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Electrohydrodynamic instability of two superposed fluids in normal electric fields

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Abstract

We consider the linear stability of two unbounded fluids that are separated by a plane interface, and stressed by initially perpendicular uniform electric field. On each side of the interface there is a Coquette flow. The fluids have different viscosities, densities, and electrical properties and surface tension acts at the interface. The linear stability of the flow is analyzed by deriving the exact dispersion relation in terms of the Airy functions and their integrals, and solving it numerically and asymptotically to find marginal stability curves. The stability of the system depends on ten parameters: the ratio of viscosities, ratio of the densities, surface tension, gravity, ratio of the permitivities, two conductivities, two equilibrium electric fields and velocity of the upper fluid in the unperturbed motion. We investigate the electric charge relaxation effects on the stability of the flow by considering various limiting cases. We also examine the effects of finite charge relaxation times.

Keywords: Electrohydrodynamic stability, Shear flow, Surface tension, Electric field

AMS classification: 76

1. Introduction

We consider the parallel flow of two fluids separated by a plane interface and stressed by perpendicular electric fields. On each side of the interface there is an unbounded Coquette flow. The fluids are assumed to have different viscosities, densities, basic velocities and electrical properties, and surface tension acts at the interface. This is an extension of the shear-flow stability problem investigated in [1] in the absence of the electric fields.

The introduction of the applied electric fields induces electromechanical effects related to the interaction of electric fields and free or polarization charges with the bulk of each fluid and their

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common interface. These effects come into play either through bulk coupling forces, or through interfacial coupling boundary conditions between the electric fields and the fluid flow quantities. The Maxwell equations lead to an exponential decay of the bulk charge density as $e^{-t/\tau}$ where the parameter τ is the electric relaxation time. In the model we develop, we assume that the charge relaxation time is sufficiently short so that the electric charge density in the bulk is essentially zero. Therefore, the bulk forces of electrical origin are negligible and, the field coupling occurs at the interface as specified by the appropriate boundary conditions.

Electrohydrodynamic instability at the interface between two fluids stressed by external electric fields has generated considerable interest due to its wide ranging scientific and engineering applications, including static and dynamic imaging [7, 11], atmospheric electrification [10], the orientation, confinement and levitation of liquids in zero gravity [4] and the separation of living and dead cells [6]. The linear electrohydrodynamic stability of the Rayleigh–Taylor problem for two inviscid dielectric superposed fluids subjected to a normal electric field has been studied by many authors including Taylor and McEwan [24], Melcher [14] and Devitt and Melcher [8]. Melcher and Smith [17] considered the viscous Rayleigh–Taylor problem and examined the dynamic interplay of the interfacial electric shear stresses and viscous stresses. The electrohydrodynamic instability of a single charge-free surface separating two semi-finite streaming invicid fluids influenced by a normal electric field was investigated by Elshehawey [9] and Mohammed et al. [19]. These problems are special cases of the shear flow electrohydrodynamic stability that is considered in this paper.

The linear stability of the flow is analyzed by deriving the exact dispersion relation in terms of the Airy functions and their integrals, and solving it numerically and asymptotically to find marginal stability curves. The stability of the system depends on ten parameters including the ratio of the viscosities, ratio of the densities, surface tension, gravity, ratio of the permitivities, two conductivities, two initial electric fields and velocity field of the upper fluid in the unperturbed motion.

In Sections 2 and 3 we formulate the stability problem and develop the dispersion relation describing the stability of the flow. In the absence of the electric fields, the dispersion relation reduces to the equivalent of the dispersion relation found in [1].

In Section 4, we consider two specific limiting cases representing configurations with no shear stresses of electrical origin. The first limit represents the configuration in which the lower fluid is highly conducting relative to the upper fluid so that the fluid interface is perfectly conducting and supports a free charge. An example of this configuration is the air–water interface which has important meteorological applications. The second limit, on the other hand, represents a class of charge interactions of purely insulating dielectrics. Here, the interface does not support any free charge and, therefore, the conduction and interface coupling are entirely due to polarization charges. This type of interaction is sometimes referred to as a dielectrophoretic phenomenon and it has applications in the orientation of cryogenic liquid propellants. For both limiting cases, we examine the effects of the streaming on the growth rates and we investigate the existence of instabilities exhibiting purely exponential growth.

In Sections 5 and 6 we consider two limiting cases in which the electromechanical effects are dominated by electrical surface shear forces. A wide range of electrohydrodynamic applications, including electro-optical image reproduction and space propulsion, involve the effects of electrical shear forces [12]. In Section 5, as in the dielectrophoretic configurations, the time scales of the surface dynamics of both fluids are short relative to the electric charge relaxation times. However, here the electromechanical interactions are dominated by free charges which relax to the interface. In Section 6 we consider the opposite case where the time scales of the surface dynamics of fluids are relatively long compared to the electric charge relaxation times so that charge relaxation is essentially instantaneous. We find that, generally, the principle of exchange of stability (i.e., the onset of a static instability exhibiting purely exponential growth), does not hold in the presence of initial streaming of the fluids. The stability of the flow in this limit is characterized by the ratio of the conductivities of the fluids.

