ELSEVIER Journal of Computational and Applied Mathematics 59 (1995) 109–128

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Equations of magnetodynamics of incompressible thermo-Bingham's fluid under the gravity effect

Jiří Nedoma*

Institute of Computer Science, Academy of Science, Pod vodárenskou věži 2, 182 07 Prague 8, Czech Republic

Received 7 October 1993; revised 22 December 1993

Abstract

Results of this contribution were obtained in the course of an investigation of a simple model problem of the global geodynamic and geomagnetic fields of the real Earth as well as model problem of astrophysics. The problem leads to an initial boundary value problem for a coupled system of equations of magnetodynamics of incompressible, electrically conducting and thermo viscous Bingham's fluid under the gravity effect. The variational formulation of the problem and the existence and uniqueness of the solution will be given. While the existence of the solution is proved for the two- and three-dimensional cases, the uniqueness can be proved for two-dimensional case only.

Keywords: Generalized magnetodynamics; Partial differential equations; Variational inequalities; Viscoplasticity; Geodynamics; Global gravity and geomagnetic models of the Earth; Astrophysics

1. Introduction

One of the important problem in the present geophysics is the connection of the geodynamic processes and the geomagnetic field in the Earth's interior and the mutual influence of the geomagnetic fields in the core and the mantle of the real Earth. So for the most part, the plastic and fluid motions in the Earth's interior consist of an axisymmetric nonuniform rotation and an axisymmetric statistical distribution of the cyclonic convention. There may be giant circulation cells in the core and the mantle of the Earth, violating this simple generalization, but we know so little about nonsymmetric circulation that we cannot include it in any meaningful way.

Contemporary models of the Earth are built isolated corresponding to the individual geophysical fields. The model presented gives a new approach how to study mutual relations between geodynamic and geomagnetic processes in the Earth's interior as well as in its separate

* E-mail: nedoma@uivt.cas.cz.

SSDI 0377-0427(94)00019-W

parts (the core, the mantle and the lithosphere). The geodynamic model of the Earth is based on assumptions that in the course of the development of the Earth's body the important factors are gravity, radioactivity, temperature, density differentiation associated with the phase transitions of the Earth's material under high temperatures, large pressures and chemical reactions under the same conditions (see [11]). Then the Earth's body can be characterized by a thermo-elasto-visco-plastic state in a relatively weak magnetic field. Consequently, for the derivation of equations describing geodynamic processes in an irregular rotating Earth's body the thermo-elasto-visco-plastic rheology and the magnetodynamic theory can be used. In the paper the Earth's body will be interpreted by a moving and heating incompressible Bingham's medium, which produces the magnetic and gravity fields. Boundary conditions on the surface of the Earth follow from our knowledge of the velocity, temperature and gravity potential fields on the Earth surface and from the presumption that the surface of the Earth is strongly electrically conductive. The model based on the thermo-Bingham rheology represents only a simple model of the real Earth. It gives, however, the idea how to solve the coupling of the geodynamic, geothermal and geomagnetic processes under the gravity effect in the core and the mantle of the Earth as well as the coupling of the magnetic field in the core and the upper parts of the Earth, i.e., the mantle and the lithosphere.

In the paper we shall assume that Earth's body as well as other planets and stars are approximated by the moving visco-plastic electrically conducting incompressible thermo-Bingham's medium which is under the gravity effect. For the case of materials processing in the space outside the Earth, similar assumptions can be considered.

The classical equations of magnetodynamics were solved during the last few years in several papers. So an initial boundary value problem for a system of equations of magnetodynamics of incompressible, electrically conducting and viscous Newtonian fluid was studied in [15]. Firstly, the incompressible electrically conducting Bingham's fluid was investigated in [2]. The mathematical model of global gravity field coupled with geodynamic processes and thermo-magnetodynamic processes in the Earth's interior based on thermo-Bingham's rheology was discussed in [11]. Štědrý and Vejvoda [16, 19] prove the existence of time-periodic solutions of equations of magnetodynamics — the incompressible case in [17, 19] and the compressible case in [16]. Various existence results have been proved in numerous papers, we mention here only Milani [8, 9] and Byhovskii [1].

The aim of this paper is to prove the existence and uniqueness of the time-dependent solutions to a system of equations appearing in generalized thermo-magnetodynamics based on the thermo-Bingham rheology which with sufficient accuracy simulate geodynamic and geomagnetic processes in the planet's and stellar interiors as well as processes in materials processing in the cosmic environment.

2. Mathematical model and classical formulation of the problem

In the sequel we shall deal with the incompressible moving thermo-Bingham fluid under the gravity and magnetic effects.

We shall suppose that $\Omega \subset \mathbb{R}^N$, $N = 2, 3$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, occupied by a moving Bingham's fluid. Let $I = (t_0, t_1)$ be a time interval. We shall use the notation $u_{i,j} = \partial u_i/\partial x_j$ and $\dot{u} = \partial u/\partial t (\equiv v)$ and $\mathrm{d}v_i/\mathrm{d}t = \partial v_i/\partial t + v_j v_{i,j}$. The Einstein summation convention over the range $1, \dots, N$ will be used. Let $\boldsymbol{n}$ denotes the unit outward normal vector at $\boldsymbol{x} \in \partial\Omega$. Let $[\cdot,\cdot]$ denote the vector product in $\mathbb{R}^N$ and $N = \{1, \dots\}$. The strain–displacement relations are

$$\varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad (i, j = 1, \dots, N). \tag{2.1}$$

We introduce in the sequel the notation $D_{ij}(u) = e_{ij}(\dot{u})$. The stress–strain relation of the thermo-Bingham medium is defined by

$$\tau_{ij} = \tau_{ij}^{\mathrm{B}} + \tau_{ij}^{\mathrm{T}} + \tau_{ij}^{\mathrm{M}}, \tag{2.2}$$

where τ_{ij}^{B} is the Bingham stress defined by

$$\tau_{ij}^{\mathrm{B}} = -p\delta_{ij} + \bar{g}D_{ij}/(D_{\mathrm{II}})^{1/2} + 2\bar{\mu}D_{ij}, \tag{2.3}$$

and $\bar{g}$ and $\bar{\mu}$ are thresholds of plasticity and viscosity, p represents a spherical part of the stress tensor τ_{ij}[1] (p has a meaning of a pressure) and δ_{ij} is the well-known Kronecker symbol, $D_{\mathrm{II}} = \frac{1}{2}D_{ij}D_{ij}$ (see, e.g., [15], [2, Ch. VI, Section 1, formula (1.9)]). Due to Nowacki [14], the thermal stress satisfies the relation

$$\tau_{ij}^{\mathrm{T}} = \beta_{ij}(T - T_0), \tag{2.4}$$

where β_{ij} is a coefficient of thermal expansion, $T_0 > 0$ an initial temperature in which the medium is in the initial stress–strain state. Furthermore, due to Landau and Lifšhitz [6] the Maxwell stress τ_{ij}^{M} is defined by

$$\tau_{ij}^{\mathrm{M}} = H_iB_j - (\boldsymbol{HB})\delta_{ij}, \tag{2.5}$$

where $\boldsymbol{H}$ and $\boldsymbol{B}$ represent the intensity of the magnetic field and the magnetic induction vector, respectively, and $\boldsymbol{B} = \mu\boldsymbol{H}$ with μ the magnetic susceptibility.

Without loss of generality of the problem studied, we can solve the following problem:

In dealing with the motion of the visco-plastic electrically conducting incompressible thermo-Bingham's medium under the gravity effect, the following system of equations for the velocity $\boldsymbol{v} = (v_i)$, the magnetic intensity $\boldsymbol{B}$, the temperature T and the perturbed gravitational potential Φ can be considered:

$$\begin{aligned} \rho\, \mathrm{d}v_i/\mathrm{d}t = {} & (-p\delta_{ij} + \bar{g}D_{ij}/(D_{\mathrm{II}})^{1/2} + 2\bar{\mu}D_{ij} + \beta_{ij}(T - T_0) \\ & + (H_iB_j - (\boldsymbol{HB})\delta_{ij}))_{,j} + f_i, \quad \boldsymbol{B} = \mu\boldsymbol{H} \quad \text{on } \Omega \times I, \end{aligned} \tag{2.6a}$$

$$\operatorname{Div} \boldsymbol{v} = 0 \quad \text{on } \Omega \times I, \tag{2.6b}$$

$$\partial\boldsymbol{B}/\partial t + \operatorname{Rot}((\mu\sigma)^{-1}\operatorname{Rot}\boldsymbol{B}) - \operatorname{Rot}[\boldsymbol{v},\boldsymbol{B}] = 0 \quad \text{on } \Omega \times I, \tag{2.6c}$$

$$\operatorname{Div}\boldsymbol{B} = 0 \quad \text{on } \Omega \times I, \tag{2.6d}$$

$$\rho\beta_{ij}T_0D_{ij} + \rho c_{\mathrm{e}}\, \mathrm{d}T/\mathrm{d}t = W - (\kappa_{ij}T_{,j})_{,i} \quad \text{on } \Omega \times I, \tag{2.6e}$$

$$\Delta\Phi = -4\pi\rho G_{\mathrm{c}}pk^{-1} + 2\omega^2 \quad \text{on } \Omega \times I, \tag{2.6f}$$

---

[1] The tensor of the second order with components τ_{ij} can be decomposed into its spherical part and its deviator τ_{ij}^{D} by the relation $\tau_{ij} = \frac{1}{3}\tau_{kk}\delta_{ij} + \tau_{ij}^{\mathrm{D}}$.

with ρ, $\bar{g}$, $\bar{\mu}$, β_{ij}, T_0, μ, σ, c_e, (κ_{ij}), G_c, K, ω given. We define

$$\Psi = -4\pi\rho G_c p K^{-1} + 2\omega^2.$$

Moreover, the following boundary value conditions,

$$\boldsymbol{v} = 0 \quad \text{on } \partial\Omega \times I, \tag{2.7a}$$

$$\boldsymbol{Bn} = 0 \quad \text{on } \partial\Omega \times I, \tag{2.7b}$$

$$[\boldsymbol{n}, \operatorname{Rot} \boldsymbol{B}] = 0 \quad \text{on } \partial\Omega \times I, \tag{2.7c}$$

$$T = 0 \quad \text{on } \partial\Omega \times I, \tag{2.7d}$$

$$\Phi = 0 \quad \text{on } \partial\Omega \times I, \tag{2.7e}$$

and the initial conditions,

$$\boldsymbol{v}(\boldsymbol{x}, t_0) = 0, \quad \boldsymbol{B}(\boldsymbol{x}, t_0) = 0, \quad T(\boldsymbol{x}, t_0) = 0 \tag{2.8}$$

are considered in the above system of equations.

The generalized model of the real Earth is much more complicated (see [11]). In this model problem the metallized state of the rocks, the volume density of the electromagnetic force, the Joule heat, the viscous force evoked by the viscous stress, the Coriolis force, diffusion processes, etc., are omitted.

3. Variational formulation of the problem

We introduce the following notation of the employed spaces, scalar products, norms and forms.

We denote by $C(\Omega)$ the set of all functions defined and continuous on Ω, by $C^k(\Omega)$, $1 \leqslant k < \infty$, the set of all functions defined on Ω which have continuous derivatives up to the order k on Ω. Moreover, we denote by $C(\bar{\Omega})$ the space of all functions in $C(\Omega)$ which are bounded and uniformly continuous on $\bar{\Omega}$ and for an integer $1 \leqslant k < \infty$, we denote by $C^k(\bar{\Omega})$ the space of all functions $u \in C^k(\Omega)$ such that $D^\alpha u \in C(\bar{\Omega})$ for all $\alpha \in M_{N,k}$, where $M_{N,k}$ is the set of all N-dimensional multiindices of length less than or equal to k. We denote $C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega)$ and $C^\infty(\bar{\Omega}) = \bigcap_{k=0}^{\infty} C^k(\bar{\Omega})$. We introduce the space $D(\Omega)$ as a space of all functions in $C^\infty(\Omega)$ with a compact support in Ω. The space is equipped with the ordinary countable system of seminorms and as usual $D(\Omega, \mathbb{R}^N) = [D(\Omega)]^N$.

