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Analytical and numerical studies of the convergence behavior of the $\mathcal{J}$ transformation

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Abstract

A new nonlinear sequence transformation, the iterative $\mathcal{J}$ transformation, was proposed recently by the author (1993). For this transformation, a derivation based on the method of hierarchical consistency, alternative recursive representations, general properties, an explicit expression for the kernel, model sequences, and its relation to other sequence transformations have been given (the author, 1994). The $\mathcal{J}$ transformation is of similar generality as the well-known E algorithm (Brezinski, 1980; Håvie, 1979). In the present contribution, some results on convergence acceleration properties of the $\mathcal{J}$ transformation are proved. Numerical test results are presented which show that the $\mathcal{J}$ transformation is a very powerful computational tool for convergence acceleration, extrapolation, and summation of divergent series.

Keywords: Convergence acceleration; Extrapolation; Summation of divergent series; Hierarchical consistency; Iterative sequence transformations; Levin-type transformations; E algorithm; Linear convergence; Logarithmic convergence; Stieltjes series

AMS classification: 65B05; 65B10

1. Introduction

There are numerous methods known to tackle the problems of convergence acceleration, of extrapolation, and of the summation of divergent series. Good general introductions to these methods have been given in [31, 25, 7].

Many of the methods mentioned above can be formulated as special cases of the E algorithm [14, 5] which is also known as the Brezinski–Håvie protocol [31, Ch. 10]. A good recent introduction to this sequence transformation can be found in [7, Section 2.1]. The kernel or model sequence of the E algorithm is formally rather simple. The E algorithm may be computed

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recursively either using the original approach of Brezinski [7, pp 58f] or, more economically, by a different approach due to Ford and Sidi [12]. Nevertheless, both approaches are computationally rather demanding.

Iterative algorithms are known which are computationally much more simple than the E algorithm. Examples are the iterated Aitken process [1] or the closely related methods studied in [26]. But quite often, they are not as general, their kernels are not known explicitly, and analytical convergence results for them are difficult to obtain.

Recently, the $\mathcal{J}$ transformation was introduced by the author [16], and its properties were studied extensively [17]. It combines many of the advantages of the E algorithm with those of the iterative methods as sketched below.

The $\mathcal{J}$ transformation can be derived by a hierarchically consistent [17] iteration of some very simple basic sequence transformation. As Levin-type methods, the $\mathcal{J}$ transformation depends on a sequence of suitable remainder estimates $\{\omega_n\}$. The important point, however, is that, in the kth step of the iteration of the basic sequence transformation mentioned above, new remainder estimates $\omega_n^{(k)}$ are computed from the remainder estimates $\omega_n^{(k-1)}$ used in the previous step, and from some auxiliary quantities $r_n^{(k-1)}$ which are related to the hierarchy of model sequences. For details, see [17]. Here, we want to stress that the flexibility of Levin-type methods — which is based upon the possibility to choose problem-adapted remainder estimates — is inherited by the $\mathcal{J}$ transformation. This method is even more flexible due to the additional freedom to choose suitable auxiliary quantities $r_n^{(k)}$. The study of these additional possibilities has just started and is very promising. It might be argued that each special case of the $\mathcal{J}$ transformation obtained by a choice of the $r_n^{(k)}$ represents an essentially new transformation, with a different range of applicability, and with different numerical properties. Viewed in this way, the $\mathcal{J}$ transformation comprises a very large class of algorithms. Consequently, the generality of the $\mathcal{J}$ transformation is comparable to that of the E algorithm.

On the other hand, all the known algorithms to compute the $\mathcal{J}$ transformation are structurally remarkably simple [17]. This fact allowed to derive an explicit, comparatively simple formula for its kernel. This formula, for instance, is valid for all the special cases mentioned above. Thus, one can hope to derive general results simultaneously for the whole class of algorithms which corresponds to special cases of the $\mathcal{J}$ transformation.

In this article, additional general results on the $\mathcal{J}$ transformation are presented. We derive analytical results regarding the convergence acceleration of linearly and logarithmically convergent and/or Stieltjes series and the summation of divergent series. Further, we will present a large number of numerical tests which show that there are variants of the $\mathcal{J}$ transformation which are among the best convergence accelerators currently known.

2. Definitions and basic relations

Generally, we use the conventions and notations in [17]. Especially, we note that the difference operator Δ defined by

$$\Delta f(n) = f(n+1) - f(n), \qquad \Delta g_n = g_{n+1} - g_n, \tag{1}$$

acts only on the index n when applied to multiply indexed quantities. The notation

$$\sum_{n>n_l>n_{l+1}>\cdots>n_{l+k-1}} = \sum_{n_l=0}^{n-1} \sum_{n_{l+1}=0}^{n_l-1} \cdots \sum_{n_{l+k-1}=0}^{n_{l+k-2}-1} \tag{2}$$

for positive k and l is used. Empty sums are assumed to be zero.

Double factorials are defined by $(-1)!! = 1, 0!! = 1, (n+1)!! = (n+1)\cdot(n-1)!!$. $\Gamma(z)$ denotes the gamma function. Pochhammer symbols $(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)$ are used for $a > 0$ and $n \in \mathbb{N}$.

Consider a sequence $\{s_n\}_{n=0}^{\infty}$ of complex numbers. If it converges, we call its limit s. Sequences satisfying

$$\lim_{n\to\infty} \frac{s_{n+1}-s}{s_n - s} = \rho \tag{3}$$

are called *linearly convergent* if $0 < |\rho| < 1$ holds. They are called *logarithmically convergent* if $\rho = 1$ holds, and they are called *hyperlinearly convergent* if $\rho = 0$ holds. For $|\rho| > 1$, the sequence diverges. In this case, s is called the antilimit of the sequence $\{s_n\}$. These definitions follow Wimp [31, p.6] and Weniger [25, p.204].

In the following, we write

$$s_n = s + R_n, \tag{4}$$

where R_n is called the remainder. When we use remainder estimates, ω_n, where ω_n does not equal 0, then they should satisfy a relation of the form

$$\lim_{n\to\infty} \frac{R_n}{\omega_n} = c, \tag{5}$$

where c is a constant with $0 < |c| < \infty$.

However, in practice one is forced to use remainder estimates where the validity of Eq. (5) cannot be guaranteed. According to Levin [18] and Smith and Ford [23] one may choose

$$\omega_n = \Delta s_{n-1}, \tag{6a}$$

$$\omega_n = \Delta s_n, \tag{6b}$$

$$\omega_n = (n+\beta)\Delta s_{n-1}, \tag{6c}$$

$$\omega_n = \frac{-\Delta s_n \Delta s_{n-1}}{\Delta^2 s_{n-1}}. \tag{6d}$$

These choices of ω_n lead to corresponding variants for any sequence transformation which is based on remainder estimates ω_n. These will be called the t variant in the case of Eq. (6a), the $\tilde{t}$ variant in the case of Eq. (6b), the u variant in the case of Eq. (6c), and the v variant in the case of Eq. (6d).

We describe briefly some properties of Stieltjes series. A Stieltjes series is a formal expansion of the form

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \mu_n z^n. \tag{7}$$

Here, the coefficient μ_n are the moments of a unique positive measure $\psi(t)$ on $t \in [0, \infty)$:

$$\mu_n = \int_0^\infty t^n \, \mathrm{d}\psi(t), \quad n \in \mathbb{N}_0. \tag{8}$$

Formally, the Stieltjes series can be identified with a Stieltjes integral of the form

$$f(z) = \int_0^\infty \frac{\mathrm{d}\psi(t)}{1+zt}, \quad |\arg(z)| < \pi. \tag{9}$$

If such a Stieltjes integral exists for a function f, then such a function is called a Stieltjes function. For every Stieltjes function there is an unique asymptotic Stieltjes series (7), uniformly in every sector $|\arg(z)| < \vartheta$ for all $\vartheta < \pi$. To every Stieltjes series, however, there can exist several different associated Stieltjes functions. Additional criteria are necessary in order to ensure uniqueness. In the context of convergence acceleration and summation of divergent sequences, an important fact is that for fixed z the remainders of Stieltjes series are bounded in magnitude by the first term of the series which is not included in the partial summation. Consequently, a suitable remainder estimate for a Stieltjes series is

$$\omega_n = (-1)^{n+1} \mu_{n+1} z^{n+1}. \tag{10}$$

This corresponds to the choice $\omega_n = \Delta s_n$, i.e., the choice of $\tilde{t}$ variants.

The $\mathcal{J}$ transformation is defined by the recursive scheme [16]

$$s_n^{(0)} = s_n, \qquad \omega_n^{(0)} = \omega_n, \tag{11a}$$

$$s_n^{(k+1)} = s_n^{(k)} - \omega_n^{(k)} \frac{\Delta s_n^{(k)}}{\Delta \omega_n^{(k)}}, \tag{11b}$$

$$\omega_n^{(k+1)} = -\frac{\omega_n^{(k)} \omega_{n+1}^{(k)}}{\Delta \omega_n^{(k)}} \delta_n^{(k)}, \tag{11c}$$

$$\mathcal{J}_n^{(k)}(\{s_n\}, \{\omega_n\}, \{r_n^{(k)}\}) = s_n^{(k)}, \tag{11d}$$

with

$$\delta_n^{(k)} = \Delta r_n^{(k)} \neq 0. \tag{12}$$

Alternatively, one may use the algorithm [17]

$$\hat{D}_n^{(0)} = \frac{1}{\omega_n}, \qquad \hat{N}_n^{(0)} = \frac{s_n}{\omega_n}, \tag{13a}$$

$$\hat{D}_n^{(k+1)} = \Phi_n^{(k)} \hat{D}_{n+1}^{(k)} - \hat{D}_n^{(k)}, \quad k \in \mathbb{N}_0, \tag{13b}$$

$$\hat{N}_n^{(k+1)} = \Phi_n^{(k)} \hat{N}_{n+1}^{(k)} - \hat{N}_n^{(k)}, \quad k \in \mathbb{N}_0, \tag{13c}$$

$$\mathcal{J}_n^{(k)}(\{s_n\}, \{\omega_n\}, \{r_n^{(k)}\}) = \frac{\hat{N}_n^{(k)}}{\hat{D}_n^{(k)}}. \tag{13d}$$

Here, we use the definitions

$$\Phi_n^{(0)} = 1, \qquad \Phi_n^{(k)} = \frac{\delta_n^{(0)} \delta_n^{(1)} \cdots \delta_n^{(k-1)}}{\delta_{n+1}^{(0)} \delta_{n+1}^{(1)} \cdots \delta_{n+1}^{(k-1)}}, \quad k \in \mathbb{N}. \tag{14}$$

More explicitly, the $\mathcal{J}$ transformation can be computed from the following data:

$$(\{s_{n+j}\}_{j=0}^{k}, \{\omega_{n+j}\}_{j=0}^{k}, \{\{\delta_{n+j}^{(l)}\}_{j=0}^{l+1}\}_{l=0}^{k-2}) \to \mathcal{J}_n^{(k)}(\{s_n\}, \{\omega_n\}, \{r_n^{(k)}\}). \tag{15}$$

For convenience, the explicit ranges are suppressed in the notation.

Let

$$\nabla_n^{(k)} = \frac{1}{\delta_n^{(k)}} \Delta \tag{16}$$

be an operator acting on n dependent sequence elements $f(n)$. This operator can obviously be regarded as a generalization of the difference operator Δ. Using this kind of operator, one may write

$$N_n^{(k)} = \frac{s_n^{(k)}}{\omega_n^{(k)}} = \nabla_n^{(k-1)} \nabla_n^{(k-2)} \cdots \nabla_n^{(0)} \left[\frac{s_n}{\omega_n} \right], \tag{17a}$$

$$D_n^{(k)} = \frac{1}{\omega_n^{(k)}} = \nabla_n^{(k-1)} \nabla_n^{(k-2)} \cdots \nabla_n^{(0)} \left[\frac{1}{\omega_n} \right], \tag{17b}$$

$$\mathcal{J}_n^{(k)}(\{s_n\}, \{\omega_n\}, \{r_n^{(k)}\}) = \frac{\nabla_n^{(k-1)} \nabla_n^{(k-2)} \cdots \nabla_n^{(0)} [s_n/\omega_n]}{\nabla_n^{(k-1)} \nabla_n^{(k-2)} \cdots \nabla_n^{(0)} [1/\omega_n]}. \tag{17c}$$

We denote the t, the $\tilde{t}$, and the u variants of the $\mathcal{J}$ transformation as

$$\mathcal{T}_n^{(k)}(\{s_n\}, \{r_n^{(k)}\}) = \mathcal{J}_n^{(k)}(\{s_n\}, \{\Delta s_{n-1}\}, \{r_n^{(k)}\}), \tag{18}$$

$$\tilde{\mathcal{T}}_n^{(k)}(\{s_n\}, \{r_n^{(k)}\}) = \mathcal{J}_n^{(k)}(\{s_n\}, \{\Delta s_n\}, \{r_n^{(k)}\}), \tag{19}$$

$$\mathcal{U}_n^{(k)}(\alpha, \{s_n\}, \{r_n^{(k)}\}) = \mathcal{J}_n^{(k)}(\{s_n\}, \{(n+\alpha)\Delta s_{n-1}\}, \{r_n^{(k)}\}). \tag{20}$$

An important special case is the ${}_pJ$ transformation which is defined by the equation

$${}_pJ_n^{(k)}(\beta, \{s_n\}, \{\omega_n\}) = \mathcal{J}_n^{(k)}(\{s_n\}, \{\omega_n\}, \{1/(n+\beta+(p-1)k)\}), \quad \beta > 0, p \geqslant 1. \tag{21}$$

The t, the $\tilde{t}$, and the u variants of the ${}_p\boldsymbol{J}$ transformation are defined as

$$ {}_p\boldsymbol{T}_n^{(k)}(\beta, \{s_n\}) = {}_p\boldsymbol{J}_n^{(k)}(\beta, \{s_n\}, \{\Delta s_{n-1}\}), \tag{22}$$

$$ {}_p\tilde{\boldsymbol{T}}_n^{(k)}(\beta, \{s_n\}) = {}_p\boldsymbol{J}_n^{(k)}(\beta, \{s_n\}, \{\Delta s_n\}), \tag{23}$$

$$ {}_p\boldsymbol{U}_n^{(k)}(\alpha, \beta, \{s_n\}) = {}_p\boldsymbol{J}_n^{(k)}(\beta, \{s_n\}, \{(n+\alpha)\Delta s_{n-1}\}). \tag{24}$$

The following theorem was proved in [17]:

Theorem 1.