In Section 7 we present a discussion of the effects of electrical charge relaxation times that are comparable to the surface dynamics time scales. These effects are likely to be important in cases involving surface free charges. Moreover, since most real fluids have some finite relaxation times, the above limiting cases are approximations only. Although these approximations have been quite successful in modeling many real systems [15], electric relaxation time effects are believed to have important implications in the modeling of electrohydrodynamic interactions involving bulk coupling of the electric fields and the fluid flow. The stability of the flow for these configurations are characterized by the ratio of the conductivities and the Hartmann number which is a measure of the relative effects of electric forces and mechanical forces due to viscosity and surface tension. For large Hartmann numbers, the threshold for static instability reduces to the threshold found for the case where the surface dynamics is short compared to the electric charge relaxation. For small Hartmann numbers, it reduces to the instantaneous charge relaxation limit. In general, a nonzero Hartmann number is destabilizing. However, the effects of the ratio of the conductivities are determined by the specific configurations. Finally, concluding remarks are presented in Section 8.

2. Formulation of the problem

We consider the two-dimensional flow configuration sketched in Fig. 1 of two homogeneous incompressible viscous fluids of constant viscosity μ_1 and μ_2, densities ρ_1 and ρ_2, permitivities ε_1^* and ε_2^* and conductivities σ_1^* and σ_2^*. In the unperturbed state, the interface $y^* = 0$ where x^* and y^* are the usual Cartesian coordinates, is stressed by uniform electric fields $\tilde{E}_1^*$ and $\tilde{E}_2^*$ in the $\boldsymbol{y}^*$ direction. Subscripts 1 and 2 refer to fluid properties and fluid flow quantities above and below the interface, respectively. Gravity g^* acts in the negative $\boldsymbol{y}^*$ direction. In the unperturbed state, the flow has the velocity field

$$\boldsymbol{u}^*(x^*, y^*) = \begin{cases} (a_1\tilde{\omega}y^*, 0) & \text{if } y^* > 0, \\ (a_2\tilde{\omega}y^*, 0) & \text{if } y^* < 0, \end{cases} \tag{1}$$

where $a_1\tilde{\omega}$ and $a_2\tilde{\omega}_2$ are constant vorticities above and below the interface, respectively, and the vorticity coefficients a_1 and a_2 are nondimensional constants. The continuity of shear stress at the interface requires that

$$\frac{a_1}{a_2} = \frac{\mu_2}{\mu_1}. \tag{2}$$

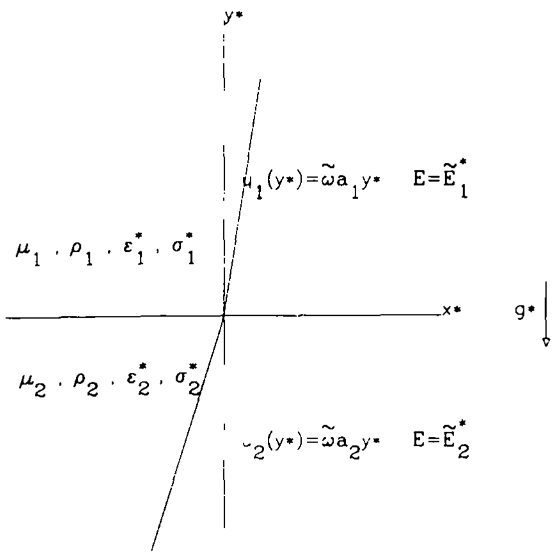

Fig. 1. Schematic representation of the problem.

2.1. The governing equations

Since large currents are not present in this flow, the effects of magnetic inductions are negligible. Hence, the electric field $\boldsymbol{E}^* = (E_x^*, E_y^*)$ is irrotational:

$$\nabla \times \boldsymbol{E}^* = 0. \tag{3}$$

The conservation of free charge requires that

$$\nabla \cdot \boldsymbol{J}^* + \frac{\partial q^*}{\partial t^*} = 0, \tag{4}$$

where J^* is the free current density and q^* is the free charge density [22]. Since the permittivity ε^* is constant, the free charge density is given by

$$q^* = \varepsilon^* \nabla \cdot \boldsymbol{E}^*. \tag{5}$$

The current density is the sum of the conduction, convection and diffusion currents. In this problem we neglect diffusion currents so that $\boldsymbol{J}^*$ can be represented by

$$\boldsymbol{J}^* = \sigma^* \boldsymbol{E}^* + \boldsymbol{u}^* q^*, \tag{6}$$

where σ^* is the electrical conductivity, which we assume to be constant, and $\boldsymbol{u}^*$ is the fluid velocity vector [25]. This is known as Ohm's conduction law. Although not obeyed by all fluids, this simplest of all conduction laws has been used to model successfully a wide range of electrohydrodynamic phenomena [15, 16].

