions of Theorem 3.3 do not hold. Then the numerical solutions are divergent and their behavior is described in Figs. 3(a) and 3(b), respectively.

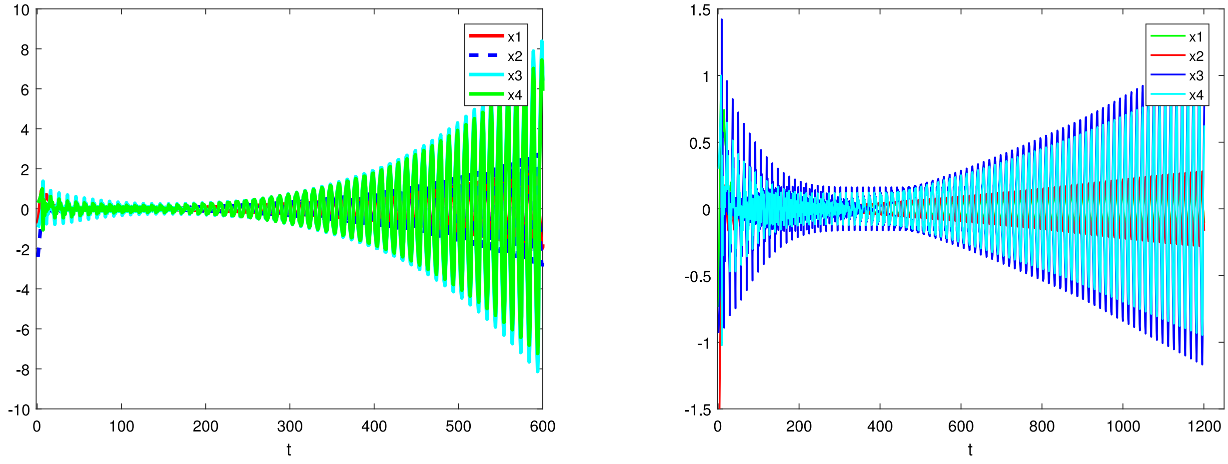
But, if we take $m = 80$, $h = \tau/m = 0.0125 < 0.4679$. By means of Algorithm 3.5, we can evaluate that

$$\frac{1}{2\pi}\Delta_\Gamma \arg P_{RK}(z) = 971.9883 \approx 972 = d(v+1)(m+1),$$

and condition (H2) in Theorem 3.3 fulfills, which ensures that the numerical solutions of the initial value problem (24)–(25) are weakly delay-dependently stable. Fig. 4(a) typifies the behavior of the numerical solutions when $m = 80$, which are converging to 0. The same case is that for $m = 200$ and the behavior is shown in Fig. 4(b).

Example 4.3. Consider the 4-dimensional DDAEs with the parameter matrices

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} -1.5 & 2 & 0 & 0 \\ 0.7 & -0.9 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$



(a) Numerical solutions with $m = 5$ (b) Numerical solutions with $m = 7$

Fig. 5. Numerical solutions are not asymptotically stable in Example 4.3.

$$M = 0.37 * \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & 0 & 2 & 1 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \tag{26}$$

and with the initial condition

$$x(t) = \begin{bmatrix} \cos 2t \\ 2t \\ \sin t \\ e^t \end{bmatrix}, \quad t \in [-\tau, 0]. \tag{27}$$

Note that

$$\|M_4\| = 0.9687 < 1, \ r = \|L_1\| + \|M_1\| + \frac{\|M_2\|\|M_3\|}{1 - \|M_4\|} \approx 21.3547.$$

Set the time-delay $\tau = 1.2$. By Lemma 2.7, we can evaluate that the DDAEs under consideration is asymptotically stable.

Algorithm 3.5 is now employed to check the stability of the numerical solutions derived by the DIRK method with the uniform step-size $h = \tau/m$. Note that the condition $\rho(a_{ii}hL_1) < 1$ means that $h < 1/2.4207a_{ii} \approx 0.3866$ for $i = 1, 2, 3$. Then we choose the step-size $h_1 = 0.0001$ to divide the boundary Γ.