We introduce the space $L_p(\Omega)$, $1 \leqslant p < \infty$, as the space of all measurable functions such that

$$\| f \|_{L_p(\Omega)} = \left(\int_\Omega | f(x)|^p \, \mathrm{d}x \right)^{1/p} < \infty.$$

By $L_\infty(\Omega)$ we denote the set all measurable functions f defined almost everywhere on Ω such that $\| f \|_\infty = \operatorname{ess\,sup}_\Omega | f(x)|$ is finite. The space is endowed with this norm.

We shall employ the Sobolev spaces of functions having generalized derivatives of the (possibly fractional) order s of the type $[H^s(\Omega)]^k \equiv H^{s,k}(\Omega)$ (see, e.g., [7, 13]). The norm will be denoted by

$\|\cdot\|_{s,k}$ and the scalar product by $(\cdot,\cdot)_s$ (for each integer k). We set $H^{0,k}(\Omega) \equiv L_2^k(\Omega)$ and denote by $(\cdot,\cdot)$ the scalar product in $L_2^k(\Omega)$ (as there is no danger of any confusion with k). Denote by $H_0^{s,k}(\Omega)$ the closure of $D(\Omega)$ in the norm of the space $H^{s,k}(\Omega)$. Moreover, we shall denote by $\|\cdot\|_E$ the equivalent norm in $H^{1,N}(\Omega)$ induced by the small strain tensor e_{ij}, i.e., $\|\boldsymbol{u}\|_E^2 = \int_\Omega e_{ij}(\boldsymbol{u})e_{ij}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} + \|\boldsymbol{u}\|_{0,N}^2$.

Moreover, we define for $s \geqslant 1$ the following spaces:

$$^1H^{s,N}(\Omega) = \{\boldsymbol{v} \mid \boldsymbol{v} \in H^{s,N}(\Omega),\ \operatorname{Div}\boldsymbol{v} = 0;\ \boldsymbol{v} = 0 \text{ on } \partial\Omega\}, \quad N = 2,3,$$

$$^2H^{s,N}(\Omega) = \{\boldsymbol{C} \mid \boldsymbol{C} \in H^{s,N}(\Omega),\ \operatorname{Div}\boldsymbol{C} = 0;\ \boldsymbol{nC} = 0 \text{ on } \partial\Omega\}, \quad N = 2,3,$$

$$^3H^{s,1}(\Omega) = \{z \mid z \in H^{s,1}(\Omega),\ z = 0 \text{ on } \partial\Omega\},$$

$$\underline{H}^s = {}^1H^{s,N}(\Omega) \times {}^2H^{s,N}(\Omega) \times {}^3H^{s,1}(\Omega).$$

We put $\underline{H}^1 = \underline{H}$. Then $^1\mathscr{H}(\Omega) = {}^1H^{1,N}(\Omega) \cap D(\Omega,\mathbb{R}^N)$. $^1H^{s,N}(\Omega)$ is a Hilbert space with the norm $\|\cdot\|_{s,N}$. For the sake of simplicity, we put $^1H^{1,N}(\Omega) = {}^1H(\Omega)$, $\|\cdot\|_1 = \|\cdot\|_{1,N}$, and in $H^{0,N}(\Omega)$, $\|\cdot\|_0 = \|\cdot\|_{0,N}$. We shall denote by w_j eigenfunctions of a canonical isomorphism $^1\Lambda_s\colon {}^1H^{s,N} \to ({}^1H^{s,N})'$, i.e., $(\boldsymbol{w},\boldsymbol{v})_s = \lambda_i(\boldsymbol{w}_j,\boldsymbol{v})$ $\forall \boldsymbol{v} \in {}^1H^{s,N}$, $\|\boldsymbol{w}_j\|_{0,1} = 1$ (see [2]). Moreover, $^2\mathscr{H}(\Omega) = {}^2H^{1,1}(\Omega) \cap D(\Omega,\mathbb{R}^N)$; $^2H^{s,N}(\Omega)$, $s \geqslant 1$ is a Hilbert space with the norm $\|\cdot\|_{s,N}$. We put $^2H(\Omega) = {}^2H^{1,N}(\Omega)$. Furthermore, $^3\mathscr{H}(\Omega) = {}^3H^{1,1}(\Omega) \cap D(\Omega)$; $^3H^{s,1}(\Omega)$, $s \geqslant 1$ is a Hilbert space with the norm $\|\cdot\|_s$. We put $^3H(\Omega) = {}^3H^{1,1}(\Omega)$. Similarly as above, we define C_j and z_j, $C_j \in {}^2H^{s,N}(\Omega)$, $z_j \in {}^3H^{s,1}(\Omega)$ eigenfunctions of canonical isomorphisms $^2\Lambda_s\colon {}^2H^{s,N} \to ({}^2H^{s,N})'$ and $^3\Lambda_s\colon {}^3H^{s,1} \to ({}^3H^{s,1})'$. We shall denote the dual space of $^1H^{s,N}(\Omega)$ by $({}^1H^{s,N}(\Omega))'$; similarly in all other cases. Furthermore, we define the following spaces:

$$^1H = \{\boldsymbol{v} \mid \boldsymbol{v} \in L_2(I;{}^1H^{s,N}(\Omega)),\ \boldsymbol{v}' \in L_2(I;{}^1H^{0,N}(\Omega)),\ \boldsymbol{v}(t_0) = 0\},$$

$$^2H = \{\boldsymbol{C} \mid \boldsymbol{C} \in L_2(I;{}^2H^{s,N}(\Omega)),\ \boldsymbol{C}' \in L_2(I;{}^2H^{0,N}(\Omega)),\ \boldsymbol{C}(t_0) = 0\},$$

$$^3H = \{z \mid z \in L_2(I;{}^3H^{s,1}(\Omega)),\ z' \in L_2(I;{}^3H^{0,1}(\Omega)),\ z(t_0) = 0\},$$

$$\underline{H} = {}^1H \times {}^2H \times {}^3H.$$

Throughout the paper we shall assume that the body forces $\boldsymbol{f}(x,t) \in L_2(I, H^{1,N}(\bar{\Omega}))$ and thermal sources $W(\boldsymbol{x},t) \in L_2(I;H^{1,1}(\Omega))$, $I = \langle t_0, t_1\rangle$. We suppose that the density ρ, the thresholds of plasticity and viscosity $\bar{g}$ and $\bar{\mu}$, the bulk modulus K, the specific heat c_e, the electric conductivity σ, and magnetic permeability μ are positive constants. G_c is the gravitational constant. The space-dependent coefficients of thermal expansion (β_{ij}) will be supposed such that $\partial\beta_{ij}/\partial x_j \in L_\infty(\Omega)$ $\forall i,j \in \{1,\ldots,N\}$.

Moreover, we suppose that $\kappa_{ij} \in L_\infty(\Omega)$ are Lipschitz on Ω, fulfil the usual symmetry condition $\kappa_{ij} = \kappa_{ji}$ on Ω for every $i,j \in \{1,\ldots,N\}$ and

$$\kappa_{ij}\xi_i\xi_j \geqslant c_T\|\xi\|_{1,N}^2, \quad x \in \Omega,\ \xi \in \mathbb{R}^N,\ c_T = \text{const.} > 0. \tag{3.1}$$

For $\boldsymbol{v}, \boldsymbol{w} \in H^{1,N}(\Omega)$, $\boldsymbol{B}, \boldsymbol{C} \in H^{1,N}(\Omega)$, $T, z \in H^{1,1}(\Omega)$, $\Phi, \varphi \in H^{1,1}(\Omega)$ we put

$$a(\boldsymbol{v}, \boldsymbol{w}) = 2 \int_\Omega D_{ij}(\boldsymbol{v}) D_{ij}(\boldsymbol{w})\, \mathrm{d}\boldsymbol{x}, \tag{3.2a}$$

$$b_{\mathrm{M}}(\boldsymbol{B}, \boldsymbol{C}) = (\sigma\mu^{-1} \operatorname{Rot} \boldsymbol{B}, \operatorname{Rot} \boldsymbol{C}), \tag{3.2b}$$

$$a_{\mathrm{T}}(T, z) = \int_\Omega \kappa_{ij} T_{,i} z_{,j}\, \mathrm{d}\boldsymbol{x}, \tag{3.2c}$$

$$a_{\mathrm{g}}(\Phi, \varphi) = \int_\Omega \Phi_{,i} \varphi_{,i}\, \mathrm{d}\boldsymbol{x}, \tag{3.2d}$$

$$j(\boldsymbol{v}) = 2 \int_\Omega (D_{\mathrm{II}}(\boldsymbol{v}))^{1/2}\, \mathrm{d}\boldsymbol{x}, \tag{3.2e}$$

$$b_0(\boldsymbol{v}, g, z) = \int_\Omega \rho c_{\mathrm{e}} v_k g_{,k} z\, \mathrm{d}\boldsymbol{x}, \tag{3.2f}$$

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \rho u_i v_{j,i} w_j\, \mathrm{d}\boldsymbol{x}, \tag{3.2g}$$

$$b_{\mathrm{s}}(T, \boldsymbol{v}) = \int_\Omega (\beta_{ij} T)_{,j} v_i\, \mathrm{d}\boldsymbol{x}, \tag{3.2h}$$

$$b_{\mathrm{p}}(\boldsymbol{v}, g) = \int_\Omega \rho T_0 \beta_{ij} v_{i,j} g\, \mathrm{d}\boldsymbol{x}. \tag{3.2i}$$

Thus, we introduce the bilinear form A on $\underline{H}$ for $\boldsymbol{a} \equiv (\boldsymbol{v}, \boldsymbol{B}, T)$, $\boldsymbol{\alpha} \equiv (\boldsymbol{w}, \boldsymbol{C}, z)$ by

$$A(\boldsymbol{a}, \boldsymbol{\alpha}) = \bar{\mu} a(\boldsymbol{v}, \boldsymbol{w}) + b_{\mathrm{M}}(\boldsymbol{B}, \boldsymbol{C}) + a_{\mathrm{T}}(T, z), \tag{3.2j}$$

the matrix

$$\begin{aligned} &\mathscr{E} = (\mathscr{E}_{ij}), \quad \mathscr{E}_{ij} = \rho\delta_{ij}, \ i = 1, \ldots, 2N+1, \ j = 1, \ldots, N, \\ &\mathscr{E}_{ij} = \mu^{-1}\delta_{ij}, \quad i = 1, \ldots, 2N+1, \ j = N+1, \ldots, 2N, \\ &\mathscr{E}_{ij} = \rho c_{\mathrm{e}} \delta_{ij}, \quad i = 1, \ldots, 2N+1, \ j = 2N+1, \end{aligned} \tag{3.2k}$$

and the right-hand side

$$\boldsymbol{P} = (\boldsymbol{f}, 0, W) \in L_2(I; H^{1,2N+1}(\Omega)). \tag{3.2l}$$

Furthermore, we have

$$D_{ij}(\boldsymbol{u}) D_{ij}(\boldsymbol{v}) \leqslant 2 D_{\mathrm{II}}^{1/2}(\boldsymbol{u}) D_{\mathrm{II}}^{1/2}(\boldsymbol{v}), \qquad D_{ij} = \frac{1}{2\bar{\mu}}(1 - \bar{g}\tau_{\mathrm{II}}^{-1/2})\tau_{ij}^{\mathrm{D}},$$

where τ_{ij}^{D} are components of the stress tensor deviator and $\tau_{\mathrm{II}} = (\bar{g} + 2\bar{\mu} D_{\mathrm{II}}^{1/2})^2$.