(J0) *For given* $\{r_n^{(k)}\}$ *the transformation* $\mathscr{J}_n^{(k)}(\{s_n\},\{\omega_n\},\{r_n^{(k)}\})$ *is a continuous mapping* $\mathscr{J}_n^{(k)} = \Gamma_n^{(k)}(s_n, s_{n+1}, \dots, s_{n+k} \,|\, \omega_n, \omega_{n+1}, \dots, \omega_{n+k})$ *on* $\mathbb{C}^{k+1} \times \boldsymbol{Y}_n^{(k)}$ *where* $\boldsymbol{Y}_n^{(k)}$ *is given by*

$$ \boldsymbol{Y}_n^{(k)} = \left\{ \boldsymbol{y} \in \mathbb{C}^{k+1} \,\middle|\, \prod_{j=1}^{k+1} y_j \neq 0 \text{ and } D_n^{(k)}\Big|_{(\omega_n,\dots,\omega_{n+k})=(y_1,\dots,y_{k+1})} \neq 0 \text{ holds} \right\}. \tag{25}$$

(J1) $\Gamma_n^{(k)}$ *is a homogeneous function of degree one in its first* $(k+1)$ *variables and a homogeneous function of degree zero in its last* $(k+1)$ *variables. Thus, for all vectors* $\boldsymbol{x} \in \mathbb{C}^{k+1}$ *and* $\boldsymbol{y} \in \mathrm{Y}_n^{(k)}$ *and for all complex constants* λ *and* $\mu \neq 0$ *we have*

$$ \Gamma_n^{(k)}(\lambda \boldsymbol{x} \,|\, \boldsymbol{y}) = \lambda \Gamma_n^{(k)}(\boldsymbol{x} \,|\, \boldsymbol{y}), \tag{26}$$

$$ \Gamma_n^{(k)}(\boldsymbol{x} \,|\, \mu \boldsymbol{y}) = \Gamma_n^{(k)}(\boldsymbol{x} \,|\, \boldsymbol{y}). \tag{27}$$

(J2) $\Gamma_n^{(k)}$ *is linear in its first* $(k+1)$ *variables. Consequently, for all vectors* $\boldsymbol{x}$ *and* $\boldsymbol{x}'$ *in* $\mathbb{C}^{k+1}$, *and* $\boldsymbol{y} \in \mathrm{Y}_n^{(k)}$, *we have*

$$ \Gamma_n^{(k)}(\boldsymbol{x} + \boldsymbol{x}' \,|\, \boldsymbol{y}) = \Gamma_n^{(k)}(\boldsymbol{x} \,|\, \boldsymbol{y}) + \Gamma_n^{(k)}(\boldsymbol{x}' \,|\, \boldsymbol{y}). \tag{28}$$

(J3) *For all constant vectors* $\boldsymbol{c} = (c, c, \dots, c) \in \mathbb{C}^{k+1}$ *and all vectors* $\boldsymbol{y} \in \mathrm{Y}_n^{(k)}$ *we have*

$$ \Gamma_n^{(k)}(\boldsymbol{c} \,|\, \boldsymbol{y}) = c. \tag{29}$$

(J4) *If we assume that for all* $k \in \mathbb{N}$ *the limits*

$$ \lim_{n\to\infty} \Phi_n^{(k)} = \lim_{n\to\infty} \frac{\delta_n^{(0)} \delta_n^{(1)} \cdots \delta_n^{(k-1)}}{\delta_{n+1}^{(0)} \delta_{n+1}^{(1)} \cdots \delta_{n+1}^{(k-1)}} = \Phi_k \tag{30}$$

exist, then a limiting transformation $\mathring{\mathscr{J}}$ *can be defined by the equations*

$$ \mathring{D}_n^{(0)} = \frac{1}{\omega_n}, \qquad \mathring{N}_n^{(0)} = \frac{s_n}{\omega_n}, \tag{31a}$$

$$ \mathring{D}_n^{(k+1)} = \Phi_k \mathring{D}_{n+1}^{(k)} - \mathring{D}_n^{(k)}, \quad k \in \mathbb{N}_0, \tag{31b}$$

$$ \mathring{N}_n^{(k+1)} = \Phi_k \mathring{N}_{n+1}^{(k)} - \mathring{N}_n^{(k)}, \quad k \in \mathbb{N}_0, \tag{31c}$$

$$ \mathring{\mathscr{J}}_n^{(k)}(\{s_n\},\{\omega_n\},\{r_n^{(k)}\}) = \frac{\mathring{N}_n^{(k)}}{\mathring{D}_n^{(k)}}. \tag{31d}$$

For given $\{\Phi_k\}$ the transformation $\mathring{\mathcal{J}}_k(\{s_n\},\{\omega_n\},\{\mathring{r}_n^{(k)}\})$ is a continuous mapping $\mathring{\mathcal{J}}_k = \mathring{\Gamma}_k(s_n, s_{n+1},\ldots, s_{n+k}\,|\,\omega_n,\omega_{n+1},\ldots,\omega_{n+k})$ on $\mathbb{C}^{k+1}\times\mathring{Y}_k$ with

$$\mathring{Y}_k = \left\{ y\in\mathbb{C}^{k+1} \,\middle|\, \prod_{j=1}^{k+1} y_j \neq 0 \text{ and } \mathring{D}_n^{(k)}\Big|_{(\omega_n,\ldots,\omega_{n+k})=(y_1,\ldots,y_{k+1})} \neq 0 \text{ holds}\right\}. \tag{32}$$

Moreover, the limiting transformation $\mathring{\Gamma}_k$ is also homogeneous and linear according to (J1) *and* (J2). *It is also exact on constant vectors according to Eq.* (29).

In the following lemma proved in [17], a property of the limiting transformation $\mathring{\Gamma}_k$ is established which is important for the treatment of linear convergence.

Lemma. *The limiting transformation $\mathring{\Gamma}_k$ satisfies*

$$\mathring{\Gamma}_k\left(0, a, \ldots, a\sum_{j=0}^{k-1} q^j \,\middle|\, 1, q, \ldots, q^k\right) = \frac{a}{1-q} \tag{33}$$

for $q \neq 1$ and $(1, q, \ldots, q^k)\in\mathring{Y}_k$.

3. Analysis of convergence properties

In this section, we give some analytical results regarding convergence acceleration using the $\mathcal{J}$ transformation. If not otherwise stated, fixed but arbitrary $\delta_n^{(k)}$ are assumed. First, two results are stated concerning the exactness of the $\mathcal{J}$ transformation. These two theorems were proved in [17].

Theorem 2. *The kernel of the $\mathcal{J}_n^{(k)}$ transformation is given by the sequences of the form*

$$\begin{aligned} s_n = s + \omega_n\Bigg(& c_0 + c_1\sum_{n_1=0}^{n-1}\delta_{n_1}^{(0)} + c_2\sum_{n_1=0}^{n-1}\delta_{n_1}^{(0)}\sum_{n_2=0}^{n_1-1}\delta_{n_2}^{(1)} + \cdots \\ & + c_{k-1}\sum_{n>n_1>n_2>\cdots>n_{k-1}} \delta_{n_1}^{(0)}\delta_{n_2}^{(1)}\cdots\delta_{n_{k-1}}^{(k-2)}\Bigg) \end{aligned} \tag{34}$$

with constants $c_0,\ldots,c_{k-1}$.

Theorem 3. *The $\mathcal{J}$ transformation is exact for the geometric series with partial sums $\{s_n\}$ if the sequence $\{\omega_n\}$ is chosen in such a way that $s_n = s + c\omega_n$ holds with $c \neq 0$. This is satisfied for $\omega_n = \Delta s_{n-1}$, for $\omega_n = \Delta s_n$, or for $\omega_n = -\Delta s_n \Delta s_{n-1}/\Delta^2 s_{n-1}$. This implies that the t, the $\tilde{t}$, and the v variants of the $\mathcal{J}$ transformation are exact for the geometric series.*

The last theorem holds for any choice of the $r_n^{(k)}$ satisfying Eq. (12) in view of Theorem 2, Eq. (34). The following two theorems are stating conditions for acceleration of linear convergence.

Theorem 4. *Assume that the sequences $\{s_n\}$ and $\{\omega_n\}$ satisfy*

$$\lim_{n\to\infty} s_n = s, \tag{35}$$

$$\lim_{n\to\infty} \frac{s_n - s}{\omega_n} = c, \quad c \neq 0, \tag{36}$$

$$\lim_{n\to\infty} \frac{\omega_{n+1}}{\omega_n} = \rho, \quad 0 < |\rho| < 1. \tag{37}$$

If the limits Φ_j defined in Eq. (30) exist for all $j \leqslant k \in \mathbb{N}$ and if $(1, \rho, \rho^2, \ldots, \rho^k) \in \mathring{Y}_k$ where $\mathring{Y}_k$ is defined in Eq. (32), then the $\mathcal{J}_n^{(k)}$ transformation accelerates the convergence of the sequence $\{s_n\}$. The condition $(1, \rho, \rho^2, \ldots, \rho^k) \in \mathring{Y}_k$ can be replaced by demanding that ρ is different from all the Φ_j for $0 \leqslant j \leqslant k$.

Proof. This follows from Theorem 1 and the Lemma in combination with Theorems 12–14 in [25, p.308]. The equivalence of the two conditions stated in the last sentence of the theorem follows from [17, Lemma 2]. □

A simple corollary is the following theorem.

Theorem 5. *Assume that the sequence $\{s_n\}$ satisfies*

$$\lim_{n\to\infty} s_n = s, \tag{38}$$

$$\lim_{n\to\infty} \frac{s_{n+1} - s}{s_n - s} = \rho, \quad 0 < |\rho| < 1. \tag{39}$$

If the limits Φ_j defined in Eq. (30) exist for all $j \leqslant k \in \mathbb{N}$ and if $(1, \rho, \rho^2, \ldots, \rho^k) \in \mathring{Y}_k$ where $\mathring{Y}_k$ is defined in Eq. (32), then the t variant $\mathcal{T}_n^{(k)}(\{s_n\}, \{r_n^{(k)}\})$, Eq. (18), and the $\tilde{t}$ variant $\tilde{\mathcal{T}}_n^{(k)}(\{s_n\}, \{r_n^{(k)}\})$, Eq. (19), of the $\mathcal{J}_n^{(k)}$ transformation accelerate the convergence of the linearly convergent sequence s_n. The condition $(1, \rho, \rho^2, \ldots, \rho^k) \in \mathring{Y}_k$ can be replaced by demanding that ρ is different from all Φ_j for $0 \leqslant j \leqslant k$.

Proof. In view of Theorem 4, one only has to show that the following holds:

$$\lim_{n\to\infty} \frac{s_n - s}{\omega_n} = c, \quad c \neq 0, \tag{40}$$

$$\lim_{n\to\infty} \frac{\omega_{n+1}}{\omega_n} = \rho, \quad 0 < |\rho| < 1. \tag{41}$$

But the second equation follows directly from [31, Theorem 1, p.6]. The first is a consequence of [31, p.6, Eq. (4)]:

$$\Delta s_n \sim (\rho - 1)(s_n - s), \quad n \to \infty,$$

which implies $c = \rho/(\rho - 1)$ for the case of the t variant, and $c = 1/(\rho - 1)$ for the case of the $\tilde{t}$ variant. □

We remark that in the preceding theorem the condition that ρ is different from all Φ_j may be dropped in the case of the t and $\tilde{t}$ variants of the ${}_p\boldsymbol{J}$ transformation because then $\Phi_j = 1$ holds for all j.

The following theorem gives a convergence result for the case of alternating signs of the ω_n, and monotone signs of the $\delta_n^{(k)}$. These assumptions are, for instance, satisfied in the case of the application of the ${}_p\boldsymbol{T}$ transformation which is defined in Eq. (22), to the partial sums

$$s_n = \sum_{j=0}^{n} (-1)^j a_j, \quad a_j > 0, \tag{42}$$

of an alternating series. Important examples of such series are Stieltjes series as discussed in the previous section.

Theorem 6. *Assume that the following holds:*

(A0) *The sequence $\{s_n\}$ has the (anti)limit s.*

(A1a) *For every n, the elements of the sequence $\{\omega_n\}$ are strictly alternating in sign and do not vanish.*

(A1b) *For all n and k, the elements of the sequence $\{\delta_n^{(k)}\} = \{\Delta r_n^{(k)}\}$ are of the same sign and do not vanish.*

(A2) *For all $n \in \mathbb{N}_0$ the ratio $(s_n - s)/\omega_n$ can be expressed as a series of the form*

$$\frac{s_n - s}{\omega_n} = c_0 + \sum_{j=1}^{\infty} c_j \sum_{n > n_1 > n_2 > \cdots > n_j} \delta_{n_1}^{(0)} \delta_{n_2}^{(1)} \cdots \delta_{n_j}^{(j-1)} \tag{43}$$

with $c_0 \neq 0$.