The conservation of momentum for the flow is then given by

$$\begin{aligned}\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} &= -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} + \frac{F_{ex}}{\rho} + \nu \nabla^2 u^*,\\ \frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} &= -\frac{1}{\rho}\frac{\partial p^*}{\partial y^*} - g^* + \frac{F_{ey}}{\rho} + \nu \nabla^2 v^*,\end{aligned} \tag{7}$$

and the conservation of mass is given by

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{8}$$

where $u^*(x^*, y^*, t^*)$ is the actual fluid velocity parallel to the x^* axis, $v^*(x^*, y^*, t^*)$ is the actual velocity parallel to the y^* axis, $p^*(x^*, y^*, t^*)$ is the pressure, ν is the kinematic viscosity and $\boldsymbol{F}_e^* = (F_{ex}, F_{ey})$ is the electric force density vector.

Combining equations (4), (5), (6) and (8) we obtain the following equation for the conservation of electric charges in the presence of charge convection:

$$\frac{\mathrm{D}q^*}{\mathrm{D}t^*} + \frac{\sigma^*}{\varepsilon^*} q^* = 0, \tag{9}$$

where the material derivative $\mathrm{D}/\mathrm{D}t^* = \partial/\partial t + \boldsymbol{u}^* \cdot \nabla$. Hence, for every fluid particle there is a charge relaxation mechanism which forces the quantity q^* to relax to zero as $e^{-t/\tau}$ where $\tau = \varepsilon^*/\sigma^*$ is the charge relaxation time associated with the relaxation of free charge density. Therefore, the free charge density in the bulk of the fluid is essentially zero regardless of the fluid motion [15].

The bulk coupling force $\boldsymbol{F}_e^*$ is composed of the Coulomb force and dielectrophoric and electrorestrictive terms. It is commonly described by the general expression

$$\boldsymbol{F}_e^* = q^* \boldsymbol{E}^* - \tfrac{1}{2} E^{*2} \nabla \varepsilon^* + \nabla \left(\rho \left(\frac{\partial \varepsilon}{\partial \rho} \right)_\theta \frac{E^{*2}}{2} \right), \tag{10}$$

where the subscript θ indicates an isothermal process [22].

Therefore, unless a net free charge is injected into the fluid, there is no free charge density in the bulk of the fluid so that the Coulomb force represented by the first term in the above expression is zero. The second term also vanishes since the fluids are assumed to be homogeneous. Moreover, the electrorestrictive term can be incorporated into the hydrodynamic pressure by defining an effective pressure π^* such that

$$\pi^* = p^* - \tfrac{1}{2} \rho \left(\frac{\partial \varepsilon^*}{\partial \rho} \right)_\theta E^{*2}. \tag{11}$$

Consequently, the effect of the electric force density in the bulk equations is absorbed by the effective pressure and the field coupling occurs only at the interface region where the conductivity and permittivity are discontinuous and charge accumulates due to conduction phenomena.

The electrohydrodynamic equations and the equations of motion are nondimensionalized with respect to the lower fluid:

$$(x, y) = \left(\frac{\rho_2 \tilde{\omega}}{\mu_2}\right)^{1/2} (x^*, y^*), \qquad (u, v) = \left(\frac{\rho_2}{\tilde{\omega}\mu_2}\right)^{1/2} (u^*, v^*),$$

$$p = \left(\frac{1}{\tilde{\omega}\mu_2}\right) p^*, \qquad g = \left(\frac{\rho_2}{\mu_2 \tilde{\omega}^3}\right)^{1/2} g^*, \qquad t = \tilde{\omega} t^*,$$

$$E = \left(\frac{\varepsilon_2^*}{\tilde{\omega}\mu_2}\right)^{1/2} E^*, \qquad \sigma = \left(\frac{1}{\tilde{\omega}\varepsilon_2^*}\right) \sigma^*,$$

$$\varepsilon = \left(\frac{\varepsilon_1^*}{\varepsilon_2^*}\right), \qquad m = \frac{\mu_1}{\mu_2}, \qquad r = \frac{\rho_1}{\rho_2}.$$

Here the nondimensionalization is carried out with respect to $\tilde{\omega}$ in order to eliminate singularities at the no initial streaming limit.

We now impose small perturbations on the basic flow as follows:

$$\begin{aligned} &u_i = \tilde{u}_i + u'(x, y, t), \qquad v_i = v'(x, y, t), \qquad \pi_i = \tilde{\pi}_i + \pi'(x, y, t), \\ &E_{x_i} = E'_{x_i}(x, y, t), \qquad E_{y_i} = \tilde{E}_i + E'_{y_i}(x, y, t), \end{aligned} \tag{12}$$

where the tilde is used to indicate quantities of the basic flow and the primed quantities denote small disturbances.