Let $\boldsymbol{v}, \boldsymbol{w} \in {}^1H^{1,N}(\Omega)$, $\boldsymbol{B}, \boldsymbol{C} \in {}^2H^{1,N}(\Omega)$. Then, according to Eq. (2.6a) and Maxwell's stresses for incompressible media, we have

(a) $(\partial/\partial x_i(\mu^{-1}|\boldsymbol{B}|^2), w_i) = 0$, since $\operatorname{Div}\boldsymbol{w} = 0$.

(b) $(\operatorname{Rot}(\eta_0 \operatorname{Rot}\boldsymbol{B}), \mu^{-1}\boldsymbol{C}) = \int_{\partial\Omega}\sigma^{-1}\mu^{-2}[\boldsymbol{n}, \operatorname{Rot}\boldsymbol{B}]\boldsymbol{C}\,\mathrm{d}s + (\eta_0 \operatorname{Rot}\boldsymbol{B}, \operatorname{Rot}(\mu^{-1}\boldsymbol{C})) = (\eta_0 \operatorname{Rot}\boldsymbol{B}, \operatorname{Rot}(\mu^{-1}\boldsymbol{C}))$ as $[\boldsymbol{n}, \operatorname{Rot}\boldsymbol{B}] = 0$ on $\partial\Omega$, where $\eta_0 = (\sigma\mu)^{-1}$.

(c) $(\partial/\partial x_i(\mu^{-1}B_i\boldsymbol{B}), \boldsymbol{w}) = (\mu^{-1}\boldsymbol{w}, B_i\partial\boldsymbol{B}/\partial x_i - \operatorname{grad}\frac{1}{2}|\boldsymbol{B}|^2) = (\mu^{-1}w_j, B_i(\partial B_j/\partial x_i - \partial B_i/\partial x_j)) = -(\mu^{-1}[\boldsymbol{w}, \boldsymbol{B}], \operatorname{Rot}\boldsymbol{B}) = -(\mu^{-1}\operatorname{Rot}[\boldsymbol{w}, \boldsymbol{B}], \boldsymbol{B})$ as $(\mu^{-1}\boldsymbol{w}, \operatorname{grad}\frac{1}{2}|\boldsymbol{B}|^2) = 0$ (for $\operatorname{Div}\boldsymbol{w} = 0$ and $\boldsymbol{w} = 0$ on $\partial\Omega$); hence $(\partial/\partial x_i(\mu^{-1}B_i\boldsymbol{B}), \boldsymbol{w}) + (\mu^{-1}\operatorname{Rot}[\boldsymbol{w}, \boldsymbol{B}], \boldsymbol{B}) = 0$ (see [2, 3]).

(d) $b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{v}) = 0$ (see [17]).

We multiply Eqs. (2.6a), (2.6c), (2.6e) and (2.6f) by $\boldsymbol{w} - \boldsymbol{v}(t)$, $\boldsymbol{C} - \boldsymbol{B}(t)$, $z - T(t)$, and $\varphi - \Phi(t)$, respectively, and we add the first three equations. We integrate both the sum and the last equation over Ω and apply the Green's theorem satisfying the boundary conditions. According to (a)–(c) the terms $(\partial/\partial x_i(\mu^{-1}|\boldsymbol{B}(t)|^2), w_i - v_i(t))$, $(\partial/\partial x_i(\mu^{-1}B_i(t)\boldsymbol{B}(t)), \boldsymbol{v}(t))$, $(\mu^{-1}\operatorname{Rot}[\boldsymbol{v}(t), \boldsymbol{B}(t)], \boldsymbol{B}(t))$ disappear. Then, after a certain modification (including among others the integration in time over the interval I and the substitution $J(\boldsymbol{w}) = \int_I j(\boldsymbol{w}(t))\,\mathrm{d}t$), we obtain the following variational formulation: Find a vector function $\boldsymbol{a} \equiv (\boldsymbol{v}, \boldsymbol{B}, T) \in \underline{H}$ and a scalar function $\Phi \in {}^3H^{s,1}(\Omega)$ such that

$$\int_I [(\mathscr{E}\boldsymbol{a}'(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}(t)) + A(\boldsymbol{a}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}(t)) + (\Xi(\boldsymbol{a}(t)), \boldsymbol{\alpha}(t) - \boldsymbol{a}(t)) + (\boldsymbol{P}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}(t))$$

$$+ \bar{g}j(\boldsymbol{w}(t)) - \bar{g}j(\boldsymbol{v}(t))]\,\mathrm{d}t \geqslant 0 \quad \forall \boldsymbol{\alpha} \equiv (\boldsymbol{w}, \boldsymbol{C}, z) \in \underline{H} \tag{3.3a}$$

and

$$a_g(\Phi(t), \varphi(t) - \Phi(t)) - (\Psi(t), \varphi(t) - \Phi(t)) \geqslant 0 \quad \forall \varphi(t) \in {}^3H^{s,1}(\Omega) \ \forall t \in I, \tag{3.3b}$$

where

$$(\Xi(\boldsymbol{a}(t)), \boldsymbol{\alpha}(t) - \boldsymbol{a}(t)) = (\Xi_0(\boldsymbol{a}(t)), \boldsymbol{a}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}(t)) + b_s(T(t) - T_0, \boldsymbol{w}(t) - \boldsymbol{v}(t))$$
$$+ b_p(\boldsymbol{v}(t), z(t) - T(t)), \tag{3.3c}$$

$$(\Xi_0(\boldsymbol{a}(t)), \boldsymbol{a}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}(t)) = b(\boldsymbol{v}(t), \boldsymbol{v}(t), \boldsymbol{w}(t) - \boldsymbol{v}(t)) + b_0(\boldsymbol{v}(t), T(t), z(t) - T(t))$$
$$+ (\partial/\partial x_i(\mu^{-1}B_i(t)\boldsymbol{B}(t)), \boldsymbol{w}(t) - \boldsymbol{v}(t))$$
$$- (\mu^{-1}\operatorname{Rot}[\boldsymbol{v}(t), \boldsymbol{B}(t)], \boldsymbol{C}(t) - \boldsymbol{B}(t)). \tag{3.3d}$$

3.1. Preliminary results and main theorem

In the sequel the following lemmas will be used.

Lemma 1. *Let $s = \frac{1}{2}N$, then $v_{i,j} \in L_N(\Omega)$, for each $\boldsymbol{v} \in {}^1H^{s,N}$, $i, j = 1, \ldots, N$.*

For the proof see [2, 7].

Lemma 2. (A corollary of the Gronwall lemma). *Let $g(t) \in C(I)$, $g(t) \geqslant 0$, $\rho(t) \in C(I)$, $\rho(t) \geqslant 0$, $g(t)$ be the nondecreasing function with increasing t. Let $\rho(t)$ be a solution of the inequality*

$$\rho(t) \leqslant c_0 \int_{t_0}^{t} \rho(\tau)\,\mathrm{d}\tau + g(t), \quad t_0 \leqslant t \leqslant t_1, \ c_0 = \text{const.}$$

Then there exists $c_1 = \text{const.}$, $c_1 = c_1(c_0, t_1, t_0)$ *such that*

$$\rho(t) \leqslant c_1 g(t) \quad \forall t \in I, \text{ where } I = \langle t_0, t_1 \rangle.$$

For the proof, see [7].

From the above given assumption the symmetry

$$a(\boldsymbol{u}, \boldsymbol{v}) = a(\boldsymbol{v}, \boldsymbol{u}), \qquad b_{\mathrm{M}}(\boldsymbol{B}, \boldsymbol{C}) = b_{\mathrm{M}}(\boldsymbol{C}, \boldsymbol{B}),$$

$$a_{\mathrm{T}}(T, z) = a_{\mathrm{T}}(z, T), \quad A(\boldsymbol{\alpha}, \boldsymbol{\gamma}) = A(\boldsymbol{\gamma}, \boldsymbol{\alpha}),$$

$$a_{\mathrm{g}}(\Phi, \varphi) = a_{\mathrm{g}}(\varphi, \Phi)$$

holds. Moreover, they yield that for $\boldsymbol{v} \in {}^1H^{1,N}(\Omega)$, $\boldsymbol{B} \in {}^2H^{1,N}(\Omega)$, $T, \Phi \in {}^3H^{1,1}(\Omega)$ there exist constants $c_{\mathrm{B}} > 0$, $c_{\mathrm{M}} > 0$, $c_{\mathrm{T}} > 0$, $c_{\Phi} > 0$ such that

$$a(\boldsymbol{v}, \boldsymbol{v}) \geqslant c_{\mathrm{B}} \| \boldsymbol{v} \|_{1,N}^2 \quad \text{for all } \boldsymbol{v} \in {}^1H^{1,N}(\Omega), \tag{3.4a}$$

$$b_{\mathrm{M}}(\boldsymbol{B}, \boldsymbol{B}) \geqslant c_{\mathrm{M}} \| \boldsymbol{B} \|_{1,N}^2 \quad \text{for all } \boldsymbol{B} \in {}^2H^{1,N}(\Omega) \tag{3.4b}$$

(which is the corollary of Theorem 6.1, Ch. 7 in [2]),

$$a_{\mathrm{T}}(T, T) \geqslant c_{\mathrm{T}} \| T \|_{1,1}^2 \quad \text{for all } T \in {}^3H^{1,1}(\Omega), \tag{3.4c}$$

$$a_{\mathrm{g}}(\Phi, \Phi) \geqslant c_{\Phi} \| \Phi |_{1,1}^2 \quad \text{for all } \Phi \in {}^3H^{1,1}(\Omega). \tag{3.4d}$$

Therefore, (3.4) yields that for $\boldsymbol{a} = (\boldsymbol{v}, \boldsymbol{B}, T)$ and $\boldsymbol{\alpha} = (\boldsymbol{w}, \boldsymbol{C}, z)$ there exists a suitable positive constant c_α independent of $\boldsymbol{\alpha}$ such that

$$A(\boldsymbol{\alpha}, \boldsymbol{\alpha}) \geqslant c_\alpha \| \boldsymbol{\alpha} \|_{1,2N+2}^2 \quad \text{for all } \boldsymbol{\alpha} \in \underline{H}. \tag{3.5}$$

From the Hölder inequality applied to $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ we have

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant c_1 \| \boldsymbol{u} \|_{L_p^N(\Omega)} \| \boldsymbol{w} \|_{L_p^N(\Omega)} \sum_{i,j} \| D_i v_j \|_{L_N^N(\Omega)}, \tag{3.6}$$

where $2/p + 1/N = 1$, i.e.,

$$p = 2N/(N-1). \tag{3.7}$$

Applying Lemma 1 we have

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant c_2 \| \boldsymbol{u} \|_{L_p^N(\Omega)} \| \boldsymbol{w} \|_{L_p^N(\Omega)} \| \boldsymbol{v} \|_{s,N}, \quad s = \tfrac{1}{2}N \tag{3.8}$$

(see [2]). Since

$$\| \boldsymbol{v} \|_{1/2,N} \leqslant c \| \boldsymbol{v} \|_{L_2^N(\Omega)}^{1/2} \| \boldsymbol{v} \|_{1,N}^{1/2} \quad \text{and} \quad H^{1/2,N}(\Omega) \subset L_p^N(\Omega), \tag{3.9}$$

for p from (3.7), then

$$\| \boldsymbol{v} \|_{L_p^N(\Omega)} \leqslant c_3 \| \boldsymbol{v} \|_{1,N}^{1/2} \| \boldsymbol{v} \|_{L_2^N(\Omega)}^{1/2} \quad \forall \boldsymbol{v} \in H_0^{1,N}(\Omega), \tag{3.10}$$

where p is defined by (3.7).