Then the following holds for $s_n^{(k)} = \mathscr{J}_n^{(k)}(\{s_n\}, \{\omega_n\}, \{r_n^{(k)}\})$:

(a) *The error $s_n^{(k)} - s$ satisfies*

$$s_n^{(k)} - s = \frac{b_n^{(k)}}{\nabla_n^{(k-1)} \nabla_n^{(k-2)} \cdots \nabla_n^{(0)} [1/\omega_n]} \tag{44}$$

with

$$b_n^{(k)} = c_k + \sum_{j=k+1}^{\infty} c_j \sum_{n > n_{k+1} > n_{k+2} > \cdots > n_j} \delta_{n_{k+1}}^{(k)} \delta_{n_{k+2}}^{(k+1)} \cdots \delta_{n_j}^{(j-1)}. \tag{45}$$

(b) *The error* $s_n^{(k)} - s$ *is bounded in magnitude according to*

$$|s_n^{(k)} - s| \leqslant |\omega_n b_n^{(k)} \delta_n^{(0)} \delta_n^{(1)} \cdots \delta_n^{(k-1)}|. \tag{46}$$

(c) *For large n the estimate*

$$\frac{s_n^{(k)} - s}{s_n - s} = \mathrm{O}(\delta_n^{(0)} \delta_n^{(1)} \cdots \delta_n^{(k-1)}) \tag{47}$$

holds if $b_n^{(k)} = \mathrm{O}(1)$ *and* $(s_n - s)/\omega_n = \mathrm{O}(1)$ *as* $n \to \infty$.

Proof. (a) This follows from [17, Lemma 4 and Eq. (17b)].

(b) This follows from Eq. (44) and the observation that assumptions (A1a) and (A1b) imply that all terms obtained by expanding

$$\sigma = \nabla_n^{(k-1)} \nabla_n^{(k-2)} \cdots \nabla_n^{(0)} \left[\frac{1}{\omega_n} \right] \tag{48}$$

have the same sign. Hence, the absolute value of σ is greater than or equal to that of any term. Taking the term involving ω_n, it follows that

$$|\sigma| \geqslant \left| \frac{1}{\omega_n \delta_n^{(0)} \delta_n^{(1)} \cdots \delta_n^{(k-1)}} \right|. \tag{49}$$

(c) This follows directly from (b). □

This theorem allows to estimate the error in the case of alternating series. The rest of this section is devoted to generalizing Theorem 6. The basis for the following discussion is the equation

$$\frac{s_n^{(k)} - s}{s_n - s} = \frac{\prod_{l=0}^{k-1} \delta_n^{(l)}}{\prod_{l=0}^{k-1} [\omega_n^{(l)}/\omega_{n+1}^{(l)} - 1]} \frac{b_n^{(k)}}{b_n^{(0)}} = \left\{ \prod_{l=0}^{k-1} \delta_n^{(l)} \right\} \left\{ \prod_{l=0}^{k-1} \left[\frac{\omega_{n+1}^{(l)}/\omega_n^{(l)}}{e_n^{(l)}} \right] \right\} \left\{ \frac{b_n^{(k)}}{b_n^{(0)}} \right\} \tag{50}$$

which is proved in Lemma A.1 in the Appendix. Here, the abbreviations $e_n^{(k)} = 1 - \omega_{n+1}^{(k)}/\omega_n^{(k)}$ and $b_n^{(k)} = (s_n^{(k)} - s)/\omega_n^{(k)}$ are used. Eq. (50) is the important formula that allows to estimate the accelerative power of the $\mathcal{J}$ transformation of large n. In Lemma A.2 which is given in the appendix, it is studied which quantities occurring in this formula have limits for $n \to \infty$.

Assume that (A2) of Theorem 6 holds. It may be noted that the formula (34) for the kernel can be rewritten as a partial sum of the infinite series in (A2). Then, Lemma 4 of [17] shows that

$$b_n^{(k)} = c_k + \sum_{j=k+1}^{\infty} c_j \sum_{n > n_{k+1} > n_{k+2} > \cdots > n_j} \delta_{n_{k+1}}^{(k)} \delta_{n_{k+2}}^{(k+1)} \cdots \delta_{n_j}^{(j-1)} \tag{51}$$

holds. Hence, assumption (52) in the following theorem is reasonable.

Theorem 7. *Assume that* (A0) *of Theorem* 6 *holds and that the following conditions are satisfied:*
(B1) *Assume that*

$$\lim_{n\to\infty} \frac{b_n^{(k)}}{b_n^{(0)}} = B_k \tag{52}$$

exists and is finite.

(B2) *Assume that*

$$\Omega_k = \lim_{n\to\infty} \frac{\omega_{n+1}^{(k)}}{\omega_n^{(k)}} \neq 0 \tag{53}$$

and

$$F_k = \lim_{n\to\infty} \frac{\delta_{n+1}^{(k)}}{\delta_n^{(k)}} \neq 0 \tag{54}$$

exist for all $k \in \mathbb{N}_0$.
Then, the following holds for $s_n^{(k)} = \mathscr{J}_n^{(k)}(\{s_n\}, \{\omega_n\}, \{r_n^{(k)}\})$:
(a) *If* $\Omega_0 \notin \{\Phi_0 = 1, \Phi_1, \ldots, \Phi_{k-1}\}$, *then*

$$\lim_{n\to\infty} \frac{s_n^{(k)} - s}{s_n - s} \left\{ \prod_{l=0}^{k-1} \delta_n^{(l)} \right\}^{-1} = B_k \frac{[\Omega_0]^k}{\prod_{l=0}^{k-1} (\Phi_l - \Omega_0)} \tag{55}$$

and, hence,

$$\frac{s_n^{(k)} - s}{s_n - s} = \mathrm{O}(\delta_n^{(0)} \delta_n^{(1)} \cdots \delta_n^{(k-1)}) \tag{56}$$

holds in the limit $n \to \infty$.
(b) *If* $\Omega_l = 1$ *for* $l \in \{0, 1, 2, \ldots, k\}$ *then*

$$\lim_{n\to\infty} \frac{s_n^{(k)} - s}{s_n - s} \left\{ \prod_{l=0}^{k-1} \frac{\delta_n^{(l)}}{e_n^{(l)}} \right\}^{-1} = B_k \tag{57}$$

and, hence,

$$\frac{s_n^{(k)} - s}{s_n - s} = \mathrm{O}\left(\prod_{l=0}^{k-1} \frac{\delta_n^{(l)}}{e_n^{(l)}} \right) \tag{58}$$

holds in the limit $n \to \infty$.

Proof. The assertions follow from Eq. (50). Additionally, one has to use Lemma A.2. The proof is completed by applying Eq. (A.8b) in the case of item (a), and Eq. (A.10) in the case of item (b). □

This theorem is the central result of this section. It holds for fixed but arbitrary sequences $\delta_n^{(k)}$ that satisfy the assumptions.

Theorem 7 can be specialized to the important case of the ${}_p\boldsymbol{J}$ transformation. This yields the following two corollaries.

Theorem 8. *Assume that the following holds:*

(C1) *Let* $\beta > 0$, $p \geqslant 1$ *and* $\delta_n^{(k)} = \Delta[(n+\beta+(p-1)k)^{-1}]$. *Thus, we deal with the* ${}_p\boldsymbol{J}$ *transformation and, hence, the equations* $F_k = \lim_{n\to\infty} \delta_{n+1}^{(k)}/\delta_n^{(k)} = 1$ *and* $\Phi_k = 1$ *hold for all* k *(compare Eq. (30)).*

(C2) *Assumptions* (A2) *of Theorem* 6 *and* (B1) *of Theorem* 7 *are satisfied for the particular choice* (C1) *for* $\delta_n^{(k)}$.

(C3) *The limit* $\Omega_0 = \lim_{n\to\infty} \omega_{n+1}/\omega_n$ *exists, and it satisfies* $\Omega_0 \notin \{0,1\}$. *Thus, according to Lemma A.2, Eq.* (A.8b), *all the limits* Ω_k *for* $k \in \mathbb{N}$ *exist and satisfy* $\Omega_k = \Omega_0$.

Then the transformation $s_n^{(k)} = {}_p\boldsymbol{J}_n^{(k)}(\beta, \{s_n\}, \{\omega_n\})$ *satisfies*

$$\lim_{n\to\infty} \frac{s_n^{(k)} - s}{s_n - s} \left\{ \prod_{l=0}^{k-1} \delta_n^{(l)} \right\}^{-1} = B_k \left\{ \frac{\Omega_0}{1-\Omega_0} \right\}^k \tag{59}$$

and, hence,

$$\frac{s_n^{(k)} - s}{s_n - s} = \mathrm{O}((n+\beta)^{-2k}) \tag{60}$$

holds in the limit $n \to \infty$.

Theorem 8 can be applied in the case of linear convergence because then $0 < |\Omega_0| < 1$ holds as shown in the proof of Theorem 5.

Theorem 9. *Assume that the following holds:*

(D1) *Let* $\beta > 0$, $p \geqslant 1$ *and* $\delta_n^{(k)} = \Delta[(n+\beta+(p-1)k)^{-1}]$. *Thus, we deal with the* ${}_p\boldsymbol{J}$ *transformation and, hence, the equations* $F_k = \lim_{n\to\infty} \delta_{n+1}^{(k)}/\delta_n^{(k)} = 1$ *and* $\Phi_k = 1$ *hold for all* k *(compare Eq. (30)).*

(D2) *Assumptions* (A2) *of Theorem* 6 *and* (B1) *of Theorem* 7 *are satisfied for the particular choice* (D1) *for* $\delta_n^{(k)}$.

(D3) *Some constants* $a_l^{(j)}$, $j = 1, 2$, *exist such that*

$$e_n^{(l)} = 1 - \frac{\omega_{n+1}^{(l)}}{\omega_n^{(l)}} = \frac{a_l^{(1)}}{n+\beta} + \frac{a_l^{(2)}}{(n+\beta)^2} + \mathrm{O}((n+\beta)^{-3}) \tag{61}$$

holds for $l = 0$. *This implies that this equation, and hence,* $\Omega_l = 1$ *holds for* $l \in \{0,1,2,\ldots,k\}$. *Assume further that* $a_l^{(1)} \neq 0$ *for* $l \in \{0,1,2,\ldots,k-1\}$.

Then the transformation $s_n^{(k)} = {}_p\boldsymbol{J}_n^{(k)}(\beta, \{s_n\}, \{\omega_n\})$ *satisfies*

$$\lim_{n\to\infty} \frac{s_n^{(k)} - s}{s_n - s} \left\{ \prod_{l=0}^{k-1} \frac{\delta_n^{(l)}}{e_n^{(l)}} \right\}^{-1} = B_k \tag{62}$$

and, hence,

$$\frac{s_n^{(k)} - s}{s_n - s} = \mathrm{O}((n+\beta)^{-k}) \tag{63}$$

holds in the limit $n \to \infty$.

Proof. The validity of Eq. (61) for $l = 0$ can be extended to $l \in \{0, 1, 2, \ldots, k\}$ with the help of Lemma A.3 in the Appendix. Then, $\Omega_l = 1$ follows from the definitions. Hence, item (b) of Theorem 7 may be applied. Eq. (63) then follows since $\delta_n^{(l)} = \mathrm{O}((n+\beta)^{-2})$, see Lemma A.3, item (b), and $e_n^{(l)} = \mathrm{O}((n+\beta)^{-1})$. $\square$

Theorem 8 and 9 can be easily generalized to the case of any variant of the $\mathscr{J}$ transformation with $\delta_n^{(k)} = \mathrm{O}((n+\beta)^{-2})$ in the limit $n \to \infty$.

Assume that some constants $u_0 \neq 0$ and u_1 exists such that

$$\omega_n = (n+\beta)^{-\alpha}\left(u_0 + \frac{u_1}{n+\beta} + \mathrm{O}((n+\beta)^{-2})\right), \quad n \to \infty \tag{64}$$

holds. Then, according to item (c) of Lemma A.3, Eq. (61) holds for $l = 0$. Hence, Theorem 9 is relevant in the case of Eq. (64) and, hence, for logarithmic convergence. Comparison of Theorems 8 and 9 shows that in this case the order of convergence acceleration, i.e., the negative exponent of $n + \beta$ in Eqs. (60) and (63), drops from $2k$ to k. Similar behaviour is known for Levin-type accelerators [25, Theorems 13.5, 13.9, 13.11, 13.12, 14.2].

Theorem 8 allows to conclude that in the case of linear convergence, the ${}_p\boldsymbol{J}$ transformations should be superior to Wynn's epsilon algorithm [32]. Consider, for instance, the case that

$$s_n \sim s + \lambda^n n^\vartheta \sum_{n=0}^{\infty} c_j/n^j, \quad c_0 \neq 0, \ n \to \infty \tag{65}$$

is an asymptotic expansion of the sequence elements s_n. Assuming $\lambda \neq 1$ and $\vartheta \notin \{0, 1, \ldots, k-1\}$ it follows that [31, p.127], [25, p.333, Eqs. (13.4–7)]

$$\frac{\varepsilon_{2k}^{(n)} - s}{s_n - s} = \mathrm{O}(n^{-2k}), \quad n \to \infty. \tag{66}$$

This is the same order of convergence acceleration as in Eq. (60). But it should be noted that for the computation of $\varepsilon_{2k}^{(n)}$ the $2k+1$ sequence elements $\{s_n, \ldots, s_{n+2k}\}$ are required. But for the computation of ${}_p\boldsymbol{J}_n^{(k)}$ only the $k+1$ sequence elements $\{s_n, \ldots, s_{n+k}\}$ are required in the case of the t and u variants, and additionally s_{n+k+1} in the case of the $\tilde{t}$ variant. Again, this is similar to Levin-type accelerators [25, p.333].