Then, by introducing Eq. (12) into (7), and by linearizing (i.e. neglecting quadratic and higher order terms in small primed quantities), we obtain a system of linear partial differential equations for the disturbances whose coefficients are functions of y only. Therefore, the equations admit sinusoidal solutions which depend on x and t of the following form

$$\begin{aligned} &(\psi_i(x, y, t)) = (\phi_i(y)) e^{i\alpha(x - ct)}, \qquad (\pi'_i(x, y, t)) = (\pi_i(y)) e^{i\alpha(x - ct)}, \\ &(E'_{x_i}(x, y, t)) = (e_{x_i}(y)) e^{i\alpha(x - ct)}, \qquad (E'_{y_i}(x, y, t)) = (e_{y_i}(y)) e^{i\alpha(x - ct)}, \end{aligned} \tag{13}$$

where ψ_i are the stream functions defined by $u'_i = -\partial\psi_i/\partial y$, $v'_i = \partial\psi_i/\partial x$ and the real parts of these expressions are taken to obtain physical quantities. Boundedness of the solutions as $|x| \to \infty$ requires the wavenumber α to be real. The wave speed $c = c_r + ic_i$ represents the wave speed with an exponential growth rate αc_i.

In terms of the complex amplitudes of Eq. (13), the partial differential equations reduce to the following ordinary differential equations:

$$(D^2 - \alpha^2)^2 \phi_1 = \frac{ir\alpha}{m} (a_1 y - c)(D^2 - \alpha^2)\phi_1, \tag{14}$$

$$(D^2 - \alpha^2)^2 \phi_2 = i\alpha\, (a_2 y - c)(D^2 - \alpha^2)\phi_2, \tag{15}$$

$$(D^2 - \alpha^2) e_{x_i} = 0, \tag{16}$$

$$\alpha e_{y_i} = -\, iDe_{x_i}, \tag{17}$$

where $D = \mathrm{d}/\mathrm{d}y$ which indicates the derivative with respect to y.

Eqs. (14) and (15) are equivalent to the Orr–Sommerfeld equations obtained in [1]. However, Eq. (16) and (17) are additional equations obtained as a result of the introduction of electric fields. As pointed out earlier, the set of equations for the electric fields are not coupled with the equations for the stream function amplitude ϕ_i. The coupling occurs when the appropriate boundary conditions are applied.

2.2. Boundary conditions

In addition to the requirement that all physical quantities must tend to zero as y tends to ∞ for $i = 1$ and as y tends to $-\infty$ for $i = 2$, we must also impose interfacial boundary conditions. The kinematic condition requires that the fluids move with the common interface and that neither fluid crosses this interface. Therefore, the normal velocity of both fluids must equal the velocity of the interface whose location is described by

$$F(x, y, t) = n(x, t) - y = 0, \tag{18}$$

where the general distortion of the interface may be represented as a superposition of normal modes given by

$$\eta(x, t) = \delta e^{i\alpha(x-ct)}, \tag{19}$$

where δ is a small parameter. The kinematic condition at the interface then implies that

$$\eta(x, t) = -\frac{\phi_1(0)}{c} e^{i\alpha(x-ct)}. \tag{20}$$

Since both fluids move together with the interface, and since there is no slip between the fluids in the direction of flow, both the normal and the tangential velocities are continuous. The continuity of the normal velocity leads to

$$\phi_1(0) = \phi_2(0) = \phi(0). \tag{21}$$

Similarly, the continuity of the tangential velocities implies that

$$D\phi_1(0) - D\phi_2(0) = \frac{a_2 - a_1}{c} \phi(0). \tag{22}$$

The stress condition at the interface is a balance between the hydrodynamic pressure, the viscous stress, the surface tension and the electrical forces. It is derived in Appendix A and in nondimensionalized variables the normal component of the boundary condition can be written as

$$\begin{aligned}&\Bigg[-\pi + 2\frac{\mu}{\mu_2}\left(\frac{\partial u}{\partial x} n_x^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)n_x n_y + \frac{\partial v}{\partial y} n_u^2\right) \\ &\quad + \frac{\varepsilon}{\varepsilon_2}(E_x^2 n_x^2 + 2E_x E_y n_x n_y + E_y^2 n_y^2) - \frac{1}{2}\frac{\varepsilon}{\varepsilon_2}(E_x^2 + E_y^2) + \frac{\rho}{\rho_2} g\eta n_y\Bigg] = -\frac{S}{R'},\end{aligned} \tag{23}$$

where R' is the nondimensionalized radius of curvature,

$$S = \left(\frac{\rho_2}{\tilde{\omega}\mu_2^3}\right)^{1/2}\gamma \tag{24}$$

and (n_x, n_y) is the unit normal to the interface. The notation [ ] indicates the jump across the interface, i.e. $[X] = X_1 - X_2$. For the basic states this condition applied at $y = 0$ gives

$$\tilde{\pi}_1 - \tilde{\pi}_2 = \tfrac{1}{2}(\varepsilon\tilde{E}_{y1}^2 - \tilde{E}_{y2}^2). \tag{25}$$