From (3.8) and (3.10) we obtain

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant c_4 \|\boldsymbol{u}\|_{1,N}^{1/2} \|\boldsymbol{u}\|_{0,N}^{1/2} \|\boldsymbol{w}\|_{1,N}^{1/2} \|\boldsymbol{w}\|_{0,N}^{1/2} \|\boldsymbol{v}\|_{s,N}, \quad s = \tfrac{1}{2}N. \tag{3.11}$$

Furthermore, if $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in {}^1\mathscr{H}(\Omega)$ then $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) + b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) = 0$. The last equality is valid also for $\boldsymbol{u} \in L_2^N(\Omega)$, $\boldsymbol{v}, \boldsymbol{w} \in {}^1H^{s,N}(\Omega)$; $b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = -b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u})$ for $u, v \in {}^1\mathscr{H}(\Omega)$ also holds.

Similarly $b_0(\boldsymbol{u}, y, z) + b_0(\boldsymbol{u}, z, y) = 0$, $\boldsymbol{u} \in {}^1H^{0,N}(\Omega)$, $y, z \in H_0^{1,1}(\Omega)$ and there exist positive constants c_5, c_6, c_7 independent of $\boldsymbol{u}, y, z$ such that

$$\begin{aligned} |b_0(\boldsymbol{u}, y, z)| &\leqslant c_5 \|\boldsymbol{u}\|_{L_p^N(\Omega)} \|y\|_{L_p(\Omega)} \|\{D_i z\}\|_{L_N^N(\Omega)} \\ &\leqslant c_6 \|\boldsymbol{u}\|_{L_p^N(\Omega)} \|y\|_{L_p(\Omega)} \|z\|_{s,1} \\ &\leqslant c_7 \|\boldsymbol{u}\|_{1,N}^{1/2} \|\boldsymbol{u}\|_{0,N}^{1/2} \|y\|_{1,1}^{1/2} \|y\|_{0,1}^{1/2} \|z\|_{s,1}, \quad s = \tfrac{1}{2}N, \end{aligned} \tag{3.12}$$

where the second and third inequalities hold for $z \in H^{s,1}(\Omega)$ only and the third requires the additional relation $\boldsymbol{u} \in H^{1,N}(\Omega)$.

Since

$$\begin{aligned} (\partial/\partial x_i(\mu^{-1}B_i\boldsymbol{B}), \boldsymbol{w}) &= \int_\Omega \mu^{-1}(\boldsymbol{B}B_{i,i} + B_i\boldsymbol{B}_{,i})\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} \\ &= (\mu^{-1}B_i\boldsymbol{B}, \partial\boldsymbol{w}/\partial x_i) \quad (\text{due to } \operatorname{Div}\boldsymbol{B} = 0) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} (\mu^{-1}\operatorname{Rot}[\boldsymbol{v}, \boldsymbol{B}], \boldsymbol{C}) &= \int_\Omega \mu^{-1}(v_{i,j}B_j - v_{j,j}B_i + v_iB_{j,j} - v_jB_{i,j})C_i\,\mathrm{d}\boldsymbol{x} \\ &= -\int_\Omega \mu^{-1}(v_iB_jC_{j,i} - v_jB_iC_{j,i} - v_iB_jC_{i,j} + v_jB_iC_{i,j})\,\mathrm{d}\boldsymbol{x} \\ &= -\int_\Omega \mu^{-1}(v_iB_j - v_jB_i)(C_{j,i} - C_{i,j})\,\mathrm{d}\boldsymbol{x} = -(\mu^{-1}[\boldsymbol{v}, \boldsymbol{B}], \operatorname{Rot}\boldsymbol{C}), \end{aligned} \tag{3.14}$$

where the Green's theorem and boundary conditions were applied, then (like in (3.5) and (3.7)) there exist positive constants c_8, c_9, c_{10} and c_{11} such that

$$\begin{aligned} |(\partial/\partial x_i(\mu^{-1}B_i\boldsymbol{B}), \boldsymbol{w})| &\leqslant c_8 \|\boldsymbol{B}\|_{L_p^N(\Omega)}^2 \|D_i\boldsymbol{w}\|_{L_N^N(\Omega)}, \quad p \text{ from (3.7)} \\ &\leqslant c_9 \|\boldsymbol{B}\|_{1,N} \|\boldsymbol{B}\|_{0,N} \|\boldsymbol{w}\|^{s,N} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} |(\mu^{-1}\operatorname{Rot}[\boldsymbol{v}, \boldsymbol{B}], \boldsymbol{C})| &\leqslant c_{10} \|\boldsymbol{v}\|_{L_p^N(\Omega)} \|\boldsymbol{B}\|_{L_p^N(\Omega)} \|\boldsymbol{C}\|_{s,N} \\ &\leqslant c_{11} \|\boldsymbol{v}\|_{1,N}^{1/2} \|\boldsymbol{v}\|_{0,N}^{1/2} \|\boldsymbol{B}\|_{1,N}^{1/2} \|\boldsymbol{B}\|_{0,N}^{1/2} \|\boldsymbol{C}\|_{s,N}, \end{aligned} \tag{3.16}$$

where the second and third inequalities are valid for $\boldsymbol{C} \in H^{s,N}(\Omega)$ and the third one is also valid for $\boldsymbol{v} \in H^{1,N}(\Omega)$.

Therefore, due to the estimates (3.6), (3.8), (3.11), (3.12), (3.15), (3.16) there exist positive constants c_{12}, c_{13} and c_{14} independent of $\boldsymbol{\alpha}, \gamma$ such that

$$\begin{aligned} |\Xi_0(\boldsymbol{\alpha}, \boldsymbol{\alpha}, \gamma)| &\leqslant c_{12} \|\boldsymbol{\alpha}\|^2_{L_p^{2N+1}(\Omega)} \sum_{i,j} \|D_i v_j\|_{L_N^{2N+1}(\Omega)} \\ &\leqslant c_{13} \|\boldsymbol{\alpha}\|^2_{L_p^{2N+1}(\Omega)} \|\gamma\|_{s,2N+1} \\ &\leqslant c_{14} \|\boldsymbol{\alpha}\|_{1,2N+1} \|\boldsymbol{\alpha}\|_{0,2N+1} \|\gamma\|^{1/2}_{s,2N+1}, \quad s = \tfrac{1}{2}N, \end{aligned} \tag{3.17}$$

where the second and third inequalities are valid for $\gamma \in H^{s,2N+1}(\Omega)$ and the third also for $\boldsymbol{\alpha} \in H^{1,2N+1}(\Omega)$.

The main result of the paper gives the following theorem.

Theorem 3. *Let $N \geqslant 2$, $s = \frac{1}{2}N$. Let*

$$f \in L_2(I; ({}^1H^{s,N}(\Omega))'), \qquad W \in L_2(I; ({}^3H^{s,N}(\Omega))'). \tag{3.18}$$

Then there exist a vector function $\boldsymbol{a} \equiv (\boldsymbol{v}, \boldsymbol{B}, T)$ and a scalar function Φ such that

$$\boldsymbol{v} \in L_2(I; {}^1H^{s,N}(\Omega)) \cap L_\infty(I; {}^1H^{0,N}(\Omega)), \qquad \boldsymbol{v}' \in L_2(I; ({}^1H^{s,N}(\Omega))'), \tag{3.19a}$$

$$\boldsymbol{B} \in L_2(I; {}^2H^{s,N}(\Omega)) \cap L_\infty(I; {}^2H^{0,N}(\Omega)), \qquad \boldsymbol{B}' \in L_2(I; ({}^2H^{s,N}(\Omega))'), \tag{3.19b}$$

$$T \in L_2(I; {}^3H^{s,1}(\Omega)) \cap L_\infty(I; {}^3H^{0,1}(\Omega)), \qquad T' \in L_2(I; ({}^3H^{s,1}(\Omega))'), \tag{3.19c}$$

$$\Phi \in L_2(I; {}^3H^{1,1}(\Omega)), \tag{3.19d}$$

$$\boldsymbol{v}(t_0) = 0, \qquad \boldsymbol{B}(t_0) = 0, \qquad T(t_0) = 0 \tag{3.20}$$

and satisfy the variational equations (3.3).

4. Proof of Theorem 3

To prove Theorem 3 the double regularization for the problem (3.3a) and the Lax–Milgram theorem for the problem (3.3b) will be used.

Let $J_\varepsilon(\boldsymbol{w})$ be a regularized functional of the function $J(\boldsymbol{w})$ defined by

$$J_\varepsilon(\boldsymbol{w}) = \int_I \bar{g} j_\varepsilon(\boldsymbol{w}(t))\,\mathrm{d}t \quad \text{where } j_\varepsilon(\boldsymbol{w}(t)) = \frac{2}{1+\varepsilon} \int_\Omega (D_{\mathrm{II}}(\boldsymbol{w}(t)))^{(1+\varepsilon)/2}\,\mathrm{d}\boldsymbol{x}, \ \varepsilon > 0, \tag{4.1}$$

for which

$$(J_\varepsilon'(\boldsymbol{v}), \boldsymbol{w}) = \int_{\Omega \times I} \bar{g} D_{\mathrm{II}}(\boldsymbol{v})^{(\varepsilon-1)/2} D_{ij}(\boldsymbol{v}) D_{ij}(\boldsymbol{w})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \tag{4.2}$$

Now for the problem (3.3a) we introduce the regularized problem $(\mathscr{P}_r)_v$: Find a vector function $\boldsymbol{a}_{\varepsilon\eta} \equiv (\boldsymbol{v}_{\varepsilon\eta}, \boldsymbol{B}_{\varepsilon\eta}, T_{\varepsilon\eta}) \in \underline{H}$ such that

$$\int_I [(\mathscr{E}\boldsymbol{a}'_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}_{\varepsilon\eta}(t)) + A(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}_{\varepsilon\eta}(t))$$
$$+ \eta(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}_{\varepsilon\eta}(t))_s + (\Xi(\boldsymbol{a}_{\varepsilon\eta}(t)), \boldsymbol{\alpha}(t) - \boldsymbol{a}_{\varepsilon\eta}(t))$$
$$- (\boldsymbol{P}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}_{\varepsilon\eta}(t))]\,\mathrm{d}t + J_\varepsilon(\boldsymbol{w}) - J_\varepsilon(\boldsymbol{v}_{\varepsilon\eta}(t)) \geqslant 0 \quad \forall\boldsymbol{\alpha} \equiv (\boldsymbol{w}, \boldsymbol{C}, z) \in \underline{H}, \tag{4.3}$$

where Ξ is defined in (3.3c), $s = \frac{1}{2}N$ and η is a positive number. For $N = 2$ we obtain $s = 1$ and the added term is of the same order as the bilinear forms and therefore can be omitted. We remark that the added term has the physical meaning of the viscosity.

The method of the proof is the following:

(a) The existence of the solution of (4.3) based on the Galerkin approximation will be proved.
(b) A priori estimates I and II independent of ε and η will be derived.
(c) Limitation processes over m, ε and η will be performed.
(d) The uniqueness of the solution of (3.3a) will be proved.
(e) The existence and uniqueness of the solution of (3.3b) for every $t \in I$ will be proved via the usual elliptic equation technique.