4. Numerical results and discussion

We now display the results of using the $\mathscr{J}$ transformation. These results were obtained using FORTRAN QUADRUPLE PRECISION which corresponds to an accuracy of approximately 32 digits on our computer, a SUN workstation under UNIX.

If not otherwise stated, we use the highest possible level of iteration and, hence, we always give results for sequence transformations of the form

$$s'_n = \mathscr{J}_0^{(n)}(\{s_n\}, \{\omega_n\}, \{r_n^{(k)}\}) \tag{67}$$

as table entries. We remind the reader of the dependencies according to Eq. (15). Thus, to compute the nth element of these transformed sequences, we use the sequence elements $\{s_\nu\}_{\nu=0}^{n}$. For $\omega_n = \Delta s_n$, i.e., for $\tilde{t}$ variants, also s_{n+1} is used.

Column headings of the form $\beta_k = f(k)$ indicate that the $\mathcal{J}$ transformation with

$$r_n^{(k)} = 1/(n + f(k)), \qquad \delta_n^{(k)} = -1/(n + f(k))_2 = -1/[(n + f(k))(n + f(k) + 1)] \tag{68}$$

has been used for the computation of the corresponding column. If $f(k)$ is linear in k, this means that some variant of the ${}_pJ$ transformation (21) has been used. In most other cases the $\mathcal{T}, \tilde{\mathcal{T}}$, and $\mathcal{U}$ transformations are applied which are defined in Eqs. (18), (19), and (20), respectively. In some cases linear variants of the $\mathcal{J}$ transformation are used which are obtained by choosing ω_n independent of the s_n. But also in these cases the $\delta_n^{(k)}$ are normally chosen according to Eq. (68).

For the present article, we consider — with a single exception — transformations based on Eq. (68). But it should be noted that other variants of the $\mathcal{J}$ transformation — originating from choosing $\delta_n^{(k)}$ which differ from Eq. (68) — are expected to be also useful. The investigation of this interesting topic will be the subject of future work.

In Tables 1–3 we investigate the numerical performance of the $\mathcal{J}$ transformation for some examples which were studied in [7, pp.273–275]. A comparison to these results is complicated by the fact that it is not completely clear which computer and which accuracy these authors used. For the purpose of discussion we assume that these data may be directly compared to our data which correspond to a computer with about 32 decimal digits (in QUADRUPLE PRECISION). In case that Brezinski and Redivo Zaglia used fewer digits in their computation accurate, contamination of their data caused by rounding errors is highly probable.

Table 1
Acceleration of $s_{n-1} = (\cos(x_n) + a\sin(x_n))^{1/x_n}\exp(-a)$ for $a = 1$, $x_n = 0.8^n$ with $s_{-1} = 1$

n	s_n	$\omega_n = (n+1)\Delta s_{n-1}$ $\beta_k = 1$	$\omega_n = (n+1)\Delta s_{n-1}$ $\beta_k = 2$	$\omega_n = x_n$ $\delta_n^{(k)} = x_{n+k}$
6	0.8306977194479	0.9741409601610	1.1225200986264	0.9996475153695
8	0.8837061177176	0.9992780180258	1.0030337856181	0.9999906939002
10	0.9218529952615	1.0009399107924	1.0007212883640	0.9999998864939
12	0.9483176856882	1.0001496110197	1.0000244503126	0.9999999993860
14	0.9661973868563	0.9999863330110	0.9999777232278	0.9999999999986
16	0.9780569274876	0.9999925816547	0.9999958948632	1.0000000000000
18	0.9858263803984	0.9999994267593	1.0000002197867	1.0000000000000
20	0.9908747138370	1.0000001945383	1.0000001889768	1.0000000000000
22	0.9941373880907	1.0000000523404	1.0000000215900	1.0000000000000
24	0.9962386775483	1.0000000007212	0.9999999960285	1.0000000000000
26	0.9975889475654	0.9999999979774	0.9999999984311	1.0000000000000
28	0.9984553630156	0.9999999996350	0.9999999999071	1.0000000000000
30	0.9990107907768	1.0000000000275	1.0000000000534	1.0000000000000
∞	1.0000000000000	1.0000000000000	1.0000000000000	1.0000000000000

Table 2
Acceleration of $s_{n-1} = (\cos(1/n) + a\sin(1/n))^n \exp(-a)$ for $a = 1$ and $\omega_n = (n+1)\Delta s_{n-1}$ with $s_{-1} = 1$

n	s_n	$\beta_k = 1$	$\beta_k = 2$
8	0.9014918397610	0.9999914737390	0.9983764638148
9	0.9103376107750	1.0000004971692	1.0002088434895
10	0.9177237957539	0.9999999745416	0.9999773398060
11	0.9239845055428	1.0000000009975	1.0000021168614
12	0.9293590040817	0.9999999999302	0.9999998275900
13	0.9340231658411	0.9999999999942	1.0000000123818
14	0.9381091811057	0.9999999999980	0.9999999992083
15	0.9417183353028	0.9999999999995	1.0000000000454
16	0.9449295472490	0.9999999999999	0.9999999999976
17	0.9478052233297	1.0000000000000	1.0000000000001
18	0.9503953635162	1.0000000000000	1.0000000000000
∞	1.0000000000000	1.0000000000000	1.0000000000000

Table 3
Acceleration of $s_{n-1} = (1 + x/n)^n \exp(-x)$ for $\omega_n = (n+1)\Delta s_{-1}$ with $s_{n-1} = 1$ and $x = 1$

n	s_n	$\beta_k = 1$	$\beta_k = 2$
8	0.9495611399413	1.0000000623405	1.0000130965297
9	0.9541845267642	1.0000000061521	0.9999985821397
10	0.9580312771961	1.0000000017113	1.0000001302724
11	0.9612819623291	1.0000000003795	0.9999999897128
12	0.9640651901688	1.0000000000880	1.0000000007052
13	0.9664750464046	1.0000000000205	0.9999999999576
14	0.9685819513499	1.0000000000048	1.0000000000022
15	0.9704396614614	1.0000000000011	0.9999999999999
16	0.9720899236857	1.0000000000003	1.0000000000000
17	0.9735656521671	1.0000000000001	1.0000000000000
18	0.9748931471696	1.0000000000000	1.0000000000000
∞	1.0000000000000	1.0000000000000	1.0000000000000

The entries in Tables 1–3 in columns labeled by $\beta_k = 1$ correspond to ${}_1\boldsymbol{U}_0^{(n)}(1, 1, \{s_n\})$, and those for $\beta_k = 2$ correspond to ${}_1\boldsymbol{U}_0^{(n)}(1, 2, \{s_n\})$. These transformations are defined in Eq. (24). In Tables 1 and 2, sequences of the form

$$s_{n-1} = (\cos(x_n) + a\sin(x_n))^{1/x_n} \exp(-a), \quad n \in \mathbb{N}, \ a \in \mathbb{R} \tag{69}$$

are studied. These sequences converge to 1 if the auxiliary sequences $\{x_n\}$ converge to zero. As a matter of fact we have $s_{n-1} - 1 = O(x_n)$ and

$$\frac{s_{n+1} - 1}{s_n - 1} = O\left(\frac{x_{n+2}}{x_{n+1}}\right) \tag{70}$$

for large n. In Table 1 the auxiliary sequence $x_n = (0.8)^n$ is treated, while in Table 2 we have $x_n = 1/n$. Thus, in Table 1 the input sequence converges linearly, while in Table 2 it converges logarithmically.

The last column in Table 1 corresponds to the exceptional case mentioned above where $\delta_n^{(k)}$ is not chosen according to Eq. (68) but according to $\delta_n^{(k)} = x_{n+k} = (0.8)^{n+k}$. Since the $\mathcal{J}$ transformation is multiplicatively invariant in $r_n^{(k)}$ [17, Theorem 4] and hence, n independent factors of $r_n^{(k)}$ are irrelevant, the same results are obtained for $\delta_n^{(k)} = (0.8)^n$. Furthermore, since $\Delta q^n = q^n(q-1)$, the same results are obtained also for $r_n^{(k)} = (0.8)^{n+k}$ or for $r_n^{(k)} = (0.8)^n$, again because the $\mathcal{J}$ transformation is multiplicatively invariant in $r_n^{(k)}$. By a similar reasoning, it can be concluded that the choice $\delta_n^{(k)} = x_{n+k+1} - x_n = (0.8)^n \cdot (0.8^{k+1} - 1)$, or, equivalently, $r_n^{(k)} = \sum_{j=0}^{k} x_{n+j} = 5x_n(1 - (0.8)^{k+1})$ also would lead to these results. But according to [17, Theorem 8], the kernel of this $\mathcal{J}_n^{(k)}(\{s_n\}, \{x_n\}, 5x_n(1 - (0.8)^{k+1}))$ transformation is given by

$$s_{n-1} = s + x_n \sum_{j=0}^{k-1} d_j (x_n)^j, \quad n \in \mathbb{N} \tag{71}$$

for some constants d_j. This kernel can be interpreted as the first terms of a power series of s_{n-1} in the variable x_n. This explains the rather rapid convergence observed in the last column of Table 1. The transforms of the other two variants converge less rapidly. But it should be observed that these two variants of the $\mathcal{J}$ transformation are only slightly less efficient than the ε algorithm which yields 10.56 digits for $n = 20$, and more efficient than Levin's t transformation which yields 7.4 digits for this value of n [7, p.275].

Repeating the calculation in DOUBLE PRECISION (which corresponds to approximately 14–15 decimal digits on out machine), loss of accuracy was observed. For $\beta_k = 1$ the best results were 7–8 digits for $n \geqslant 19$, and 6–8 digits for $n \geqslant 18$ in the case $\beta_k = 2$. For $n > 25$ the accuracy deteriorated again. In the case $\omega_n = x_n$ and $\delta_n^{(k)} = x_{n+k}$, the best result was 11–12 digits for $n \geqslant 14$ deteriorating slowly for larger n to 9 digits for $n = 30$.

In Table 2 it is shown that these two u variants are even more efficient for $x_n = 1/n$. The one with $\beta_k = 1$ is slightly superior to the other one. It should be noted that for smaller n the variant with $\beta_k = 1$ ranges between the ρ and the Θ algorithm, while several other algorithms perform much worse [7, p.275]. But for higher n, our results are better than those for the ρ and the Θ algorithms which for $n = 18$ yield 13.47 and 9.86 digits, respectively [7, p.275].

In DOUBLE PRECISION the best results were 10 decimal digits for $n = 11$ for $\beta_k = 1$ and 8 decimal digits for $n = 13$ for $\beta_k = 2$.

In Table 3 the sequence

$$s_{n-1} = \left(1 + \frac{x}{n}\right)^n \exp(-x) \tag{72}$$

is studied for $x = 1$. As is well known, its limit is 1. The data treated in Table 3 provide an example of a logarithmically convergent sequence, with $s_{n-1} - 1 = -x^2/(2n) + O(1/n^2) = O(1/n)$. Similarly as in Table 2, the performance of the different u variants of the $\mathscr{J}$ transformation ranges between the ρ and Θ algorithms [7, p.274] for lower values of n. For higher values of n, the convergence is quite satisfactory, especially for the case $\beta_k = 2$.

In DOUBLE PRECISION the best results were 9 digits for $n = 10$ and $\beta_k = 1$, and 8 digits for $n = 11$ and $\beta_k = 2$.

In Table 4, we transform the partial sums

$$s_n = \sum_{j=0}^{n} (j+1)^{-2} \tag{73}$$

of the infinite series

$$\zeta(2) = \sum_{j=0}^{\infty} (j+1)^{-2} = \tfrac{1}{6}\pi^2. \tag{74}$$

The sequence of partial sums is logarithmically convergent and its remainders $s_n - \zeta(2)$ decay as n^{-1} for large n [25, p.345]. This is a special case of the series

$$\zeta(z) = \sum_{j=0}^{\infty} (j+1)^{-z} \tag{75}$$

defining the Riemann zeta function which has been used by many researchers as a test case for logarithmic convergence. The series is very slowly convergent if $\mathrm{Re}(z)$ is only slightly larger than 1 (see, for instance, [2, p.379]). In fact, its remainders are of the order n^{1-z} for large n.

The entries in Table 4 correspond to the use of the transformation ${}_p U$ defined in Eq. (24). Entries in the column labeled $\beta_k = 1$ correspond to ${}_1U_0^{(n)}(1, 1, \{s_n\})$, those for $\beta_k = 2$ correspond

Table 4
Acceleration of the $\zeta(2)$ series using $\omega_n = (n+1)\Delta s_{n-1}$

n	s_n	$\beta_k = 1$	$\beta_k = 2$	$\beta_k = 1 + k$	$\beta_k = 2 + k$
5	1.4914	1.6449152542373	1.6449307397583	1.6449358283971	1.6447858317612
6	1.5118	1.6449322143318	1.6449337653464	1.6449339851725	1.6449706252376
7	1.5274	1.6449338790187	1.6449340381402	1.6449340707114	1.6449242762514
8	1.5398	1.6449340473956	1.6449340640174	1.6449340666632	1.6449368570545
9	1.5498	1.6449340648035	1.6449340665620	1.6449340668572	1.6449332323647
10	1.5580	1.6449340666310	1.6449340668188	1.6449340668478	1.6449343262121
11	1.5650	1.6449340668250	1.6449340668452	1.6449340668482	1.6449339836640
12	1.5709	1.6449340668457	1.6449340668479	1.6449340668482	1.6449340942338
13	1.5760	1.6449340668480	1.6449340668482	1.6449340668482	1.6449340576310
14	1.5804	1.6449340668482	1.6449340668482	1.6449340668482	1.6449340700098
15	1.5843	1.6449340668482	1.6449340668482	1.6449340668482	1.6449340657459
∞	1.6449	1.6449340668482	1.6449340668482	1.6449340668482	1.6449340668482

to ${}_1\boldsymbol{U}_0^{(n)}(1,2,\{s_n\})$, those for $\beta_k = 1+k$ to ${}_2\boldsymbol{U}_0^{(n)}(1,1,\{s_n\})$, and those for $\beta_k = 2+k$ to ${}_2\boldsymbol{U}_0^{(n)}(1,2,\{s_n\})$.