When we apply the linearized form of the electric field we get the following linearized condition at $y = 0$:

$$\begin{aligned} &- i\alpha(1 - r)(cD\phi_2(0) + a_2\phi_2(0)) + i\alpha(S\alpha^2 + (1 - r)g)\,\frac{D\phi_1(0) - D\phi_2(0)}{a_2 - a_1} \\ &\quad = - m(D^3 - 3\alpha^2 D)\phi_1(0) + (D^3 - 3\alpha^2 D)\phi_2(0) + i(\varepsilon\tilde{E}_1 e_{y1} - \tilde{E}_2 e_{y2}). \end{aligned} \tag{26}$$

Similarly, the tangential component of the boundary condition is derived in Appendix A and the linearized form at $y = 0$ can be written as

$$m\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right) - \left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x}\right) + \frac{\partial \eta}{\partial x}(\varepsilon\tilde{E}_1^2 - \tilde{E}_2^2) + \varepsilon\tilde{E}_{y1}e_{x1} - \tilde{E}_{y2}e_{x2} = 0. \tag{27}$$

In terms of the complex amplitudes, this reduces to

$$\begin{aligned} &m(D^2\phi_1(0) + \alpha^2\phi_1(0)) - D^2\phi_2(0) + \alpha^2\phi_2(0) \\ &\quad = - \tilde{E}_{y2}e_{x2} + \varepsilon\tilde{E}_{y1}e_{x1} + \frac{\phi_1(0)\alpha}{ic}(\varepsilon\tilde{E}_1^2 - \tilde{E}_2^2) \end{aligned} \tag{28}$$

which is another coupling equation.

Furthermore, the integration of Eq. (3)–(6) across the interface yields the following conditions

$$\boldsymbol{n} \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0, \tag{29}$$

$$\boldsymbol{n} \cdot (\sigma_1 \boldsymbol{E}_1 - \sigma_2 \boldsymbol{E}_2) + \nabla_\Sigma \cdot (\rho_f^* \boldsymbol{u}) + \frac{\partial \rho_f^*}{\partial t^*} = 0, \tag{30}$$

where

$$\rho_f^* = \boldsymbol{n} \cdot (\varepsilon \boldsymbol{E}_1 - \boldsymbol{E}_2) \tag{31}$$

and $\nabla_\Sigma \cdot (\rho_f^* \boldsymbol{u})$ is the surface divergence of the current density $\rho_f^* \boldsymbol{u}$. These equations along with the stress conditions provide the coupling mechanism between the fluid flow quantities and the electrical quantities.

Eq. (29) can be written as

$$n_x(E_{y1} - E_{y2}) = n_y(E_{x1} - E_{x2}) \tag{32}$$

which by linearization becomes

$$- \eta_x P = e_{x1} - e_{x2}, \tag{33}$$

where

$$P = \tilde{E}_1 - \tilde{E}_2 \tag{34}$$

so that we get

$$e_{x1} - e_{x2} = \frac{\phi_1(0)P}{c}. \tag{35}$$

Similarly, from Eq. (30)

$$n_x(\sigma_1 E_{x1} - \sigma_2 E_{x2}) + n_y(\sigma_1 E_{y1} - \sigma_2 E_{y2}) + \nabla_\Sigma \cdot (n_x(\varepsilon E_{x1} - E_{x2}) + n_y(\varepsilon E_{y1} - E_{y2}))$$
$$+ \frac{\partial}{\partial t}(n_x(\varepsilon E_{x1} - E_{x2}) + n_y(\varepsilon E_{y1} - E_{y2})) = 0 \tag{36}$$

which can be linearized about $y = 0$ to give

$$(\sigma_1 \tilde{E}_1 - \sigma_2 \tilde{E}_2) + \sigma_1 e_{y1} - \sigma_2 e_{y2} + Q\frac{\partial \tilde{u}}{\partial x} + Q\frac{\partial u'}{\partial x} + Q\frac{\partial v'}{\partial x} + \frac{\partial}{\partial t}e_{y2} - \varepsilon\frac{\partial}{\partial t}e_{y1} = 0, \tag{37}$$

where

$$Q = \varepsilon \tilde{E}_1 - \tilde{E}_2. \tag{38}$$

The tilde terms add to zero and the v' term vanishes so that we get

$$\sigma_1 e_{y1} - \sigma_2 e_{y2} - \alpha \mathrm{i} Q D\phi_1 = -\frac{\partial e_{y2}}{\partial t} + \varepsilon\frac{\partial e_{y1}}{\partial t}. \tag{39}$$

3. The dispersion relation

Eqs. (14) and (15) can be solved exactly in terms of the Airy functions [13]. In order to solve the differential equations given in the last section we make the following changes of variables:

$$z_1 = m^{-1/3} a_1^{1/3} r^{1/3} \alpha^{1/3} \mathrm{e}^{-\mathrm{i}\pi/2}(y - ca_1^{-1} - \mathrm{i}\alpha r^{-1} m a_1 - 1),$$
$$z_2 = \alpha^{1/3} a_2^{1/3} \mathrm{e}^{-\mathrm{i}\pi/2}\left(y - \frac{c}{a_2} - \mathrm{i}\alpha a_2^{-1}\right), \tag{40}$$
$$\xi_i(z_i) = (\alpha^2 - D^2)\phi_i(y).$$