The existence of $\boldsymbol{a}_{\varepsilon\eta}$ will be proved by means of the finite-dimensional approximation. Let $M_H = \{\boldsymbol{\alpha}_i\}$ be a countable basis of the space $\underline{H}^s(\Omega)$, i.e., each finite subset of M_H is linearly independent and span $\{\boldsymbol{\alpha}_i \mid i = 1, 2, \ldots\}$ is dense in $\underline{H}^s(\Omega)$, as $\underline{H}^s(\Omega)$ is a separable space. Let $\mathscr{W}^m$ be the space spanned by $\{\boldsymbol{\alpha}_j \mid 1 \leqslant j, k \leqslant m\}$. Then the approximate solution $\boldsymbol{a}_m$ of the order m satisfies

$$(\mathscr{E}\boldsymbol{a}'_m(t), \boldsymbol{\alpha}_j) + A(\boldsymbol{a}_m(t), \boldsymbol{\alpha}_j) + \eta(\boldsymbol{a}_m(t), \boldsymbol{\alpha}_j)_s + (\Xi(\boldsymbol{a}_m(t)), \boldsymbol{\alpha}_j) + \bar{g}(j'_\varepsilon(\boldsymbol{a}_m(t), \alpha_j)$$
$$- (\boldsymbol{P}(t), \boldsymbol{\alpha}_j) = 0, \quad 1 \leqslant j, k \leqslant m, \tag{4.4}$$
$$\boldsymbol{a}_m(\boldsymbol{x}, t_0) = 0. \tag{4.5}$$

Since $\{\boldsymbol{\alpha}_j\}_{j=1}^m$ are linearly independent, the system (4.4), (4.5) is a regular system of ordinary differential equations of the first order and therefore (4.4), (4.5) uniquely define $\boldsymbol{a}_m$ on the interval $I_m = \langle t_0, t_m \rangle$. Therefore, (4.4) is valid for every test function $\gamma(t) = \sum_{i=1}^m c_i(t)\alpha_i$, $t \in I_m$ where c_i are continuously differentiable functions on I_m, $i = 1, \ldots, m$. Particularly, it holds for $\gamma(t) = \boldsymbol{a}_m(t)$, $t \in I_m$.

A priori estimates I and II show that $t_m = t_1$.

4.1. A priori estimate I

Using (c) and (d) mentioned above we have

$$(\partial/\partial x_i(\mu^{-1}B_{mi}(t)\boldsymbol{B}_m(t)), \boldsymbol{v}_m(t)) + (\mu^{-1}\operatorname{Rot}[\boldsymbol{v}_m(t), \boldsymbol{B}_m(t)], \boldsymbol{B}_m(t)) = 0,$$
$$b(\boldsymbol{v}_m(t), \boldsymbol{v}_m(t), \boldsymbol{v}_m(t)) = 0, \qquad b_0(\boldsymbol{v}_m(t), T_m(t), T_m(t)) = 0. \tag{4.6}$$

Let us introduce the notation

$$X_m = b_s(T_m - T_0, \boldsymbol{v}_m) + b_p(\boldsymbol{v}_m, T_m). \tag{4.7}$$

Let $\partial\beta_{ij}/\partial x_j \in L^\infty(\Omega)\ \forall i,j$. Then there exists a positive constant c independent of m (and also of ε and η) such that

$$\begin{aligned}|X_m| &= |b_s(T_m - T_0, \boldsymbol{v}_m) + \mathrm{b}_p(\boldsymbol{v}_m, T_m)|\\ &= \left|\int_\Omega (\partial/\partial x_j(\beta_{ij}(T_m(t) - T_0))\boldsymbol{v}_i(t) + \rho\, T_0 \beta_{ij}\partial v_i(t)/\partial x_j T_m(t))\,\mathrm{d}\boldsymbol{x}\right|\\ &\leqslant c(1 + \|T_m(t)\|_{1,1}\,\|\boldsymbol{v}_m(t)\|_{0,N} + \|T_m(t)\|_{0,1}\,\|\boldsymbol{v}_m(t)\|_{1,N}).\end{aligned} \tag{4.8}$$

Via the integration of (4.4) (with $\gamma(t) = \boldsymbol{a}_m(t)$) in time over $I_m = (t_0, t_m)$ we obtain

$$\begin{aligned}&\int_{I_m} [(\mathscr{E}\boldsymbol{a}'_m(t), \boldsymbol{a}_m(t)) + A(\boldsymbol{a}_m(t), \boldsymbol{a}_m(t)) + (\Xi(\boldsymbol{a}_m(t)), \boldsymbol{a}_m(t))\\ &\qquad + \eta(\boldsymbol{a}_m(t), \boldsymbol{a}_m(t))_s - (\boldsymbol{P}(t), \boldsymbol{a}_m(t))]\,\mathrm{d}t + (J'_\varepsilon(\boldsymbol{v}_m(t), \boldsymbol{a}_m(t)) = 0.\end{aligned} \tag{4.9}$$

Since $(j'_\varepsilon(\boldsymbol{v}), \boldsymbol{v}) \geqslant 0$, according to the ellipticity (3.4) of the bilinear form $A(\boldsymbol{a}, \boldsymbol{\alpha})$ and due to (4.6) and (4.8) after some modification we obtain

$$\begin{aligned}&e_0\,\|\boldsymbol{a}_m(t)\|^2_{0,N} + 2c_\alpha \int_{I_m} \|\boldsymbol{a}_m(\tau)\|^2_{1,N}\,\mathrm{d}\tau + 2\eta \int_{I_m} \|\boldsymbol{a}_m(\tau)\|_s\,\mathrm{d}\tau\\ &\quad\leqslant 2c \int_{I_m} (1 + \|T_m(\tau)\|_{1,1}\,\|\boldsymbol{v}_m(\tau)\|_{0,N} + \|T_m(\tau)\|_{0,1}\,\|\boldsymbol{v}_m(\tau)\|_{1,N})\,\mathrm{d}\tau\\ &\qquad + 2\int_{I_m} \|\boldsymbol{P}(\tau)\|_*\,\|\boldsymbol{a}_m(\tau)\|_{1,N}\,\mathrm{d}\tau,\end{aligned} \tag{4.10}$$

where $e_0 = \min\{\rho^{1/2}, \mu^{-1/2}, (\rho c_e)^{1/2}\}$. According to the Gronwall lemma, after some modification, we have the following estimates:

$$\|\boldsymbol{a}_m(t)\|_{0,N} \leqslant c, \quad t \in I, \qquad \int_I \|\boldsymbol{a}_m(\tau)\|^2_{1,N}\,\mathrm{d}\tau \leqslant c, \qquad \eta\int_I \|\boldsymbol{a}_m(\tau)\|^2_s\,\mathrm{d}\tau \leqslant c, \tag{4.11}$$

where $c = \text{const.} > 0$ independent of m.

From these estimates we obtain

$$\{\boldsymbol{a}_m(t);\ m \in \mathbb{N}\} \text{ is a bounded subset in } L_2(I; \underline{H}), \tag{4.12a}$$

$$\{\eta^{1/2}\boldsymbol{a}_m(t);\ m \in \mathbb{N}\} \text{ is a bounded subset in } L_2(I; \underline{H}^s). \tag{4.12b}$$

4.2. *A priori estimate II*

We shall show that

$$\boldsymbol{a}'_m(t) \text{ is bounded subset of } L_2(I; (\underline{H}^s)'). \tag{4.13}$$

By virtue of (3.17) and (4.12) we obtain

$$\Xi_0(\boldsymbol{a}_m(t), \boldsymbol{a}_m(t), \boldsymbol{\alpha}(t)) = (\boldsymbol{h}_m(t), \boldsymbol{\alpha}(t)), \quad \boldsymbol{\alpha}(t) \in L_2(I; H^{s,2N+1}(\Omega)),$$
$$\|\boldsymbol{h}_m(t)\|_{L_2(I;({}^1H^{s,N}(\Omega))')} \leqslant c, \tag{4.14}$$

where c is a positive constant independent of m, ε, η and where the separate terms of $\Xi_0(\boldsymbol{a}_m(t), \boldsymbol{a}_m(t), \boldsymbol{\alpha}(t))$ have a sense due to the estimates (3.7), (3.9), (3.10), (3.13) and (3.14).

Linear form $\boldsymbol{a} \to A(\boldsymbol{a}, \boldsymbol{\alpha})$ for a fixed $\alpha \in \underline{H}$ is continuous on $\underline{H}$ so that

$$A(\boldsymbol{a}(t), \boldsymbol{\alpha}) = (A_{\mathrm{B}}\boldsymbol{a}(t), \boldsymbol{\alpha}), \quad A_{\mathrm{B}} \in \mathscr{L}(\underline{H}, (\underline{H})'), \tag{4.15a}$$

$$(\boldsymbol{a}(t), \boldsymbol{\alpha})_s = (\Lambda_s \boldsymbol{a}(t), \boldsymbol{\alpha}), \quad \Lambda_s \in \mathscr{L}(\underline{H}, (\underline{H})'). \tag{4.15b}$$

Then (4.5), (4.6) are equivalent to

$$(\mathscr{E}\boldsymbol{a}_m' + A_{\mathrm{B}}\boldsymbol{a}_m + \eta\Lambda_s\boldsymbol{a}_m + \bar{g}j_\varepsilon'(\boldsymbol{v}_m) + \boldsymbol{h}_m - \boldsymbol{P}, \boldsymbol{\alpha}_j) = 0, \quad 1 \leqslant j \leqslant m. \tag{4.16}$$

Let S_m be orthogonal projections $\underline{H}^0 \to \mathscr{W}^m$, where $\mathscr{W}^m$ was defined above, then

$$S_m\boldsymbol{h} = \sum_{j=1}^{m} (\boldsymbol{h}, \boldsymbol{w}_j)\boldsymbol{w}_j.$$

Then from (4.16) and from the fact that $S_m\boldsymbol{a}_m' = \boldsymbol{a}_m$ we obtain

$$\mathscr{E}\boldsymbol{a}' = S_m(\boldsymbol{P} - A_{\mathrm{B}}\boldsymbol{a}_m - \eta\Lambda_s\boldsymbol{a}_m - \bar{g}j_\varepsilon'(\boldsymbol{v}_m) - \boldsymbol{h}_m). \tag{4.17}$$

Then according to (4.12a) and (4.15b) $A_{\mathrm{B}}\boldsymbol{a}_m$ is a bounded subset of $L_2(I; (\underline{H})') \cap L_2(I; (\underline{H}^s)')$, and therefore from (4.12b),

$$\|\eta\Lambda_s\boldsymbol{a}_m\|_{L_2(I;(\underline{H})')} = \mathrm{O}(\eta^{1/2}).$$

But due to (4.2),

$$\|j_\varepsilon'(\boldsymbol{v})\|_* \leqslant c\left(\int_\Omega (D_{\mathrm{II}}(\boldsymbol{v}))^\varepsilon \,\mathrm{d}\boldsymbol{x}\right)^{1/2},$$

where $\|\cdot\|_*$ is the norm in $({}^1H^{s,N}(\Omega))'$, then $j_\varepsilon'(\boldsymbol{v}_m)$ is a bounded subset of $L_2(I; ({}^1H^{1,N}(\Omega))')$. Thus, (4.17) indicates that $\boldsymbol{a}_m' = S_m p_m$, where $p_m \in \mathscr{P}_P \subset L_2(I; (\underline{H}^s)')$, and where $\mathscr{P}_P$ is a bounded subset of $L_2(I; (\underline{H}^s)')$. Since $\lambda^{1/2}\boldsymbol{\alpha}_j$ is an orthogonal system in $(\underline{H}^s)'$ (with respect to the norm $\|\boldsymbol{p}\|_{\underline{H}^s}^2 = \|\Lambda_s^{-1}\boldsymbol{p}\|_{\underline{H}^s}$) and since $\|\boldsymbol{p}\|_{(\underline{H}^s)'}^2 = \sum_{j=1}^m (\boldsymbol{p}, \alpha_j\lambda_j^{1/2})$, $\|S_m\boldsymbol{p}\|_{(\underline{H}^s)'}^2 = \sum_{j=1}^m (\boldsymbol{p}, \alpha_j\lambda_j^{1/2})$, then

$$\|S_m\boldsymbol{p}\|_{(\underline{H}^s)'} \leqslant c\,\|\boldsymbol{p}\|_{(\underline{H}^s)'}, \tag{4.18}$$

which proves (4.13).

4.3. Passages to the limit over m

At first the limitation process with m, i.e., the convergence of the finite-dimensional approximation will be proved for ε, η being fixed.