Winner in this table is the transformation ${}_2\boldsymbol{U}_0^{(n)}(1,1,\{s_n\})$.

When the calculations for this table were repeated in DOUBLE PRECISION some loss of accuracy was observed. This is to be expected for the case of logarithmic convergence. For $\beta_k = 1$ we obtained 11 digits for $n = 11$, while 12 decimal digits were reproduced for $\beta_k = 2$. The corresponding results for $\beta_k = 1+k$ were 11 digits for $n = 9$, while only 8 decimal digits were accurate for $\beta_k = 2+k$. For larger values of n the accuracy deteriorated again. Also, it may be noted that the DOUBLE PRECISION results are dependent on the choice of the computational algorithm for the $\mathscr{J}$ transformation, i.e., whether Eq. (11) or whether Eq. (13) is used.

Comparing the above results to results in the literature [25, Table 14-1, p.351], [26, Table 1] it is seen that for the series (74) the transformed sequence ${}_2\boldsymbol{U}_0^{(n)}(1,1,\{s_n\})$ converges faster than the iterated ρ_2 transformation which was — according to [25, p.351] — together with the standard form of Wynn's ρ algorithm [33] regarded as the best accelerator for the series (74). However, it should be noted that ${}_2\boldsymbol{U}_0^{(n)}(1,1,\{s_n\})$ seems to be slightly more susceptible to rounding errors than the iterated ρ_2 transformation because the latter could produce 13 decimal digits for $n = 12$ in DOUBLE PRECISION [26].

If the dominant behavior of the error of partial sums of a series can be obtained, faster convergence can be achieved by subtracting known series with the same behavior of the remainders (compare, for instance, [10, p.152f]). For example, in the case of $\zeta(2)$, one may subtract

$$\sum_{n=1}^{\infty} \frac{4}{(4n^2-1)} = 2 \tag{76}$$

and obtain

$$\tfrac{1}{6}\pi^2 = 2 - \sum_{n=1}^{\infty} \frac{1}{[n^2(4n^2-1)]}. \tag{77}$$

The remainder of the partial sums

$$\tilde{s}_n = 2 - \sum_{m=1}^{n} \frac{1}{[m^2(4m^2-1)]} \tag{78}$$

then is $\mathrm{O}(n^{-3})$. Besides accelerating the convergence in this way, this technique often has a beneficial effect on round-off errors [10]. In the example given above, the best numbers in DOUBLE PRECISION are as follows:

$${}_1\boldsymbol{U}_0^{(12)}(1,1,\{s_n\}) = 1.6449340668401, \qquad {}_1\boldsymbol{U}_0^{(12)}(1,2,\{s_n\}) = 1.6449340668458,$$

$${}_2\boldsymbol{U}_0^{(9)}(1,1,\{s_n\}) = 1.6449340668613, \qquad {}_2\boldsymbol{U}_0^{(14)}(1,2,\{s_n\}) = 1.6449340738669,$$

and

$${}_1\boldsymbol{U}_0^{(11)}(1,1,\{\tilde{s}_n\}) = 1.6449340668485, \qquad {}_1\boldsymbol{U}_0^{(12)}(1,2,\{\tilde{s}_n\}) = 1.6449340668481,$$

$${}_2\boldsymbol{U}_0^{(14)}(1,1,\{\tilde{s}_n\}) = 1.6449340664834, \qquad {}_2\boldsymbol{U}_0^{(16)}(1,2,\{\tilde{s}_n\}) = 1.6449340654318.$$

Thus, it is possible to gain 1–2 digits in accuracy. The stability is also increased. This may be seen from the following facts (DOUBLE PRECISION): For $n = 20$ and application of the algorithms to s_n, we find only 5 digit accuracy for $\beta_k = 1 + k$ and $\beta_k = 2 + k$, and 7 digit accuracy for $\beta_k = 1$ and $\beta_k = 2$. When applied to $\tilde{s}_n$. for the same value of n, the accuracy is 7 digits for $\beta_k = 1 + k$ and $\beta_k = 2 + k$, 10 digits for $\beta_k = 1$, and 9 digits for $\beta_k = 2$.

In order to ease comparison to literature data, where in most cases this technique is not applied, it will also not be applied further in the present article.

In Tables 5 and 6 a particularly difficult example of logarithmic convergence is treated, namely the so-called $1/z$ expansion which is given below. It is a series expansion of $1/z$ in terms of reduced Bessel functions [24]

$$\hat{k}_\nu(z) = \left(\frac{2}{\pi}\right)^{1/2} z^\nu K_\nu(z). \tag{79}$$

Here, $K_\nu(z)$ is a modified Bessel function of the second kind [19, p.66]. The most recent detailed discussion of reduced Bessel functions can be found in [15, Section 3].

The $1/z$ expansion given by (see, e.g., [15, Eq. (3.2–32), p.30])

$$\frac{1}{z} = \sum_{j=0}^{\infty} \frac{\hat{k}_{j-1/2}(z)}{[2^j j!]} \tag{80}$$

is an extremely slowly convergent series. Its terms decay for large j as $1/(j + 1)^{3/2}$ which implies that the truncation errors of the partial sums

$$s_n = \sum_{j=0}^{\infty} \frac{\hat{k}_{j-1/2}(z)}{[2^j j!]} \tag{81}$$

Table 5
Acceleration of the $1/z$ expansion for $z = \frac{1}{2}$ using ${}_1\mathcal{J}_0^{(n)}(\frac{1}{2}, \{s_n\}, \{\omega_n\})$.

		ω_n		
n	s_n	$(n+1)\Delta s_{n-1}$	$(n+1)^{-1/2}$	$(2n-1)!!/(2n)!!$
10	1.8241884943835	1.9999881696571	2.0000000603100	2.0000000486355
11	1.8321449539650	2.0000024401806	1.9999999944846	1.9999999931683
12	1.8391112148225	1.9999999024935	2.0000000002443	1.9999999994553
13	1.8452768640345	1.9999998776094	2.0000000003953	2.0000000001831
14	1.8507843002336	2.0000000055899	2.0000000000432	1.9999999999963
15	1.8557428278275	1.9999999998206	2.0000000000093	1.9999999999969
16	1.8602379276877	1.9999999991435	2.0000000000036	2.0000000000003
17	1.8643375404571	1.9999999999243	2.0000000000008	2.0000000000000
18	1.8680964369459	1.9999999999851	2.0000000000002	2.0000000000000
19	1.8715593276470	1.9999999999927	2.0000000000001	2.0000000000000
20	1.8747631197850	1.9999999999985	2.0000000000000	2.0000000000000
∞	2.0000000000000	2.0000000000000	2.0000000000000	2.0000000000000

Table 6
Acceleration of the $1/z$ expansion for $z = \frac{4}{5}$ using ${}_1\mathcal{J}_0^{(n)}(\frac{1}{2}, \{s_n\}, \{\omega_n\})$.

		ω_n		
n	s_n	$(n+1)\Delta s_{n-1}$	$(n+1)^{-1/2}$	$(2n-1)!!/(2n)!!$
10	1.0747865667307	1.2499682133207	1.2500002099507	1.2500001534578
11	1.0826618965033	1.2499936918374	1.2500000455466	1.2500000421434
12	1.0895638413456	1.2500020835907	1.2499999926654	1.2499999931519
13	1.0956774851981	1.2500000290267	1.2499999994014	1.2499999991662
14	1.1011421634246	1.2499998954210	1.2500000003150	1.2500000002378
15	1.1060650318428	1.2500000061421	1.2500000000173	1.2500000000069
16	1.1105300244656	1.2500000024541	1.2499999999964	1.2499999999936
17	1.1146039429560	1.2499999993682	1.2500000000012	1.2500000000003
18	1.1183407028756	1.2499999999381	1.2500000000003	1.2500000000001
19	1.1217843613599	1.2500000000142	1.2500000000000	1.2500000000000
20	1.1249713188304	1.2499999999977	1.2500000000000	1.2500000000000
∞	1.2500000000000	1.2500000000000	1.2500000000000	1.2500000000000

— i.e., the quantities $(s_n - 1/z)$ — behave as $n^{-1/2}$ in the limit $n \to \infty$ [25, p.349]. The estimate for the truncation error mentioned above is based on the fact that [25, Eq. (14.3-18)]

$$\hat{k}_{n+1/2}(z) = 2^n(\tfrac{1}{2})_n[1 + \mathrm{O}(n^{-1})] \tag{82}$$

for large n and, hence,

$$\sum_{j=n+1}^{\infty} \frac{\hat{k}_{j-1/2}(z)}{[2^j j!]} \sim \sum_{j=n+1}^{\infty} \frac{2^{j-1}(1/2)_{j-1}}{2^j j!} = \frac{1}{2}\sum_{j=n}^{\infty} \frac{(1/2)_j}{(j+1)!} = \frac{(2n-1)!!}{(2n)!!} = \mathrm{O}(n^{-1/2}) \tag{83}$$

for large n. Here, we used [25, Eq. (14.3–21)]

$$\sum_{j=n}^{\infty} \frac{(2j-1)!!}{(2j+2)!!} = \frac{(1/2)_n}{(n)!} = \frac{(2n-1)!!}{(2n)!!}. \tag{84}$$

Compare also the discussion in [25, Ch. 14]. However, for larger values of z, the acceleration of the $1/z$ expansion is definitely much more difficult than that of the series for the lemniscate constant A which also has remainders of order $n^{-1/2}$ for large n [25, p.350]. The reason is that the asymptotic form (82) of the terms is reached only for large n values if z is relatively large. Because the $1/z$ expansion is more difficult to accelerate, it is a more demanding test case for the ability of a sequence transformation of accelerating logarithmic convergence than the more simple examples of Riemann's function $\zeta(2)$ or the lemniscate constant A [25, p.346].

One of the two best sequence transformations known for the $1/z$ expansion is, according to [26], a transformation generalizing Wynn's ρ algorithm for which different algorithms have been derived independently in [9] and [20]. The other very successful sequence transformation was derived in [3] by modifying Aitken's iterated Δ^2 process. It can also be obtained [26] by iterating Osada's transformation $\bar\rho_2^{(n)}$ of [20]. These two sequence transformations depend explicitly on a parameter α. Its significance can be seen from the fact that for sequences of the form

$$s_n = s + \sum_{j=0}^{\infty} \frac{c_j}{(n+1)^{\alpha+j}} \tag{85}$$

the error of these algorithms is $\mathrm{O}(n^{-\alpha-2k})$ for large n.

In the case of the $1/z$ expansion one has to choose $\alpha = \frac{1}{2}$ as a result of Eq. (83): This value of α corresponds to the $\mathrm{O}(n^{-1/2})$ behavior of the remainders. Consequently, a reasonable choice for the remainder estimates is $\omega_n = (n+1)^{-1/2}$. Eq. (83) motivates also the alternative choice $\omega_n = (2n-1)!!/(2n)!!$. This remainder estimate is of order $\mathrm{O}(n^{-1/2})$ for large n, too. If either one of these two choices for the remainder estimates is used, and if the $\delta_n^{(k)}$ do not depend on the s_n, the $\mathscr{J}$ transformation is a linear function of the partial sums.

In Table 5, the case $z = \frac{1}{2}$ is treated. In Table 6, we chose the slightly more difficult value $z = \frac{4}{5}$. In each table, the nonlinear u variant ${}_1\boldsymbol{J}_0^{(n)}(\beta, \{s_n\}, \{(n+1)\Delta s_{n-1}\})$ and the two linear transformations ${}_1\boldsymbol{J}_0^{(n)}(\beta, \{s_n\}, \{(n+1)^{-1/2}\})$ and ${}_1\boldsymbol{J}_0^{(n)}(\beta, \{s_n\}, \{(2n-1)!!/(2n)!!\}$ are displayed. For the definition of the ${}_1\boldsymbol{J}$ transformation see Eq. (21). For each value of z, additional computation with $\beta = \frac{3}{2}$ were done that yielded rather similar results to the case $\beta = \frac{1}{2}$.

In Tables 5 and 6, the linear transformations perform better than the nonlinear u variant. The choice $\omega_n = (n+1)^{-1/2}$ is slightly inferior to the choice $\omega_n = (2n-1)!!/(2n)!!$.

In DOUBLE PRECISION the best results for the u variant were 9 digits for $n = 16$, $\beta = \frac{1}{2}$, $z = \frac{4}{5}$, 8 digits for $n = 17$, $\beta = \frac{3}{2}$, $z = \frac{4}{5}$, 10 digits for $n = 15$, $\beta = \frac{1}{2}$, $z = \frac{1}{2}$, and 8 digits for $n = 15$, $\beta = \frac{3}{2}$, $z = \frac{1}{2}$. For the variant with $\omega_n = (n+1)^{-1/2}$ the corresponding results were 10 digits for $n = 14$, $\beta = \frac{1}{2}$, $z = \frac{4}{5}$, 9 digits for $n = 13$, $\beta = \frac{3}{2}$, $z = \frac{4}{5}$, 10 digits for $n = 12$, $\beta = \frac{1}{2}$, $z = \frac{1}{2}$, and 9 digits for $n = 12$, $\beta = \frac{3}{2}$, $z = \frac{1}{2}$. For the variant with $\omega_n = (2n-1)!!/(2n)!!$ the best results were 9 digits for $n = 13$, $\beta = \frac{1}{2}$, $z = \frac{4}{5}$, 9 digits for $n = 14$, $\beta = \frac{3}{2}$, $z = \frac{4}{5}$, 10 digits for $n = 13$, $\beta = \frac{1}{2}$, $z = \frac{1}{2}$, and 9 digits for $n = 12$, $\beta = \frac{3}{2}$, $z = \frac{1}{2}$. For larger n the accuracy deteriorated again.