Hence, ξ_i represents the complex amplitude of the disturbance vorticity. Then, in terms of ξ_i, Eqs. (14) and (15) become

$$\frac{\mathrm{d}^2\xi_1}{\mathrm{d}z_1^2} - z_1\xi_1 = 0, \tag{41}$$

$$\frac{\mathrm{d}^2\xi_2}{\mathrm{d}z_2^2} - z_2\xi_2 = 0. \tag{42}$$

These equations are in the form of the Airy equation and therefore their solutions are given by

$$\xi_1 = b_1\, Ai(z_1) + c_1\, Ai(z_1 e^{\theta_1}), \tag{43}$$

$$\xi_2 = b_2\, Ai(z_2) + c_2\, Ai(z_2 e^{\theta_2}), \tag{44}$$

where Ai denotes the Airy function and $\theta_i = 2\pi/3$ or $-2\pi/3$ [2]. Then, the boundary conditions at infinity imply that the vorticities must tend to zero as $y \to \infty$ or as $y \to -\infty$, so that $b_1 = b_2 = 0$, $\theta_1 = 2\pi/3$ and $\theta_2 = -2\pi/3$. Therefore,

$$\xi_1 = c_1 A_1(y), \tag{45}$$

$$\xi_2 = c_2 A_2(y), \tag{46}$$

where

$$A_1(y) = Ai(z_1 e^{2\pi/3}) = Ai\left(m^{-1/3} a_1^{1/3} r^{1/3} \alpha^{1/3}\left(y - \frac{c}{a_1} - \mathrm{i}\alpha r^{-1} m a_1^{-1}\right) e^{\mathrm{i}\pi/6}\right),$$

$$A_2(y) = Ai(z_2 e^{-2\pi/3}) = Ai\left(a_2^{1/3} \alpha^{1/3}\left(y - \frac{c}{a_2} - \mathrm{i}\alpha a_2^{-1}\right) e^{5\mathrm{i}\pi/6}\right).$$

Consequently, we obtain the following equations for ϕ_1 and ϕ_2:

$$(D^2 - \alpha^2)\phi_1 = -c_1 A_1(y), \tag{47}$$

$$(D^2 - \alpha^2)\phi_2 = -c_2 A_2(y). \tag{48}$$

After solving these second order linear differential equations with the boundary conditions at infinity we obtain the following expressions for the stream functions:

$$\phi_1 = c_3 e^{-\alpha y} + c_1' \alpha\left(e^{-\alpha y} \int_0^y e^{\alpha s} A_1(s)\, ds + e^{\alpha y} \int_y^{\infty} e^{-\alpha s} A_1(s)\, ds\right), \tag{49}$$

$$\phi_2 = c_4 e^{\alpha y} + c_2' \alpha\left(e^{\alpha y} \int_0^y e^{-\alpha s} A_2(s)\, ds + e^{-\alpha y} \int_y^{-\infty} e^{\alpha s} A_2(s)\, ds\right), \tag{50}$$

where c_3 and c_4 are constants, and

$$c_1' = -\frac{c_1}{2\alpha^2}, \qquad c_2' = \frac{c_2}{2\alpha^2}.$$

Similarly, by solving (16) and (17) with conditions at infinity we obtain

$$e_{x1} = c_5 e^{-\alpha y}, \tag{51}$$

$$e_{x2} = c_6 e^{\alpha y}, \tag{52}$$

$$e_{y1} = \mathrm{i} c_5 e^{-\alpha y}, \tag{53}$$

$$e_{y2} = -\mathrm{i} c_6 e^{\alpha y}. \tag{54}$$

Finally, applying the remaining six boundary conditions given by the Eqs. (19)–(24), we obtain six linear equations for the six unknown constants c_1, c_2, c_3, c_4, c_5 and c_6. The linear homogeneous system of equations can then be written as

$$\boldsymbol{A}\boldsymbol{h} = 0, \tag{55}$$

where $\boldsymbol{h}^{\mathrm{T}} = (c_3, c_4, c'_1, c'_2, c'_5, c'_6)$, $c'_5 = -c_5/\alpha$ and $c'_6 = -c_6/\alpha$.

The matrix A is given by

$$A = \begin{pmatrix} 1 & -1 & J_1 & J_2 & 0 & 0 \\ -\alpha c + a_1 & -\alpha c - a_2 & (\alpha c + a_1)J_1 & (-\alpha c + a_2)J_2 & 0 & 0 \\ \phi_{3,1} & -2\alpha c & \phi_{3,3} & (2J_2 - 2A_2)\alpha c & \varepsilon c\tilde{E}_1 & -c\tilde{E}_2 \\ \phi_{4,1} & \phi_{4,2} & \phi_{4,3} & \phi_{4,4} & (a_2 - a_1)\varepsilon\tilde{E}_1 & (a_2 - a_1)\tilde{E}_2 \\ P\mathrm{i} & 0 & PJ_1\mathrm{i} & 0 & c & -c \\ -\mathrm{i}\alpha Q & 0 & \mathrm{i}\alpha J_1 Q & 0 & c\varepsilon + \mathrm{i}\sigma_1 & c + \mathrm{i}\sigma_2 \end{pmatrix}.$$