From the estimates I and II as well as from (4.11) and (4.12) the subsequence $\{\boldsymbol{a}_\mu(t), \mu \in \mathbb{N}\}$ of the sequence $\{\boldsymbol{a}_m, m \in \mathbb{N}\}$, can be taken such that

$$\begin{aligned} &\boldsymbol{a}_\mu \to \boldsymbol{a}_{\varepsilon\eta} \quad \text{in } L_\infty(I; \underline{H}^0) \text{ *-weakly}^2, \\ &\boldsymbol{a}_\mu \to \boldsymbol{a}_{\varepsilon\eta} \quad \text{in } L_2(I; \underline{H}^1) \text{ weakly}, \\ &\boldsymbol{a}_\mu \to \boldsymbol{a}_{\varepsilon\eta} \quad \text{in } L_2(I; (\underline{H}^s)') \text{ weakly}, \\ &\boldsymbol{a}_\mu \to \boldsymbol{a}_{\varepsilon\eta} \quad \text{in } L_2(I; \underline{H}^0) \text{ strongly}. \end{aligned} \tag{4.19}$$

Thus, for the appropriate components $\boldsymbol{v}_\mu$ and $\boldsymbol{B}_\mu$ of $\boldsymbol{\alpha}_\mu$ we have

$$\begin{aligned} &v_{i\mu} \to v_{i\varepsilon\eta} \quad \text{weakly a.e. in } G \times I, \\ &B_{i\mu} \to B_{i\varepsilon\eta} \quad \text{weakly a.e. in } G \times I \end{aligned} \tag{4.20}$$

(where $v_{i\mu}$, $v_{i\varepsilon\eta}$, $B_{i\mu}$, $B_{i\varepsilon\eta}$ represent the ith component of v_μ, $v_{\mu\eta}$, B_μ, $B_{\mu\eta}$, respectively) since $\boldsymbol{v}_\mu \to \boldsymbol{v}_{\varepsilon\eta}$, $\boldsymbol{B}_\mu \to \boldsymbol{B}_{\varepsilon\eta}$ in $L_2(I; {}^1H^{0,N}(\Omega))$ strongly or in $L_2(I; {}^2H^{0,N}(\Omega))$ strongly, respectively. Furthermore, $\{j'_\varepsilon(\boldsymbol{v}_m)\}$, $\{v_{i\mu}v_{j\mu}\}$, $\{B_{i\mu}B_{j\mu}\}$, due to (3.10) and (4.12a), are bounded subsets of spaces $L_2(I; {}^1(H^{1,N}(\Omega))')$ and $L_2(I; L^N_{p/2}(\Omega))$, respectively. Then we can also assume that

$$j'_\varepsilon(\boldsymbol{v}_\mu) \to \chi \quad \text{weakly in } L_2(I; ({}^1H^{1,N}(\Omega))'), \tag{4.21a}$$

$$v_{i\mu}v_{j\mu} \to \Theta_{ij} \quad \text{weakly in } L_2(I; L_{p/2}(\Omega)), \tag{4.21b}$$

$$v_{i\mu}B_{j\mu} \to {}^1\Theta_{ij} \quad \text{weakly in } L_2(I; L_{p/2}(\Omega)), \tag{4.21c}$$

$$B_{i\mu}B_{j\mu} \to {}^2\Theta_{ij} \quad \text{weakly in } L_2(I; L_{p/2}(\Omega)). \tag{4.21d}$$

From (4.20), $v_{i\mu}v_{j\mu} \to v_{i\varepsilon\eta}v_{j\varepsilon\eta}$, $v_{i\mu}B_{j\mu} \to v_{i\varepsilon\eta}B_{j\varepsilon\eta}$, $B_{i\mu}B_{j\mu} \to B_{i\varepsilon\eta}B_{j\varepsilon\eta}$ in the sense of distributions in $\Omega \times I$, which comparing with (4.21b)–(4.21d) gives

$$\Theta_{ij} = v_{i\varepsilon\eta}v_{j\varepsilon\eta}, \quad {}^1\Theta_{ij} = v_{i\varepsilon\eta}B_{j\varepsilon\eta}, \quad {}^2\Theta_{ij} = B_{i\varepsilon\eta}B_{j\varepsilon\eta}.$$

But

$$b(\boldsymbol{v}_\mu, \boldsymbol{v}_\mu, \boldsymbol{w}_j) = -b(\boldsymbol{v}_\mu, \boldsymbol{w}_j, \boldsymbol{v}_\mu) \to -b(\boldsymbol{v}_{\varepsilon\eta}, \boldsymbol{w}_j, \boldsymbol{v}_{\varepsilon\eta}) \quad \text{weakly in } L_2(I)\ \forall \boldsymbol{w}_j$$

(see [2]). Similarly for the other trilinear forms,

$$\Xi_0(\boldsymbol{a}_\mu, \boldsymbol{a}_\mu, \boldsymbol{\alpha}_j) \to -\Xi_0(\boldsymbol{a}_{\varepsilon\eta}, \boldsymbol{\alpha}_j, \boldsymbol{a}_{\varepsilon\eta}) \quad \text{weakly in } L_2(I)\ \forall \boldsymbol{\alpha}_j.$$

From (4.3a) for $m = \mu$ we obtain

$$\begin{aligned} &\int_I [(\mathscr{E}\boldsymbol{a}'_{\varepsilon\eta}(t), \boldsymbol{\alpha}_j(t)) + A(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}_j(t)) + \Xi(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}_j(t)) + \eta(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}_j)_s]\, \mathrm{d}t \\ &\qquad + (\chi(t), \boldsymbol{w}_j(t)) = 0 \quad \forall j, \end{aligned} \tag{4.22}$$

where we denote by $\chi = J'_\varepsilon(\boldsymbol{v}_{\varepsilon\eta}(t))$.

[2] The function $f_j \to f$ *-weakly in $L_\infty(I; H^{0,N}(\Omega))$ if $\int_{t_0}^{t_1}(f_j(t), \varphi)\,\mathrm{d}t \underset{\text{weakly}}{\to} \int_{t_0}^{t_1}(f(t), \varphi(t))\,\mathrm{d}t\ \forall \varphi \in L_1(I; H^{0,N}(\Omega))$.

But the system of functions $\{\boldsymbol{\alpha}_j\}$ is complete in $\underline{H}^s$, so from (4.22) it follows that

$$\int_I [(\mathscr{E}\boldsymbol{a}'_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t)) + A(t; \boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t)) + \Xi(t; \boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t)) + \eta(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t))_s]\,\mathrm{d}t$$
$$+ (\chi(t), \boldsymbol{w}(t)) = 0 \quad \forall \boldsymbol{\alpha} \in \underline{H}^s.$$

Since (4.3) and (4.11) are satisfied it is sufficient to prove that

$$\chi(t) = J'_\varepsilon(\boldsymbol{v}_{\varepsilon\eta}(t)). \tag{4.23}$$

To prove (4.23), similarly as in [2], the property of monotonicity will be used. Let

$$\boldsymbol{a} \in L_2(I; \underline{H}^s) \quad \text{such that } \boldsymbol{a} \in L_2(I; (\underline{H}^s)'), \quad \boldsymbol{a}(t_0) = 0.$$

Let us put

$$X_\mu = (J'_\varepsilon(\boldsymbol{a}_\mu(t)) - J'_\varepsilon(\boldsymbol{\alpha}(t)), \boldsymbol{a}_\mu(t) - \boldsymbol{\alpha}(t)) + \int_I A(t; \boldsymbol{a}_\mu(t) - \boldsymbol{\alpha}(t), \boldsymbol{a}_\mu(t) - \alpha(t))\,\mathrm{d}t$$
$$+ \eta \int_I \|\boldsymbol{a}_\mu(t) - \boldsymbol{\alpha}(t)\|^2_{s,N}\,\mathrm{d}t + \int_I (\mathscr{E}\boldsymbol{a}'_\mu(t) - \mathscr{E}\boldsymbol{\alpha}'(t), \boldsymbol{a}_\mu(t) - \alpha(t))\,\mathrm{d}t.$$

From (4.3) we have

$$X_\mu = \int_I (\boldsymbol{P}(t), \boldsymbol{a}_\mu(t))\,\mathrm{d}t - (J'_\varepsilon(\boldsymbol{a}_\mu(t), \boldsymbol{\alpha}(t)) - (J'_\varepsilon(\boldsymbol{\alpha}(t), \boldsymbol{a}_\mu(t) - \boldsymbol{\alpha}(t))$$
$$- \int_I [A(t; \boldsymbol{a}_\mu(t), \boldsymbol{\alpha}(t)) + A(t; \boldsymbol{\alpha}(t), \boldsymbol{a}_\mu(t) - \boldsymbol{\alpha}(t))]\,\mathrm{d}t$$
$$- \eta \int_I [(\boldsymbol{a}_\mu(t), \boldsymbol{\alpha}(t))_s + (\boldsymbol{\alpha}(t), \boldsymbol{a}_\mu(t) - \boldsymbol{\alpha}(t))_s]\,\mathrm{d}t$$
$$- \int_I [(\mathscr{E}\boldsymbol{a}'_m(t), \boldsymbol{\alpha}(t)) - (\mathscr{E}\boldsymbol{\alpha}'(t), \boldsymbol{a}_m(t) - \boldsymbol{\alpha}(t))]\,\mathrm{d}t.$$

Hence $X_\mu \to X$, where

$$X = \int_I (\boldsymbol{P}(t), \boldsymbol{a}_{\varepsilon\eta}(t))\,\mathrm{d}t - [(\chi(t), \boldsymbol{\alpha}(t)) + J'_\varepsilon(\boldsymbol{\alpha}(t), \boldsymbol{a}_{\varepsilon\eta}(t) - \boldsymbol{\alpha}(t))]$$
$$- \int_I [A(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t)) + A(\boldsymbol{\alpha}(t), a_{\varepsilon\eta}(t) - \boldsymbol{\alpha}(t))]\,\mathrm{d}t$$
$$- \eta \int_I [(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t))_s + (\boldsymbol{\alpha}(t), \boldsymbol{a}_{\varepsilon\eta}(t) - \boldsymbol{\alpha}(t))_s]\,\mathrm{d}t$$
$$- \int_I [(\mathscr{E}\boldsymbol{a}'_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t)) - (\mathscr{E}\boldsymbol{\alpha}'(t), \boldsymbol{a}_{\varepsilon\eta}(t) - \alpha(t))]\,\mathrm{d}t.$$

Since $X_\mu(t) \geqslant 0$ for all μ then $X(t) \geqslant 0$.

Let us put $\boldsymbol{\alpha} = \boldsymbol{a}_{\varepsilon\eta} - \lambda \boldsymbol{u}$, $\lambda \geqslant 0$, where

$\boldsymbol{u} \in L_2(I; \underline{H}^s)$ such that $\boldsymbol{u}' \in L_2(I; (\underline{H}^s)')$, $\boldsymbol{u}(t_0) = 0$.