The results for $z = \frac{1}{2}$ can be compared directly to the results in [26, Table 3]. These data indicate that the method of [3] is slightly superior to Osada's modified ρ algorithm [20] for $n < 14$ and slightly worse for $n \geqslant 14$. The entries in the last column of Table 5, which are our best results for $z = \frac{1}{2}$, are almost as good as the results for Osada's algorithm with $\alpha = \frac{1}{2}$. For instance, the absolute error of Osada's method for $n = 10$ is 2.83×10^{-8}, and for $n = 15$ it is 2×10^{-13}. However, it should be noted that for the example under consideration, Osada's method was somewhat more stable then our method.

For $z = \frac{4}{5}$ our best results are those in the last column of Table 6. They can be compared directly with the data in [25, Table 14-5, p.358]. This shows that our best results are much better than those obtained by using Brezinski's ϑ algorithm [4] and the iterated ϑ_2 transformation [25, Section 10.3]. All the data in Table 6 can be compared directly to [25, Table 14-6, p.360]. In the latter table, Levin's transformation [18] in the form $\mathscr{L}_n^{(0)}$ with $\beta = \frac{1}{2}$, i.e., the transformation (compare [25, p.238, Eq. (7.1-7)])

$$\mathscr{L}_n^{(0)}(\tfrac{1}{2}, s_0, \omega_0) = \frac{\sum_{j=0}^{n} (-1)^j \binom{n}{j} \dfrac{(j+\frac{1}{2})^{n-1}}{(n+\frac{1}{2})^{n-1}} \dfrac{s_j}{\omega_j}}{\sum_{j=0}^{n} (-1)^j \binom{n}{j} \dfrac{(j+\frac{1}{2})^{n-1}}{(n+\frac{1}{2})^{n-1}} \dfrac{1}{\omega_j}}, \quad n \in \mathbb{N}_0, \tag{86}$$

Table 7
Summation of the ${}_2F_0(1,1;\,-1/z)$ series for $z=3$ using $\omega_n = \Delta s_{n-1}$

n	s_n	$\beta_k = 2$	$\beta_k = 1 + k$	$\beta_k = 2 + k$
10	$0.48316 \cdot 10^{2}$	0.78625122105973	0.78625122076092	0.78625122072187
11	$-0.17702 \cdot 10^{3}$	0.78625122053322	0.78625122078671	0.78625122076540
12	$0.72431 \cdot 10^{3}$	0.78625122082790	0.78625122076556	0.78625122076700
13	$-0.31814 \cdot 10^{4}$	0.78625122075800	0.78625122076545	0.78625122076597
14	$0.15045 \cdot 10^{5}$	0.78625122076502	0.78625122076597	0.78625122076593
15	$-0.76089 \cdot 10^{5}$	0.78625122076677	0.78625122076597	0.78625122076595
16	$0.40996 \cdot 10^{6}$	0.78625122076573	0.78625122076596	0.78625122076596
17	$-0.23443 \cdot 10^{7}$	0.78625122076598	0.78625122076596	0.78625122076596
18	$0.14181 \cdot 10^{8}$	0.78625122076596	0.78625122076596	0.78625122076596
19	$-0.90481 \cdot 10^{8}$	0.78625122076595	0.78625122076596	0.78625122076596
20	$0.60727 \cdot 10^{9}$	0.78625122076596	0.78625122076596	0.78625122076596
∞		0.78625122076596	0.78625122076596	0.78625122076596

Table 8
Summation of the ${}_2F_0(1,1;\,-1/z)$ series for $z=3$ using $\omega_n = \Delta s_n$

n	s_n	$\beta_k = 2$	$\beta_k = 1 + k$	$\beta_k = 2 + k$
10	$0.483 \cdot 10^{2}$	0.78625122248647	0.78625122061800	0.78625122079313
11	$-0.177 \cdot 10^{3}$	0.78625122062189	0.78625122075743	0.78625122076011
12	$0.724 \cdot 10^{3}$	0.78625122072420	0.78625122076954	0.78625122076542
13	$-0.318 \cdot 10^{4}$	0.78625122078909	0.78625122076610	0.78625122076611
14	$0.150 \cdot 10^{5}$	0.78625122076015	0.78625122076586	0.78625122076597
15	$-0.761 \cdot 10^{5}$	0.78625122076659	0.78625122076595	0.78625122076595
16	$0.410 \cdot 10^{6}$	0.78625122076611	0.78625122076596	0.78625122076595
17	$-0.234 \cdot 10^{7}$	0.78625122076586	0.78625122076596	0.78625122076596
18	$0.142 \cdot 10^{8}$	0.78625122076598	0.78625122076596	0.78625122076596
19	$-0.905 \cdot 10^{8}$	0.78625122076595	0.78625122076596	0.78625122076596
20	$0.607 \cdot 10^{9}$	0.78625122076595	0.78625122076596	0.78625122076596
∞		0.78625122076596	0.78625122076596	0.78625122076596

is used instead of our transformation ${}_1\mathbf{J}_0^{(n)}(\frac{1}{2},\{s_n\},\{\omega_n\})$ with the same choices for ω_n. The comparison shows that each of our methods is superior to the corresponding Levin transformation with the same ω_n for this example.

In Tables 7–9 we treat the Euler series. It is a divergent Stieltjes series of the form (7). The Euler series is the asymptotic expansion

$$E(z) \sim {}_2F_0(1,1;\,-z) = \sum_{n=0}^{\infty} (-1)^n n!\, z^n, \quad z \to 0, \tag{87}$$

Table 9
Summation of the ${}_2F_0(1,1;\,-1/z)$ series for $z = \frac{1}{2}$ using $\omega_n = \Delta s_n$

n	s_n	$\beta_k = 1$	$\beta_k = 1 + k$	$\beta_k = 1 + 2k$
15	-0.41471×10^{17}	0.46145534818892	0.46145531777895	0.46145531958535
16	0.13297×10^{19}	0.46145532179262	0.46145531653531	0.46145531701552
17	-0.45291×10^{20}	0.46145530928911	0.46145531617413	0.46145531625982
18	0.16331×10^{22}	0.46145531718043	0.46145531617643	0.46145531613493
19	-0.62144×10^{23}	0.46145531722130	0.46145531622219	0.46145531616450
20	0.24889×10^{25}	0.46145531581628	0.46145531624179	0.46145531620445
21	-0.10466×10^{27}	0.46145531617051	0.46145531624454	0.46145531622787
22	0.46097×10^{28}	0.46145531633945	0.46145531624315	0.46145531623807
23	-0.21225×10^{30}	0.46145531623155	0.46145531624210	0.46145531624153
24	0.10197×10^{32}	0.46145531622546	0.46145531624179	0.46145531624231
25	-0.51027×10^{33}	0.46145531624774	0.46145531624179	0.46145531624227
26	0.26554×10^{35}	0.46145531624378	0.46145531624184	0.46145531624210
27	-0.14349×10^{37}	0.46145531624025	0.46145531624186	0.46145531624197
28	0.80408×10^{38}	0.46145531624183	0.46145531624187	0.46145531624191
29	-0.46665×10^{40}	0.46145531624221	0.46145531624187	0.46145531624188
30	0.28015×10^{42}	0.46145531624180	0.46145531624187	0.46145531624187
∞		0.46145531624187	0.46145531624187	0.46145531624187

of the Euler integral

$$E(z) = \int_0^\infty \frac{\exp(-t)}{1+zt}\,\mathrm{d}t. \tag{88}$$

The terms of the Euler series are essentially given by the moments $\mu_n = n!$ of the positive measure $\mathrm{d}\psi(t) = \exp(-t)\mathrm{d}t$. Hence, the Euler integral is a Stieltjes function. It is related to the exponential integral

$$E_1(z) = \int_0^\infty \frac{\exp(-t)}{t}\,\mathrm{d}t \tag{89}$$

by the equation [11, p.144, Eq. (14)]

$$z\exp(z)E_1(z) = E\left(\frac{1}{z}\right). \tag{90}$$

The exponential integral $E_1(z)$, however, is for $z > 0$ readily computed using the routine S13AAF of the NAG library. This means that the exact value of the summed series is available. For this reason, we study in our examples the Euler series in the variable $1/z$, i.e., the hypergeometric series ${}_2F_0(1,1;\,-1/z)$ with partial sums

$$s_n = \sum_{j=0}^{n} (-1)^j j!\, z^{-j}. \tag{91}$$

This hypergeometric series has zero radius of convergence and is rapidly divergent for all $|z| < \infty$. The divergence is the faster the smaller z is. The aim is to sum this divergent series. The uniqueness of the result is ensured by Carleman's theorem [21, p.39].

In Tables 7 and 8, the case $z = 3$ is treated using the ${}_p\boldsymbol{T}$ transformation designed in Eq. (22), and the ${}_p\tilde{\boldsymbol{T}}$ transformation defined in Eq. (23). Compared are ${}_1\boldsymbol{T}_0^{(n)}(2, \{s_n\})$ corresponding to $\beta_k = 2$, and ${}_2\boldsymbol{T}_0^{(n)}(\beta, \{s_n\})$ for $\beta = 1$ and $\beta = 2$ corresponding to $\beta_k = 1 + k$ and $\beta_k = 2 + k$ in Table 7, with ${}_1\tilde{\boldsymbol{T}}_0^{(n)}(2, \{s_n\})$ corresponding to $\beta_k = 2$, and ${}_2\tilde{\boldsymbol{T}}_0^{(n)}(\beta, \{s_n\})$ for $\beta = 1$ and $\beta = 2$ corresponding to $\beta_k = 1 + k$ and $\beta_k = 2 + k$ in Table 8. One would expect the $\tilde{t}$ variants to perform better, but it turns out that the corresponding t variants have a tiny advantage. The ${}_1\boldsymbol{T}$ and ${}_1\tilde{\boldsymbol{T}}$ transformations are not as efficient as the t and $\tilde{t}$ variants of the ${}_2\boldsymbol{J}$ transformation. The latter perform nearly identically in this example.

Repeating the calculation in DOUBLE PRECISION, it was observed that at most the two last digits for $\beta_k = 2$, and the last digit for $\beta_k = 1 + k$ and $\beta_k = 2 + k$, disagreed with the data presented in Tables 7 and 8. Thus, numerical instabilities are not important in the case of this alternating series for an argument as large as $z = 3$.

These results can be compared directly to the data given in [25, Tables 13-1, p.328; 13-2, p.329]. This comparison shows that both the t and $\tilde{t}$ variants for $\beta_k = 2$ perform very similar to the $\tilde{t}$ variant of Levin's transformation. Furthermore, the t and $\tilde{t}$ variants for $\beta_k = 1 + k$ and $\beta_k = 2 + k$ perform very similar to the $\tilde{t}$ variant of the Weniger transformation $\mathscr{S}_n^{(0)}$. This is based upon factorial series and defined by (see [25, Section 8]).

$$\mathscr{S}_k^{(n)}(\beta, s_n, \omega_n) = \frac{\Delta^k\left[\dfrac{(\beta+n)_{k-1}\, s_n}{\omega_n}\right]}{\Delta^k\left[\dfrac{(\beta+n)_{k-1}}{\omega_n}\right]}. \tag{92}$$

This is a remarkable success of the ${}_2\boldsymbol{J}$ transformation because the Weniger transformation is apparently able to sum strongly divergent Stieltjes series quite efficiently [28–30].

In Table 9, results for the case $z = \frac{1}{2}$ are presented. Compared are the $\tilde{t}$ variants ${}_p\tilde{\boldsymbol{T}}_0^{(n)}(1, \{s_n\})$, Eq. (23), for $p = 1$ corresponding to $\beta_k = 1$, for $p = 2$ corresponding to $\beta_k = 1 + k$, and for $p = 3$ corresponding to $\beta_k = 1 + 2k$.

It should be noted that the choices $\beta_k = \beta + 2k$ which correspond to the ${}_3\boldsymbol{J}$ transformation, lead to corresponding t or $\tilde{t}$ variants of Weniger's transformation (92) according to [17]. A comparison of Table 9 with Table 13-3 in [25, p.330] shows that indeed identical numerical results are obtained.

Additional calculations of the t variants ${}_p\boldsymbol{T}_0^{(n)}(2, \{s_n\})$, Eq. (22), for $p = 1$ corresponding to $\beta_k = 2$, for $p = 2$ corresponding to $\beta_k = 2 + k$, and for $p = 3$ corresponding to $\beta_k = 2 + 2k$ with rather similar results were performed. This leads to the conclusion that, again, the difference between corresponding t and $\tilde{t}$ variants is not large. The choice of β_k is more important than this difference that is characterized by an index shift by one in ω_n. The choices $\beta_k = 1$ and $\beta_k = 2$ are of similar performance as the $\tilde{t}$ variant of Levin's transformation [25, Table 13-3, p.330].