Here

$$\phi_{3,1} = 2m\alpha c + \mathrm{i}(\varepsilon\tilde{E}_1^2 - \tilde{E}_2^2),$$

$$\phi_{3,3} = m(2J_1 - 2A_1)\alpha c + \mathrm{i}J_1(\varepsilon\tilde{E}_1^2 - \tilde{E}_2^2),$$

$$\phi_{4,1} = 2m(a_2 - a_1)\alpha - \mathrm{i}(S\alpha^2 + (1-r)g),$$

$$\phi_{4,2} = 2(a_2 - a_1)\alpha - \mathrm{i}(S\alpha^2 + (1-r)g) - \mathrm{i}(a_2 - a_1)\alpha^{-1}(1-r)(\alpha c + a_2),$$

$$\phi_{4,3} = -2m(J_1 + A'_1)(a_2 - a_1)\alpha + \mathrm{i}J_1(S\alpha^2 + (1-r)g),$$

$$\phi_{4,4} = 2(J_2 - A'_2)(a_2 - a_1)\alpha - \mathrm{i}J_2(S\alpha^2 + (1-r)g) - \mathrm{i}(a_2 - a_1)\alpha^{-1}(1-r)(\alpha c - a_2)J_2,$$

and

$$J_1 = \alpha\int_0^{\infty} \mathrm{e}^{-\alpha s}A_1(s)\,\mathrm{d}s, \qquad J_2 = \alpha\int_0^{\infty} \mathrm{e}^{-\alpha s}A_2(-s)\,\mathrm{d}s,$$

$$A_1 = A_1(0), \qquad A_2 = A_2(0), \qquad A'_1 = \frac{1}{\alpha}\frac{\mathrm{d}A_1(y)}{\mathrm{d}y}\bigg|_{y=0}, \qquad A'_2 = \frac{1}{\alpha}\frac{\mathrm{d}A_2(y)}{\mathrm{d}y}\bigg|_{y=0}.$$

For a nontrivial solution of Eq. (55) we require the determinant of A to vanish. This gives the following dispersion relation relating the eigenvalues c and the nondimensional quantities α, a_1, a_2, m, r, g, S, ε, σ_1, σ_2, $\tilde{E}_1$ and $\tilde{E}_2$:

$$F(\alpha, c, a_1, a_2, m, r, g, S, \varepsilon, \sigma_1, \sigma_2, \tilde{E}_1, \tilde{E}_2) = 0, \tag{56}$$

where

$$F = F_1 + \frac{F_2}{\alpha c} + \frac{PF_3 + \mathrm{i}QF_4}{\alpha c(\varepsilon + 1) + \mathrm{i}(\sigma_1 + \sigma_2)} + \frac{\varepsilon\tilde{E}_1^2 - \tilde{E}_2^2}{\alpha c}F_5, \tag{57}$$

and

$$F_1 = 2(m-1)J_1(A_2 + A_2') + 2m(A_1'A_2 - A_2'A_1) + 2m(m-1)(A_1' - A_1)J_2 + 4(1-m)^2 J_1J_2 + \mathrm{i}(1-r)\alpha^{-2}((2(m-1)J_1J_2 + (\alpha c + a_2)A_2J_1 + (a_1 - \alpha c)A_1J_2)),$$

$$F_2 = 2(a_2 - a_1)m(J_1A_2 + A_1J_2) - \mathrm{i}\alpha(S + \alpha^{-2}(1-r)g)(mJ_2A_1 + J_1A_2) + m(a_2 - a_1)A_1'A_2 - m(a_2 - a_1)A_1'A_1,$$

$$F_3 = (A_2J_1 + mA_1J_2)\left(\frac{\tilde{E}_2\sigma_1 - \sigma_2\varepsilon\tilde{E}_1}{\alpha c} + \mathrm{i}\varepsilon P\right) + (mA_1'J_2 + A_2'J_1) + 2(m-1)J_1J_2 + \mathrm{i}c\alpha^{-1}(1-r)J_1J_2)\left(\frac{\varepsilon\tilde{E}_1\sigma_2 + \tilde{E}_2\sigma_1}{\alpha c} - \mathrm{i}\varepsilon(\tilde{E}_1 + \tilde{E}_2)\right),$$

$$F_4 = \left(\left(1 + \frac{a_2 - a_1}{\alpha c}\right)J_1A_2 + 2(m-1)J_1J_2 - mJ_2A_1\right)(\varepsilon\tilde{E}_1 + \tilde{E}_2)) - Q\left(\left(1 + \frac{a_2 - a_1}{\alpha c}\right)A_2'J_1 + \frac{\mathrm{i}\alpha S + \alpha^{-1}(1-r)g}{\alpha c}J_1J_2 - \frac{2(a_2 - a_1)}{\alpha c}J_1J_2 - mA_1'J_2 - \mathrm{i}a_1\alpha^{-2}(1-r)J_1J_2 - \mathrm{i}\frac{\varepsilon\tilde{E}_1^2 + \tilde{E}_2^2}{\alpha c}J_1J_2\right),$$

$$F_5 = \mathrm{i}(A_2'J_1 + mA_1'J_2 + 2(m-1)J_1J_2) - \alpha^{-1}(1-r)J_1J_2c.$$

Note that, in the limit of no electric fields, this equation reduces to the dispersion relation discussed in [1], and in the limit of no streaming, it reduces to the dispersion relation discussed in [1] and in the limit of no streaming, it reduces to the dispersion relation discussed in [17].