Substituting for $\boldsymbol{\alpha}$ dividing by λ we then successively have

$$(\chi(t) - J'_\varepsilon(\boldsymbol{a}_{\varepsilon\eta}(t) - \lambda \boldsymbol{u}(t)), \boldsymbol{u}(t)) + \lambda \int_I \{A(\boldsymbol{u}(t), \boldsymbol{u}(t)) + \eta \, \| \boldsymbol{u}(t) \|_{s,N} + (\boldsymbol{u}'(t), \boldsymbol{u}(t))\} \, \mathrm{d}t \geqslant 0.$$

Hence, in the limit $\lambda \to 0$ we obtain

$$(\chi(t) - J'_\varepsilon(\boldsymbol{a}_{\varepsilon\eta}(t)), \boldsymbol{u}(t)) \geqslant 0 \quad \forall \boldsymbol{u},$$

from which

$$\chi(t) = J'_\varepsilon(\boldsymbol{a}_{\varepsilon\eta}(t)).$$

Therefore, we proved the existence of the vector function $\boldsymbol{a}_{\varepsilon\eta}$ satisfying (4.3) and the conditions

$$\begin{aligned} &\boldsymbol{a}_{\varepsilon\eta}(t) \text{ is bounded in } L_2(I; \underline{H}^1) \cap L_\infty(I; \underline{H}^0), \\ &\boldsymbol{a}'_{\varepsilon\eta}(t) \text{ is bounded in } L_2(I; (\underline{H}^s)'), \\ &\eta^{1/2} \boldsymbol{a}_{\varepsilon\eta}(t) \text{ is bounded in } L_2(I; \underline{H}^s). \end{aligned} \tag{4.24}$$

Now the limitation process $\varepsilon, \eta \to 0$ will be investigated. Let us put for $\boldsymbol{\alpha} \in \underline{H}$ fixed,

$$\begin{aligned} Y_{\varepsilon\eta} = \int_I \, & [(\mathscr{E}\boldsymbol{\alpha}'(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}_{\varepsilon\eta}(t)) + A(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}_{\varepsilon\eta}(t)) \\ & + \Xi(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}_{\varepsilon\eta}(t)) + \eta(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}_{\varepsilon\eta}(t))_s \\ & - (\boldsymbol{P}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}_{\varepsilon\eta}(t))] \, \mathrm{d}t + J_\varepsilon(\boldsymbol{w}) - J_\varepsilon(\boldsymbol{v}_{\varepsilon\eta}). \end{aligned}$$

By virtue of (4.3),

$$Y_{\varepsilon\eta} = \int_I [(\mathscr{E}(\boldsymbol{\alpha}'(t) - \boldsymbol{a}'_{\varepsilon\eta}(t)), \boldsymbol{\alpha} - \boldsymbol{a}_{\varepsilon\eta}(t))] \, \mathrm{d}t + J_\varepsilon(\boldsymbol{w}) - J_\varepsilon(\boldsymbol{v}_{\varepsilon\eta}) - (J'_\varepsilon(\boldsymbol{v}_{\varepsilon\eta}), \boldsymbol{w} - \boldsymbol{v}_{\varepsilon\eta}).$$

According to the initial conditions the first term is equivalent to $\frac{1}{2} |\mathscr{E}(\boldsymbol{\alpha}(t_1) - \boldsymbol{a}_{\varepsilon\eta}(t_1))|^2$ and since the functional $\boldsymbol{w} \to j_\varepsilon(\boldsymbol{w})$ is convex, the second term is $\geqslant 0$. Thus $Y_{\varepsilon\eta} \geqslant 0$, i.e.,

$$\begin{aligned} &\int_I [(\mathscr{E}\boldsymbol{\alpha}'(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}_{\varepsilon\eta}(t)) + A(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t)) - \Xi(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t), \boldsymbol{a}_{\varepsilon\eta}(t)) + \eta(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{\alpha}(t))_s \\ &\quad - (\boldsymbol{P}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}_{\varepsilon\eta}(t))] \, \mathrm{d}t + J_\varepsilon(\boldsymbol{w}) \geqslant \int_I [A(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{a}_{\varepsilon\eta}(t)) + \eta \, \| \boldsymbol{a}_{\varepsilon\eta}(t) \|^2] \, \mathrm{d}t + J_\varepsilon(\boldsymbol{v}_{\varepsilon\eta}(t)) \\ &\quad \geqslant \int_I [A(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{a}_{\varepsilon\eta}(t))] \, \mathrm{d}t + J_\varepsilon(\boldsymbol{v}_{\varepsilon\eta}(t)). \end{aligned} \tag{4.25}$$

Due to (4.24) there exists a subsequence, we denote it also by $\{a_{\varepsilon\eta}\}$ such that

$$\begin{aligned} &\boldsymbol{a}_{\varepsilon\eta}(t) \to \boldsymbol{a}(t) \quad \text{in } L_\infty(I; \underline{H}^0) \ast\text{-weakly and weakly in } L_2(I; \underline{H}^1), \\ &\boldsymbol{a}'_{\varepsilon\eta}(t) \to \boldsymbol{a}'(t) \quad \text{weakly in } L_2(I; (\underline{H}^s)'). \end{aligned} \tag{4.26}$$

From (4.25), (4.26) it follows that

$$\begin{aligned} &\int_I [(\mathscr{E}\boldsymbol{a}'(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}(t)) + A(\boldsymbol{a}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}(t)) - \Xi(\boldsymbol{a}(t), \boldsymbol{\alpha}(t), \boldsymbol{a}(t)) - (\boldsymbol{P}(t), \boldsymbol{\alpha}(t) - \boldsymbol{a}(t))]\, \mathrm{d}t + J(\boldsymbol{w}) \\ &\quad \geqslant \liminf \int_I [A(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{a}_{\varepsilon\eta}(t)) + \bar{g} j_\varepsilon(\boldsymbol{v}_{\varepsilon\eta})]\, \mathrm{d}t \geqslant \int_I [A(\boldsymbol{a}(t), \boldsymbol{a}(t))]\, \mathrm{d}t + J(\boldsymbol{v}(t)) \end{aligned} \tag{4.27}$$

as

$$\liminf \int_I A(\boldsymbol{a}_{\varepsilon\eta}(t), \boldsymbol{a}_{\varepsilon\eta}(t))\, \mathrm{d}t \geqslant \int_I A(\boldsymbol{a}(t), \boldsymbol{a}(t))\, \mathrm{d}t \tag{4.28}$$

and since (see [2])

$$\int_I j(\boldsymbol{v}(t))\, \mathrm{d}t \leqslant \left(\int_{G \times I} D_{\mathrm{II}}(\boldsymbol{v}(t))^{(1+\varepsilon)/2}\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \right)^{1/(1+\varepsilon)} \left(\int_{G \times I} \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \right)^{\varepsilon/(1+\varepsilon)}, \tag{4.29}$$

hence

$$\int_I j_\varepsilon(\boldsymbol{v}_{\varepsilon\eta}(t))\, \mathrm{d}t \geqslant c \left(\int_I j(\boldsymbol{v}_{\varepsilon\eta}(t))\, \mathrm{d}t \right)^{(1+\varepsilon)}, \quad c = c(\varepsilon), \tag{4.30}$$

$$\liminf \int_I j_\varepsilon(\boldsymbol{v}_{\varepsilon\eta}(t))\, \mathrm{d}t \geqslant \liminf \int_I j(\boldsymbol{v}_{\varepsilon\eta}(t))\, \mathrm{d}t. \tag{4.31}$$

Also since the function $\boldsymbol{w} \to \int_I j(\boldsymbol{w}(t))\, \mathrm{d}t$ is convex and continuous on $L_2(I; {}^1H^{1,N}(\Omega))$, then it is l.s.c. in the weak topology of the space $L_2(I; {}^1H^{1,N}(\Omega))$ and thus

$$\liminf \int_I j(\boldsymbol{v}_{\varepsilon\eta}(t))\, \mathrm{d}t \geqslant \int_I j(\boldsymbol{v}(t))\, \mathrm{d}t$$

which together with (4.28) proves

$$\liminf \int_I j_\varepsilon(\boldsymbol{v}_{\varepsilon\eta}(t))\, \mathrm{d}t \geqslant \int_I j(\boldsymbol{v}(t))\, \mathrm{d}t. \tag{4.32}$$

Then (4.26)–(4.31) prove that $(\boldsymbol{v}(t), \boldsymbol{B}(t), T(t))$ satisfies (3.3a).

The existence of $\Phi(t)$ follows from the Lax–Milgram theorem for every $t \in I$. Thus, we proved that $(\boldsymbol{a}(t), \Phi(t))$ is the solution of the problem given.

4.4. Uniqueness of the solution

To prove the uniqueness, we shall assume that $(\boldsymbol{v}(t), \boldsymbol{B}(t), T(t), \Phi(t))$ and $(\boldsymbol{v}_1(t), \boldsymbol{B}_1(t), T_1(t), \Phi_1(t))$ are two solutions of the problem discussed and let $N = 2$. Then $s = 1$ and ${}^1H^{s,N}(\Omega) = {}^1H^{1,N}(\Omega)$,

${}^2H^{s,N}(\Omega) = {}^2H^{1,N}(\Omega)$ and ${}^3H^{s,1}(\Omega) = {}^3H^{1,1}(\Omega)$ and therefore terms of viscosity regularization $\eta(\boldsymbol{v}(t), \boldsymbol{w})_s$, $\eta(\boldsymbol{B}(t), \boldsymbol{C})_s$, $\eta(T(t), z)_s$ are of the same order as $\bar{\mu}a(\boldsymbol{v}(t), \boldsymbol{w})$, $b_{\mathrm{M}}(\boldsymbol{B}(t), \boldsymbol{C})$, $A(T(t), z)$ and therefore can be neglected. Let us denote $\boldsymbol{U}(t) = \boldsymbol{v}(t) - \boldsymbol{v}_1(t)$, $\boldsymbol{B}^*(t) = \boldsymbol{B}(t) - \boldsymbol{B}_1(T)$, $T^*(t) = T(t) - T_1(t)$, $\Phi^*(t) = \Phi(T) - \Phi_1(t)$, then $(\boldsymbol{a}^*(t), \Phi^*(T)) = (\boldsymbol{a}(t) - \boldsymbol{a}_1(t), \Phi(t) - \Phi_1(t))$.

Let us put $\boldsymbol{\alpha} = \boldsymbol{a}(t)$ or $\boldsymbol{\alpha} = \boldsymbol{a}_1(t)$, respectively, and let us put them into (3.3a). Then

$$\int_I \left\{ -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{|\mathscr{E}\boldsymbol{a}^*(t)|^2\} - A(\boldsymbol{a}^*(t), \boldsymbol{a}^*(t)) - \Xi_0(\boldsymbol{a}(t), \boldsymbol{a}(t), \boldsymbol{a}^*(t)) + \Xi_0(\boldsymbol{a}_1(t), \boldsymbol{a}_1(t), \boldsymbol{a}^*(t)) \right.$$
$$\left. - [b_s(T^*(t), \boldsymbol{U}(t)) + b_p(\boldsymbol{U}(t), T^*(t))] \right\} \mathrm{d}t \geqslant 0.$$

Hence,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\mathscr{E}\,|\boldsymbol{a}^*(t)|^2\} + A(\boldsymbol{a}^*(t), \boldsymbol{a}(t)) \leqslant -b(\boldsymbol{v}(t) - \boldsymbol{U}(t), \boldsymbol{v}(t) - \boldsymbol{U}(t), \boldsymbol{U}(t))$$
$$- b(\boldsymbol{v}(t), \boldsymbol{v}(t), \boldsymbol{U}(t)) + F + G, \tag{4.33}$$

where

$$F = (\partial/\partial x_i(\mu^{-1}B_i(t)\boldsymbol{B}(t)), \boldsymbol{U}(t)) - (\partial/\partial x_i(\mu^{-1}B_{1i}(t)\boldsymbol{B}_1(t)), \boldsymbol{U}(t))$$
$$+ (\mu^{-1}\operatorname{Rot}[\boldsymbol{v}(t), \boldsymbol{B}(t)], \boldsymbol{B}^*(t)) - (\mu^{-1}\operatorname{Rot}[\boldsymbol{v}_1(t), \boldsymbol{B}_1(t)], \boldsymbol{B}(t)),$$
$$G = b_0(\boldsymbol{U}(t), T^*(t), T^*(t)) + c(\|\boldsymbol{U}(t)\|_{1,1}\,\|T^*(t)\|_{0,N} + \|\boldsymbol{U}(t)\|_{1,N}\,\|T^*(t)\|_{0,1})$$

and where

$$|b_s(T^*(t), \boldsymbol{U}(t)) + b_p(\boldsymbol{U}(t), T^*(t))| \leqslant c(\|T^*(t)\|_{1,1}\,\|\boldsymbol{U}(t)\|_{0,N} + \|T^*(t)\|_{0,1}\,\|\boldsymbol{U}(t)\|_{1,N}).$$