Repeating the calculations in DOUBLE PRECISION, some loss of accuracy was observed. For $\beta_k = 2$, the best results were 7 decimal digits for $n = 14$, 10 decimal digits for $\beta_k = 2 + k$ and $n = 19$, and for $\beta_k = 2 + 2k$ and $n = 18$, 7 decimal digits for $n = 15$ and $\beta_k = 1$, 10 decimal digits for $\beta_k = 1 + k$ and $n = 16$, and also for $\beta_k = 1 + 2k$ and $n = 18$. For higher n, the accuracy deteriorated again.

In Table 9, the choice $\beta_k = \beta + k$, i.e., the ${}_2\boldsymbol{J}$ transformation is slightly superior to the transformation $\Delta_n^{(0)}(29, s_0)$ which is based on an expansion of the remainder in terms of Pochhammer symbols $1/(-\gamma - n)_j$ [25, Section 9.2]. One should note that this $\Delta_k^{(n)}$ transformation seems to be one of the best summation methods for the Euler series known so far. See [25, Table 13-3, p.330].

In Table 9, the choice $\beta_k = \beta + k$ is seen to be significantly better than the choice $\beta_k = \beta + 2k$. Thus, the ${}_2\boldsymbol{J}$ allows an improvement of Weniger's transformation (92) in this case. This is in some sense similar to the results in [27]. There, convergence acceleration methods were applied to strongly divergent perturbation expansions of the ground state energy of anharmonic oscillators. It was shown that the transformation (92) was not the most efficient method, but a sequence transformation which can be interpreted as an interpolation between Levin's transformation and Weniger's transformation (92). According to [17], Levin's transformation [18] corresponds to

$$\delta_n^{(k)} = \frac{1}{(n+\beta)(n+\beta+k+1)}, \tag{93}$$

and Weniger's transformation (92) corresponds to

$$\delta_n^{(k)} = \frac{1}{(n+\beta+2k)(n+\beta+2k+1)}. \tag{94}$$

Hence, one observes that the quantities

$$\delta_n^{(k)} = \frac{1}{(n+\beta+k)(n+\beta+k+1)} \tag{95}$$

which correspond to the ${}_2\boldsymbol{J}$ transformation, range in between Levin's and Weniger's transformations as far as the decay of the $\delta_n^{(k)}$ for large n (or large k) is concerned.

In Tables 10–12 the Stieltjes series

$$\ln(1+z) = \sum_{j=0}^{\infty} \frac{(-1)^j z^{j+1}}{j+1} = z\,{}_2F_1(1,1;2;\,-z), \tag{96}$$

Table 10
Acceleration of the $z\,{}_2F_1(1,1;2;-z)$ series for $z = -0.9$ and $\omega_n = \Delta s_{n-1}$

n	s_n	$\beta_k = 1$	$\beta_k = 1 + k$	$\beta_k = 1 + 2k$
20	−2.269	−2.30258500431671	−2.30258509185778	−2.30258509296883
21	−2.273	−2.30258505436952	−2.30258509261089	−2.30258509298723
22	−2.277	−2.30258507617057	−2.30258509286486	−2.30258509299220
23	−2.280	−2.30258508566631	−2.30258509295049	−2.30258509299355
24	−2.283	−2.30258508980232	−2.30258509297937	−2.30258509299391
25	−2.286	−2.30258509160383	−2.30258509298910	−2.30258509299401
26	−2.288	−2.30258509238851	−2.30258509299238	−2.30258509299404
27	−2.290	−2.30258509273029	−2.30258509299348	−2.30258509299404
28	−2.291	−2.30258509287916	−2.30258509299386	−2.30258509299404
29	−2.293	−2.30258509294401	−2.30258509299398	−2.30258509299405
30	−2.294	−2.30258509297225	−2.30258509299402	−2.30258509299405
∞	−2.303	−2.30258509299405	−2.30258509299405	−2.30258509299405

Table 11
Acceleration of the $z\,{}_2F_1(1,1;2;-z)$ series for $z = 1$ and $\omega_n = \Delta s_{n-1}$

n	s_n	$\beta_k = 1$	$\beta_k = 1 + k$	$\beta_k = 1 + 2k$
3	0.583	0.69312169312169	0.69318181818182	0.69321533923304
4	0.783	0.69314489928525	0.69314592545799	0.69314971751412
5	0.617	0.69314752228759	0.69314712706300	0.69314726571364
6	0.760	0.69314715292958	0.69314718127769	0.69314718328808
7	0.635	0.69314718212288	0.69314718062868	0.69314718064517
8	0.746	0.69314718051087	0.69314718056011	0.69314718056257
9	0.646	0.69314718055803	0.69314718055987	0.69314718056003
10	0.737	0.69314718056042	0.69314718055994	0.69314718055995
11	0.653	0.69314718055990	0.69314718055995	0.69314718055995
12	0.730	0.69314718055995	0.69314718055995	0.69314718055995
13	0.659	0.69314718055995	0.69314718055995	0.69314718055995
∞	0.693	0.69314718055995	0.69314718055995	0.69314718055995

Table 12
Summation of the $z\,{}_2F_1(1,1;2;-z)$ series for $z = 5$ and $\omega_n = \Delta s_{n-1}$

n	s_n	$\beta_k = 1$	$\beta_k = 1 + k$	$\beta_k = 1 + 2k$
10	0.364×10^{07}	1.79175940186339	1.79175947649056	1.79175959220168
11	-0.167×10^{08}	1.79175947401768	1.79175947012676	1.79175949178480
12	0.772×10^{08}	1.79175947060006	1.79175946924325	1.79175947333854
13	-0.359×10^{09}	1.79175946873775	1.79175946921405	1.79175946997338
14	0.168×10^{10}	1.79175946929034	1.79175946922545	1.79175946936268
15	-0.786×10^{10}	1.79175946923181	1.79175946922788	1.79175946925230
16	0.370×10^{11}	1.79175946922487	1.79175496922807	1.79175946923241
17	-0.175×10^{12}	1.79175946922866	1.79175946922806	1.79175946922884
18	0.829×10^{12}	1.79175946922803	1.79175946922806	1.79175946922819
19	-0.394×10^{13}	1.79175946922804	1.79175946922806	1.79175946922808
20	0.188×10^{14}	1.79175946922806	1.79175946922805	1.79175946922806
∞		1.79175946922806	1.79175946922806	1.79175946922806

with partial sums

$$s_n = \sum_{j=0}^{n} \frac{(-1)^j z^{j+1}}{j+1} \tag{97}$$

is investigated. The corresponding Stieltjes function is

$$\frac{\ln(1+z)}{z} = \int_0^1 \frac{1}{1+zt}\,\mathrm{d}t. \tag{98}$$

The radius of convergence of the series is 1. For $|z| > 1$ the series diverges but can be summed for z not on the cut $-\infty < z \leqslant -1$.

For $z = -0.9$ the series is absolutely convergent, and all its terms have the same sign. This case is treated in Table 10. There, results are presented for the t variants ${}_p\boldsymbol{T}_0^{(n)}(1, \{s_n\})$, Eq. (22), for $p = 1$ corresponding to $\beta_k = 1$, $p = 2$ corresponding to $\beta_k = 1 + k$, and $p = 3$ corresponding to $\beta_k = 1 + 2k$. The latter is identical to the t variant of Weniger's transformation (92). The data may be compared directly to [25, Table 13-6, p.337]. Then, one obtains the result that ${}_1\boldsymbol{T}_0^{(n)}(1, \{s_n\})$ performs better than Wynn's ε algorithm [32] but worse than Levin's t transformation. The latter is inferior to ${}_2\boldsymbol{T}_0^{(n)}(1, \{s_n\})$ which corresponds to $\beta_k = 1 + k$. The winner for this example is the transformation ${}_3\boldsymbol{T}_0^{(n)}(1, \{s_n\})$, i.e., Weniger's t transformation.

In DOUBLE PRECISION, the best result were 9 decimal digits for $\beta_k = 1$ and $n = 24$, 8 digits for $\beta_k = 1 + k$ and $n = 17$, and 8 digits for $\beta_k = 1 + 2k$ and $n = 16$. This corresponds to a heavy loss of accuracy due to the single sign of the terms.

For $z = 1$, the series (96) is alternating and conditionally convergent. Due to the alternating signs of the terms, it is expected that the numerical stability of the sequence transformation is quite high. In Table 11, this case is treated. Again, results are presented for the t variants ${}_p\boldsymbol{T}_0^{(n)}(1, \{s_n\})$, Eq. (22), for $p = 1$ corresponding to $\beta_k = 1$, $p = 2$ corresponding to $\beta_k = 1 + k$, and $p = 3$ corresponding to $\beta_k = 1 + 2k$. Contrary to Table 10, it is seen that the performance of the two variants ${}_2\boldsymbol{T}_0^{(n)}(1, \{s_n\})$ and ${}_3\boldsymbol{T}_0^{(n)}(1, \{s_n\})$ is nearly identical in this case. The transformation ${}_1\boldsymbol{T}_0^{(n)}(1, \{s_n\})$ is inferior to these two variants. But comparing Table 13-5 in [25, p.335], it is seen that the ${}_1\boldsymbol{T}_0^{(n)}(1, \{s_n\})$ transformation clearly performs better than Wynn's ε algorithm [32]. Levin's t transformation performs slightly worse than Weniger's t transformation which is identical to ${}_3\boldsymbol{T}_0^{(n)}(1, \{s_n\})$. As expected, in DOUBLE PRECISION no loss of accuracy was observed apart from an occasional deviation in the last digit for $\beta_k = 1$.

For $z = 5$, the series (96) is alternating and divergent. Its summation is the topic of Table 12. Again, the t variants ${}_p\boldsymbol{T}_0^{(n)}(1, \{s_n\})$, Eq. (22), for $p = 1$ corresponding to $\beta_k = 1$, $p = 2$ corresponding to $\beta_k = 1 + k$, and $p = 3$ corresponding to $\beta_k = 1 + 2k$ are compared. It is seen that in the case of this divergent series the variant $\beta_k = 1 + k$, i.e., the ${}_2\boldsymbol{T}$ transformation, performs better than the variants $\beta_k = 1$ and $\beta_k = 1 + 2k$. The latter variants perform very similar.

In DOUBLE PRECISION, best results were 10 decimal digits for $\beta_k = 1$ and $n = 13$, 13 digits for $\beta_k = 1 + k$ and $n = 14$, and 13 digits for $\beta_k = 1 + 2k$ and $n = 16$. Thus, at least for the latter two variants, numerical stability is not critical.

Important results of these numerical studies are summarized as follows:

(a) The $\mathscr{J}$ transformation can be combined profitably with the remainder estimates in [18, 23]. Especially suitable u variants are seen to be powerful general purpose accelerators similar to Levin's u transformation. For linearly convergent, and also for divergent sequences, the t and $\tilde{t}$ variants can be applied successfully. The latter is not always superior to the former. For special examples, also linear variants are useful if the asymptotic behavior of the remainders can be derived analytically.

(b) The ${}_1\boldsymbol{J}_n^{(k)}(\beta, \{s_n\}, \{\omega_n\})$ transformation corresponding to $\beta_k = \beta$ and even more the transformation ${}_2\boldsymbol{J}_n^{(k)}(\beta, \{s_n\}, \{\omega_n\})$ corresponding to $\beta_k = \beta + k$ are very useful convergence accelerators. For logarithmically convergent sequences the u variants of these two transformations have similar properties as Levin's u transformation [18]. For alternating divergent series, the t and $\tilde{t}$ variants of the ${}_2\boldsymbol{J}$ transformation are comparable or superior to Weniger's transformation (92).

(c) The choice of the hierarchy, i.e., of $\delta_n^{(k)}$ seems to be more important than the difference between t and $\tilde{t}$ variants.

(d) The numerical stability of the methods is similar to Levin-type methods. Logarithmic convergence is demanding in this respect. Stability in this case can be improved by subtraction of known series with similar remainders. The stability is high for alternating series.

(e) The flexibility of the approach is very useful. Thus, it is easy to obtain rather powerful transformations by a suitable, heuristic combination of proper remainder estimates and hierarchies. These transformations can be computed at very low costs. The numerical implementation of these methods can be based on a single very simple subroutine which calls a function subprogram to compute the $\delta_n^{(k)}$. These programs will be made available in the future.