When $m = 5$, $h = \tau/m < 0.3866$, we obtain that

$$\frac{1}{2\pi} \Delta_\Gamma \arg P_{RK}(z) = 86.0003 \neq 96 = d(v + 1)(m + 1).$$

When $m = 7$, $h = \tau/m < 0.3866$, we obtain that

$$\frac{1}{2\pi} \Delta_\Gamma \arg P_{RK}(z) = 114.0016 \neq 128 = d(v + 1)(m + 1).$$

In such two cases, the theorem does not fulfill. Then the numerical solutions are not guaranteed to be asymptotically stable. And their figures are given in Figs. 5(a) and 5(b), respectively.

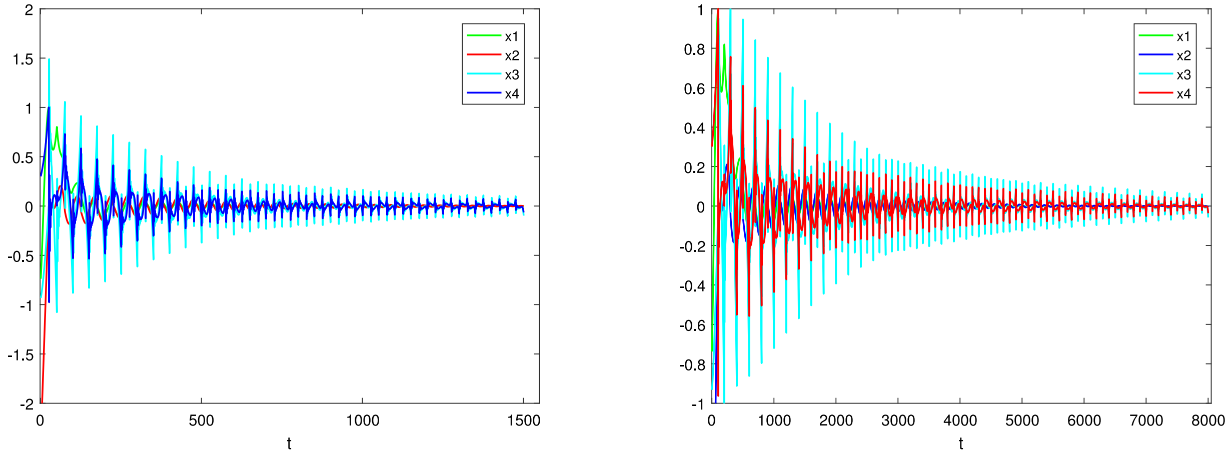
Conversely, when $m = 25$ and $m = 100$, Algorithm 3.5 evaluates that

$$\frac{1}{2\pi} \Delta_\Gamma \arg P_{RK}(z) = 415.9032 \approx 416 = d(v + 1)(m + 1),$$

and

$$\frac{1}{2\pi} \Delta_\Gamma \arg P_{RK}(z) = 1615.9576 \approx 1616 = d(v + 1)(m + 1),$$

respectively. So the conditions of Theorem 3.3 are satisfied. It follows that the numerical solutions are convergent, which are depicted in Figs. 6(a) and 6(b), respectively.



(a) Numerical solutions with $m = 25$

(b) Numerical solutions with $m = 100$

Fig. 6. Numerical solutions are not asymptotically stable in Example 4.3.

Remark 4.4. In our numerical experiments, the computational procedure is repeated every time when the coefficients of the DDAEs change. In the sense of the weak delay-dependent stability, there exists a sufficient large integer m such that the difference system obtained by RK methods is asymptotically stable. However, it is still an open problem to estimate whether integers greater than the found integer m are all good or not.

5. Conclusions

In the present paper, we focus on the delay-dependent stability of numerical solutions obtained by implicit RK methods for linear delay differential–algebraic equations. By means of the argument principle, computable stability criteria of numerical solutions for DDAEs (1) are established. In addition, a practical algorithm is provided to check delay-dependent stability of numerical solutions. Using the provided algorithm, three numerical examples are given to demonstrate the obtained results. It indicates that our theoretical results work well in the practical computations. However it is a remarkable fact that the stability criteria established in this paper are just sufficient conditions, not necessary conditions.

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