Furthermore, as above

$$(\partial/\partial x_i(\mu^{-1}B_i(t)\boldsymbol{B}(t)), \boldsymbol{U}(t)) + (\mu^{-1}\operatorname{Rot}[\boldsymbol{U}(t), \boldsymbol{B}(t)], \boldsymbol{B}(t)) = 0,$$

then

$$F = (\partial/\partial x_i(\mu^{-1}(B_i(t)\boldsymbol{B}^*(t) + B_i^*(t)\boldsymbol{B}(t))), \boldsymbol{U}(t))$$
$$+ (\mu^{-1}\operatorname{Rot}[\boldsymbol{v}(t), \boldsymbol{B}(t)], \boldsymbol{B}(t)) + (\mu^{-1}\operatorname{Rot}[\boldsymbol{U}(t), \boldsymbol{B}(t)], \boldsymbol{B}(t))$$
$$= -(\mu^{-1}(B_i(t)\boldsymbol{B}^*(t) + B_i^*(t)\boldsymbol{B}(t)), \partial\boldsymbol{U}(t)/\partial x_i)$$
$$+ (\mu^{-1}([\boldsymbol{v}(t), \boldsymbol{B}(t)] + [\boldsymbol{U}(t), \boldsymbol{B}(t)]), \operatorname{Rot}\boldsymbol{B}^*(t)).$$

Since as above

$$|(\partial/\partial x_i(\mu^{-1}B_i(t)\boldsymbol{B}(t)), \boldsymbol{w})| \leqslant c_0\,\|\boldsymbol{B}(t)\|_{1,N}\,\|\boldsymbol{B}(t)\|_{0,N}\,\|\boldsymbol{w}\|_{s,N}$$
$$\leqslant c_0\,\|\boldsymbol{B}(t)\|_{1,N}\,\|\boldsymbol{w}\|_{s,N},$$

$$|(\mu^{-1}\,\mathrm{Rot}[\boldsymbol{v}(t), \boldsymbol{B}(t)], \boldsymbol{C})| \leqslant c_1 \|\boldsymbol{v}(t)\|_{1,N}^{1/2} \|\boldsymbol{v}(t)\|_{0,N}^{1/2} \|\boldsymbol{B}(t)\|_{1,N}^{1/2} \|\boldsymbol{B}(t)\|_{0,N}^{1/2} \|\boldsymbol{C}\|_{s,N}$$
$$\leqslant c_1 \|\boldsymbol{v}(t)\|_{1,N}^{1/2} \|\boldsymbol{B}(t)\|_{1,N}^{1/2} \|\boldsymbol{C}\|_{s,N},$$
$$|b_s(t; \boldsymbol{w}, z) + b_p(t; \boldsymbol{w}, z)| \leqslant c_s(\|z\|_{1,1} \|\boldsymbol{w}\|_{0,N} + \|z\|_{0,1} \|\boldsymbol{w}\|_{1,N}),$$
$$|b(t, \boldsymbol{w}, \boldsymbol{v}(t), \boldsymbol{w})| \leqslant c_3 \|\boldsymbol{w}\|_{1,N} \|\boldsymbol{v}(t)\|_{0,N} \|\boldsymbol{w}\|_{s,N} \leqslant c_1 \|\boldsymbol{w}\|_{1,N}^2 + c_2 \|\boldsymbol{v}(t)\|_{0,N}^2 \|\boldsymbol{w}\|_{s,N}^2,$$

and therefore

$$|F(t)| \leqslant c_0(\|\boldsymbol{B}(t)\|_{1,N}^{1/2} \|\boldsymbol{B}(t)\|_{0,N}^{1/2} \|\boldsymbol{B}^*(t)\|_{1,N}^{1/2} \|\boldsymbol{B}^*(t)\|_{0,N}^{1/2} \|\boldsymbol{U}(t)\|_{1,N}$$
$$+ \|\boldsymbol{v}(t)\|_{1,N}^{1/2} \|\boldsymbol{v}(t)\|_{0,N}^{1/2} \|\boldsymbol{B}^*(T)\|_{1,N}^{1/2} \|\boldsymbol{B}^*(t)\|_{0,N}^{1/2}$$
$$+ \|\boldsymbol{U}(t)\|_{1,N}^{1/2} \|\boldsymbol{U}(t)\|_{0,N}^{1/2} \|\boldsymbol{B}(t)\|_{1,N}^{1/2} \|\boldsymbol{B}(t)\|_{0,N}^{1/2} \|\boldsymbol{B}^*(t)\|_{1,N})$$
$$\leqslant c_{p1}(\|\boldsymbol{U}(t)\|_{1,N}^2 + \|\boldsymbol{B}(t)\|_{1,N}^2)$$
$$+ K(c_{p1}) \|\boldsymbol{B}(t)\|_{1,N}^2 \|\boldsymbol{B}(t)\|_{0,N}^2 \|\boldsymbol{B}(t)_{0,N}^2 + c_{p2} \|\boldsymbol{B}(t)\|_{1,N}^2$$
$$+ K(c_{p2}) \|\boldsymbol{v}(t)\|_{1,N}^2 \|\boldsymbol{v}(t)\|_{0,N}^2 \|\boldsymbol{B}^*(t)\|_{1,N}^2$$
$$+ c_{p3}(\|\boldsymbol{U}(t)\|_{1,N}^2 + \|\boldsymbol{B}^*(t)\|_{1,N}^2)$$
$$+ K(c_{p3}) \|\boldsymbol{B}(t)\|_{1,N}^2 \|\boldsymbol{B}(t)\|_{0,N}^2 \|\boldsymbol{U}(t)\|_{0,N}^2,$$
$$|b(\boldsymbol{U}(t), \boldsymbol{w}, \boldsymbol{U}(t))| + |F(t)| \leqslant c_p(\|\boldsymbol{U}(t)\|_{1,N}^2 + \|\boldsymbol{B}(t)\|_{1,N}^2)$$
$$+ K(c_p)(\|\boldsymbol{w}\|_{1,N}^2 + \|\boldsymbol{B}(t)\|_{1,N}^2)(\|\boldsymbol{U}(t)\|_{0,N}^2 + \|\boldsymbol{B}(t)\|_{0,N}^2),$$
$$A(\boldsymbol{a}^*(t), \boldsymbol{a}^*(t)) \geqslant c_\alpha \|\boldsymbol{a}^*(t)\|_{1,2N+2}^2.$$

Then (4.33) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\mathscr{E}\boldsymbol{a}^*(t)|^2) \leqslant c_\alpha(\|\boldsymbol{a}^*(t)\|_{1,N2+2}^2) \|\boldsymbol{a}^*(t)\|_{0,2N+2}^2$$
$$= c_\alpha g(t)(\|\boldsymbol{a}^*(t)\|_{0,2N+2}^2),$$

where $g(t)$ is an integrable function in variable t and thus

$$\|\boldsymbol{a}^*(t)\|_{0,2N+1}^2 \leqslant \int_{t_0}^{t} g(\tau)(\|\boldsymbol{a}^*(\tau)\|_{0,2N+1}^2)\,\mathrm{d}\tau.$$

Hence $\boldsymbol{a}(t) = 0$, i.e., $\boldsymbol{U}(t) = 0$, $\boldsymbol{B}^*(t) = 0$, $T^*(t) = 0$.

Let $\Phi(t)$ and $\Phi_1(t)$ from ${}^3H^{1,1}(\Omega) \times I$ be two solutions of (3.3b). Then

$$a_g(\Phi(t), \varphi(t) - \Phi(t)) - (\Psi(t), \varphi(t) - \Phi_1(t)) \geqslant 0, \tag{4.34a}$$
$$a_g(\Phi_1(t), \varphi(t) - \Phi_1(t)) - (\Psi(t), \varphi(t) - \Phi_1(t)) \geqslant 0 \quad \forall \varphi(t) \in {}^3H^{1,1}(\Omega), \quad \forall t \in I. \tag{4.34b}$$

Putting $\varphi = \Phi_1$ in (4.34a) and $\varphi = \Phi$ in (4.34b) and adding them, we then obtain

$$0 \geqslant a_g(\Phi(t) - \Phi_1(t), \Phi(t) - \Phi_1(t)) \geqslant c_\Phi \|\Phi(t) - \Phi_1(t)\|_{1,1}^2,$$

hence $\Phi(t) = \Phi_1(t)$, which completes the proof.

References

[1] N.B. Byhovskii, A solution of a mixed problem for a system of Maxwell's equations in the case of ideally conducting boundary, *Vestnik Leningrad. Univ.* (13) (1957) 50–66 (in Russian).

[2] G. Duvaut and J.L. Lions, *Les Inéquations en Mécanique et en Physique* (Dunod, Paris, 1972).

[3] G. Duvaut and J.L. Lions, Inéquations en thermoélasticité et magnétohydrodynamique, *Arch. Rational Mech. Anal.* **46** (1972) 241–279.

[4] O.A. Ladyženskaja and V.A. Solonnikov, Solutions of some nonstationary problems of magnetohydrodynamics for incompressible fluid, *Trudy Mat. Inst. Steklova* **29** (1960) 115–173 (in Russian).

[5] O.A. Ladyženskaja and V.A. Solonnikov, On the principle of linearization and invariant manifolds in problems of magnetohydrodynamics, *Zapiski Naučnych Seminarov LOMI* **38** (1973) 46–93 (in Russian).

[6] L.D. Landau and E.M. Lifshitz, *Course of Theoretical Physics. Fluid Mechanics* (Pergamon Press, New York, 1959).

[7] J.L. Lions and E. Magenes, *Problèmes aux Limites Non Homogénes et Applications, Vols. 1–3* (Dunod, Paris, 1968, 1970).

[8] A. Milani, On a singular perturbation problem for the linear Maxwell equations, *Quaderni di Matematika* **20** (1980) 11–16.

[9] A. Milani, On a singular perturbation problem for the Maxwell equations in a multiply connected domain, *Rend. Sem. Mat. Univ. Politech. Torino* **38** (1980) 1, 123–132.

[10] J. Nečas, *Les Méthodes en Théorie des Equations Elliptiques* (Academia, Prague, 1967).

[11] J. Nedoma, On the geodynamic model of the Earth and the plate tectonic hypothesis, Parts I–III, *Gerlands Beitr. Geophys.* **95** (1986) 1, 36–62; **95** (1986) 2, 89–105.

[12] J. Nedoma, On the new problem in the potential gravity theory, in: J. Král et al., Eds., *Potential Theory* (Plenum Press, New York and London, 1988) 255–262.

[13] S.M. Nikol'skij, *Približenije Funkcij Mnogich Peremennych i Teoremy Vloženija* (Nauka, Moscow, 1969).

[14] W. Nowacki, *Dinamičeskiji Zadači Termouprugosti* (Mir, Moscow, 1970).

[15] W. Prager, *Vvedenije v Mechaniku Splošnych Sred* (IL, Moscow, 1963).

[16] M. Štědrý and O. Vejvoda, Small time-periodic solutions of equations of magnetohydrodynamics as a singularity perturbed problem, *Apl. Mat.* **28** (1983) 5, 344–356.

[17] M. Štědrý and O. Vejvoda, Equations of magnetohydrodynamics of compressible fluid: Periodic solutions, *Apl. Mat.* **30** (1985) 2, 72–91.

[18] M. Štědrý and O. Vejvoda, Equations of magnetohydrodynamics: Periodic solutions, *Časopis Pěst. Mat.* **111** (1986) 177–184.

[19] L. Stupjalis, On solvability of an initial-boundary value problem of magnetohydrodynamics, *Zapiski Naučnych Seminarov LOMI* **69** (1977) 219–239.

[20] R. Temam, *Navier–Stokes Equations, Theory and Numerical Analysis* (North-Holland, Amsterdam, New York, Oxford, 1979).