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Appendix

Lemma A.1. *Put for $n \in \mathbb{N}_0$*

$$q_n^{(k)} = \frac{s_n^{(k+1)} - s}{s_n^{(k)} - s}, \quad k \in \mathbb{N}_0, \tag{A.1a}$$

$$b_n^{(k)} = \frac{s_n^{(k)} - s}{\omega_n^{(k)}}, \quad k \in \mathbb{N}_0, \tag{A.1b}$$

$$e_n^{(k)} = 1 - \frac{\omega_{n+1}^{(k)}}{\omega_n^{(k)}}, \quad k \in \mathbb{N}_0. \tag{A.1c}$$

Then the following relations hold:

(a) *We have $b_n^{(k)} = N_n^{(k)} - sD_n^{(k)}$. In addition,*

$$b_n^{(k+1)} = \nabla_n^{(k)} b_n^{(k)} = (\Delta b_n^{(k)})/\delta_n^{(k)}. \tag{A.2}$$

(b) *The $q_n^{(k)}$ may be computed from the following formulas:*

$$q_n^{(k)} = \frac{-\omega_{n+1}^{(k)}}{\Delta \omega_n^{(k)}} \frac{b_n^{(k+1)}}{b_n^{(k)}} \delta_n^{(k)} = \frac{1}{\omega_n^{(k)}/\omega_{n+1}^{(k)} - 1} \frac{b_n^{(k+1)}}{b_n^{(k)}} \delta_n^{(k)} = \frac{\omega_{n+1}^{(k)}/\omega_n^{(k)}}{e_n^{(k)}} \frac{b_n^{(k+1)}}{b_n^{(k)}} \delta_n^{(k)}. \tag{A.3}$$

(c) *We have*

$$\frac{\omega_{n+1}^{(k+1)}}{\omega_n^{(k+1)}} = \frac{\delta_{n+1}^{(k)}}{\delta_n^{(k)}} \frac{\omega_{n+2}^{(k)}}{\omega_{n+1}^{(k)}} \frac{\omega_{n+1}^{(k)}}{\omega_n^{(k)}} \frac{\omega_{n+1}^{(k)} - \omega_n^{(k)}}{\omega_{n+2}^{(k)} - \omega_{n+1}^{(k)}} \tag{A.4a}$$

$$= \frac{\omega_{n+2}^{(k)}}{\omega_{n+1}^{(k)}} \frac{\delta_{n+1}^{(k)}}{\delta_n^{(k)}} \frac{e_n^{(k)}}{e_{n+1}^{(k)}} \tag{A.4b}$$

and, hence,

$$e_n^{(k+1)} = 1 - \frac{\delta_{n+1}^{(k)}}{\delta_n^{(k)}} \frac{e_n^{(k)}}{e_{n+1}^{(k)}} (1 - e_{n+1}^{(k)}). \tag{A.5}$$

(d) *Eq.* (50) *holds.*

Proof. (a) As a direct consequence of Eq. (17) and the definition (A.1b), $b_n^{(k)} = N_n^{(k)} - sD_n^{(k)}$ holds. Then, Eq. (A.2) follows, since both $N_n^{(k)}$ and $D_n^{(k)}$ satisfy the recursion $X_n^{(k+1)} = \nabla_n^{(k)} X_n^{(k)}$ as implied by Eq. (17).

(b) In definition (A.1a) one substitutes $s_n^{(k)} - s = b_n^{(k)}\omega_n^{(k)}$ and $s_n^{(k+1)} - s = b_n^{(k+1)}\omega_n^{(k+1)}$. In the resulting expression, $q_n^{(k)} = [b_n^{(k+1)}/b_n^{(k)}][\omega_n^{(k+1)}/\omega_n^{(k)}]$, Eq. (11c) is used to evaluate $\omega_n^{(k+1)}/\omega_n^{(k)}$. Eq. (A.3) follows.

(c) Eqs. (A.4a) and (A.4b) are a direct consequence of Eq. (11c) and the definitions.

(d) Application of the formula

$$\frac{u_k}{u_0} = \prod_{l=0}^{k-1} \frac{u_{l+1}}{u_l} \tag{A.6}$$

with $u_l = s_n^{(l)} - s$ allows to represent $[s_n^{(k)} - s]/[s_n - s]$ as a product of the $q_n^{(l)}$, whence item(b) can be used. The resulting formula can then be simplified by a second application of Eq. (A.6) with $u_l = b_n^{(l)}$. □

Lemma A.2. *Assume that* (B2) *of Theorem* 7 *holds. Then,*

$$E_k = \lim_{n\to\infty} e_n^{(k)} = 1 - \Omega_k \neq 1 \tag{A.7}$$

exists, and the following holds:

(a) *If* $\Omega_k \neq 1$ *and, hence,* $E_k \neq 0$ *for all* k *then*

$$\Omega_k = F_{k-1}\Omega_{k=1} = \Omega_0/\Phi_k, \tag{A.8a}$$

$$\lim_{n\to\infty} \prod_{l=0}^{k-1} \frac{\omega_{n+1}^{(l)}/\omega_n^{(l)}}{e_n^{(l)}} = \frac{[\Omega_0]^k}{\prod_{l=0}^{k-1}(\Phi_l - \Omega_0)}. \tag{A.8b}$$

Eq. (A.8b) *also follows from* $\Omega_0 \notin \{\Phi_0 = 1, \Phi_1, \ldots, \Phi_{k-1}\}$.

(b) *If* $\Omega_k = 1$ *and, hence,* $E_k = 0$ *for all* k *then*

$$\lim_{n\to\infty} \frac{e_{n+1}^{(k)}}{e_n^{(k)}} = F_k, \quad k \in \mathbb{N}_0. \tag{A.9}$$

Conversely, if Eq. (A.9) holds then $\Omega_0 = 1$ *implies* $\Omega_k = 1$ *and, hence,* $E_k = 0$ *for all* k.

(c) *If* $\Omega_k = 1$ *for all* k, *then*

$$\left\{\prod_{l=0}^{k-1}\left[\frac{\omega_{n+1}^{(l)}/\omega_n^{(l)}}{e_n^{(l)}}\right]\right\} \to \left\{\prod_{l=0}^{k-1} e_n^{(l)}\right\}^{-1} \tag{A.10}$$

for $n \to \infty$.

Proof. (a) The assumptions imply that $e_n^{(k)}/e_{n+1}^{(k)}$ approaches $E_k/E_k = 1$ for $n \to \infty$. Then, the first equality in Eq. (A.8a) is a direct consequence of Eq. (A.4b). The second equality follows from the iteration of the first part of Eq. (A.8a) which yields

$$\Omega_k = F_{k-1}F_{k-2}\cdots F_0\Omega_0 \tag{A.11}$$

and definition (30) for the Φ_k. Eq. (A.8a) implies

$$\lim_{n\to\infty} \frac{\omega_{n+1}^{(l)}/\omega_n^{(l)}}{e_n^{(l)}} = \frac{\Omega_l}{1-\Omega_l} = \frac{\Omega_0}{\Phi_l - \Omega_0} \tag{A.12}$$

Then, Eq. (A.8b) follows by taking the product over l. The last sentence of the assertion follows because $\Omega_0 \notin \{\Phi_0 = 1, \Phi_1, \ldots, \Phi_{k-1}\}$ and Eq. (A.8a) imply that $\Omega_l \neq 1$, and hence, $E_l \neq 0$ for $l \in \{0, 1, \ldots, k-1\}$. It follows that one may take limits as in Eq. (A.8b).

(b) This follows by taking the limit $n \to \infty$ in Eq. (A.4b)

(c) This follows from the definition of Ω_k. □

Lemma A.3. (a) *If Eq. (61) holds for* $l = k$ *with* $a_k^{(1)} \neq 0$ *and if*

$$\frac{\delta_{n+1}^{(k)}}{\delta_n^{(k)}} = 1 + \frac{d_k^{(1)}}{n+\beta} + \frac{d_k^{(2)}}{(n+\beta)^2} + \mathrm{O}((n+\beta)^{-3}), \quad n \to \infty, \tag{A.13}$$

holds then

$$e_n^{(k+1)} = \frac{a_k^{(1)} - 1 - d_k^{(1)}}{n+\beta} + \frac{a_k^{(2)} + a_k^{(1)}d_k^{(1)} - d_k^{(1)} - d_k^{(2)} - a_k^{(2)}/a_k^{(1)}}{(n+\beta)^2} + \mathrm{O}((n+\beta)^{-3}) \tag{A.14}$$

holds for large n *and, hence, we have*

$$a_{k+1}^{(1)} = a_k^{(1)} - 1 - d_k^{(1)}, \qquad a_{k+1}^{(2)} = a_k^{(2)} + a_k^{(1)}d_k^{(1)} - d_k^{(1)} - d_k^{(2)} - a_k^{(2)}/a_k^{(1)}. \tag{A.15}$$

(b) *If*

$$\delta_n^{(k)} = \Delta(n + \beta + (p-1)k)^{-1}, \tag{A.16}$$

then

$$d_k^{(1)} = -2, \qquad d_k^{(2)} = 2(2 + (p-1)k), \tag{A.17}$$

and

$$a_k^{(1)} = a_0^{(1)} + k. \tag{A.18}$$

(c) *If for some constants* $u_k^{(0)} \neq 0$ *and* $u_k^{(1)}$,

$$\omega_n^{(k)} = (n+\beta)^{-\alpha_k}\left(u_k^{(0)} + \frac{u_k^{(1)}}{n+\beta} + \mathrm{O}((n+\beta)^{-2})\right), \quad n \to \infty \tag{A.19}$$

holds then

$$e_n^{(k)} = \frac{\alpha_k}{n+\beta} - \frac{\alpha_k(\alpha_k+1)/2 - u_k^{(1)}/u_k^{(0)}}{(n+\beta)^2} + \mathrm{O}((n+\beta)^{-3}) \tag{A.20}$$

for large n and, hence, we have

$$a_k^{(1)} = \alpha_k, \qquad a_k^{(2)} = \frac{\alpha_k(\alpha_k+1)}{2} - \frac{u_k^{(1)}}{u_k^{(0)}}. \tag{A.21}$$

Proof. (a) This follows by straightforward algebra from Eq. (A.5).

(b) This follows by series expansion of $\delta_{n+1}^{(k)}/\delta_n^{(k)}$ in $(n+\beta)^{-1}$, and from Eq. (A.15).

(c) This follows from the definition of $e_n^{(k)}$ by series expansion in $(n+\beta)^{-1}$. □

References

[1] A. C. Aitken, On Bernoulli's numerical solution of algebraic equations, *Proc. R. Soc. Edinburgh* **46** (1926) 289–305.

[2] C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).

[3] P. Bjørstad, G. Dahlquist and E. Grosse, Extrapolation of asymptotic expansions by a modified δ^2 formula, *BIT* **21** (1981) 56–65.

[4] C. Brezinski, Accélération de suites à convergence logarithmique, *C. R. Acad. Sci Paris A* **273** (1971) 727–730.

[5] C. Brezinski, A general extrapolation algorithm, *Numer. Math.* **35** (1980) 175–180.

[6] C. Brezinski, A new approach to convergence acceleration methods, in: A. Cuyt, Ed., *Nonlinear Numerical Methods and Rational Approximation* (Reidel, Dordrecht, 1988) 373–405.

[7] C. Brezinski and M. Redivo Zaglia, *Extrapolation Methods. Theory and Practice* (North-Holland, Amsterdam, 1991).

[8] J.E. Drummond, A formula for accelerating the convergence of a general series, *Bull. Austral. Math. Soc.* **6** (1972) 69–74.

[9] J.E. Drummond, Summing a common type of slowly convergent series of positive terms, *J. Austral. Math. Soc. B* **19** (1976) 416–421.

[10] J.E. Drummond, Convergence speeding, convergence and summability, *J. Comput. Appl. Math.* **11** (1984) 145–159.

[11] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions, Vol. II* (McGraw-Hill, New York, 1953).

[12] W.F. Ford and A. Sidi, An algorithm for a generalization of the Richardson extrapolation process, *SIAM J. Numer. Anal.* **24** (1988) 1212–1232.

[13] B. Germain-Bonne, Transformations de suites, *Rev. Française Automat. Rech. Operat.* **7(R-1)** (1973) 84–90.

[14] T. Håvie, Generalized Neville type extrapolation schemes, *BIT* **19** (1979) 204–213.

[15] H.H.H. Homeier, *Integraltransformationsmethoden und Quadraturverfahren für Molekülintegrale mit B-Funktionen* (Roderer, Regensburg, 1990).

[16] H.H.H. Homeier, Some applications of nonlinear convergence accelerators, *Int. J. Quantum Chem.* **45** (1993) 545–562.

[17] H.H.H. Homeier, A hierarchically consistent, iterative sequence transformation, *Numer. Algo.* **8** (1994) 47–81.

[18] D. Levin, Development of non-linear transformations for improving convergence of sequences, *Int. J. Comput. Math. B* **3** (1973) 371–388.

[19] W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer, New York, 1966).

[20] N. Osada, A convergence acceleration method for some logarithmically convergent sequences, *SIAM J. Numer. Anal.* **27** (1990) 178–189.

[21] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators* (Academic Press, New York, 1978).

[22] A. Sidi, A new method for deriving Padé approximants for some hypergeometric functions, *J. Comput. Appl. Math.* **7** (1981) 37–40.

[23] D.A. Smith and W.F. Ford, Acceleration of linear and logarithmic convergence, *SIAM J. Numer. Anal.* **16** (1979) 233–240.

[24] E.O. Steinborn and E. Filter, Translations of fields represented by spherical-harmonic expansions for molecular calculations III. Translations of reduced Bessel function, Slater-type s-orbitals, and other functions, *Theoret. Chim. Acta* **38** (1975) 273–281.

[25] E.J. Weniger, Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series, *Comput. Phys. Rep.* **10** (1989) 189–373.

[26] E.J. Weniger, On the derivation of iterated sequence transformations for the acceleration of convergence and the summation of divergent series, *Comput. Phys. Commun.* **64** (1991) 19–45.

[27] E.J. Weniger, Interpolation between sequence transformations, *Numer. Algo.* **3** (1992) 477–486.

[28] E.J. Weniger and J. Čižek, Rational approximations for the modified Bessel function of the second kind, *Comput. Phys. Commun.* **59** (1990) 471–493.

[29] E.J. Weniger, J. Čižek and F. Vinette, Very accurate summation for the infinite coupling limit of the perturbation series expansions of anharmonic oscillators, *Phys. Lett. A* **156** (1991) 169–174.

[30] E.J. Weniger, J. Čižek and F. Vinette, The summation of the ordinary and renormalized perturbation series for the ground state energy of the quartic, sextic, and octic anharmonic oscillators using nonlinear sequence transformations, *J. Math. Phys.* **34** (1993) 571–609.

[31] J. Wimp. *Sequence Transformations and their Applications* (Academic Press, New York, 1981).

[32] P. Wynn, On a device for computing the $e_m(S_n)$ transformation, *Math. Tables Aids Comput.* **10** (1956) 91–96.

[33] P. Wynn, On a Procrustean technique for the numerical transformation of slowly convergent sequences and series, *Proc. Camb. Phil. Soc.* **52** (1956) 663–671.