To analyze the stability of this problem we investigate the dependence of the eigenvalues c on the various stability parameters. Since the effects of m, r, g and S have been studied in [1], we will examine the solutions of this dispersion relation and investigate the stability of the flow with respect to the electrical stability parameters such as ε, σ_i and $\tilde{E}_i$ in the following sections.

4. Free charge (EH-If) and polarization charge (EH-Ip) configurations

In this section we consider two configurations, each representing a specific type of charge interaction phenomena, in which there are no shear stresses of electrical origin. The free charge configurations (EH-If) represent the limiting cases in which the fluid interface is perfectly conducting and supports a free charge Q which may be induced on a conducting film at the interface by externally applied electric fields. In practice, this configuration represents cases in which one fluid has much greater conductivity than the other. If the lower fluid is highly conducting relative to the upper fluid, then the electric field is confined to region 1 and $\tilde{E}_2 = 0$. An important example of this case is the air–water interface which has attached so much interest due to its meteorological applications [10]. In this limit the dispersion relation reduces to

$$F_1 + \frac{F_2}{\alpha c} + \frac{\varepsilon \mathrm{i}\tilde{E}_1^2(A_2J_1 + mA_1J_2)}{\alpha c} = 0. \tag{58}$$

On the other hand, in the polarization charge configuration (EH-Ip), sometimes termed dielectrophoretic phenomenon [20] there is no free charge on the interface ($Q = 0$). Therefore, there are no effects of free charge and hence of conduction and the coupling is entirely due to polarization. Regardless of the interfacial deformations, the shear forces of electrical origin represented by the right-hand side of Eq. (28) are zero in these important classes of interactions. Therefore, the surface forces of electrical origin always act perpendicular to the interface. The physical mechanisms of the interactions are discussed in [15]. In addition to its application in the separation of living and dead cells [6], and in understanding ferrohydrodynamic phenomena in ferrofluids [21], this class of polarization interaction has important applications because of its possibilities for solving orientation problems of cryogenic liquid propellants in the zero-gravity environment of space [4].

We further assume that both fluids are perfectly insulating and that the time scales of the surface dynamics is relatively short compared to the electric charge relaxation times which are given by

$$\tau_1 = \frac{\varepsilon}{\sigma_1}, \qquad \tau_2 = \frac{1}{\sigma_2}. \tag{59}$$

In our stability analysis where we assumed an instability dynamics of the form $\mathrm{e}^{\mathrm{i}\alpha(x-ct)}$, the time scale τ_s is given by

$$\tau_s = 1/\alpha|c|.$$

Since $\tau_s \ll \tau_1$ and $\tau_s \ll \tau_2$ the dispersion relation reduces to

$$F_1 + \frac{F_2}{\alpha c} + \frac{\mathrm{i}\tilde{E}_1\tilde{E}_2(1-\varepsilon)^2(A_2J_1 + mA_1J_2)}{(\varepsilon+1)c\alpha} = 0. \tag{60}$$

Eqs. (58) and (60) are analogous to (20) and (22), respectively, [17]. The latter equations were obtained by assuming that there is no streaming of the fluids. In general, the dispersion relations cannot be solved analytically. However, we can obtain an asymptotic expression for c as $\alpha \to \infty$. Following the asymptotic methods described in [1], we assume that

$$c = c_0 + c_1\alpha^{-1} + c_2\alpha^{-2} + o(\alpha^{-2}). \tag{61}$$

Then the Airy functions and their integrals can be approximated by

$$\begin{aligned}
\frac{A_1'}{A_1} &\approx -1 + \frac{\mathrm{i}c_0 r}{2m\alpha} + \frac{4rc_1\mathrm{i} - c_0^2r^2 - 2a_1r\mathrm{i}}{8m\alpha^2},\\
\frac{A_2'}{A_2} &\approx 1 - \frac{\mathrm{i}c_0}{2\alpha} - \frac{4c_1\mathrm{i} - c_0^2 + 2a_2\mathrm{i}}{8\alpha^2},\\
\frac{J_1}{A_1} &\approx \frac{1}{2} + \frac{\mathrm{i}c_0 r}{8\alpha} + \frac{2rc_1\mathrm{i} - c_0^2r^2 - 2a_1r\mathrm{i}}{16m\alpha^2},\\
\frac{J_2}{A_2} &\approx \frac{1}{2} + \frac{\mathrm{i}c_0}{8\alpha} + \frac{2c_1\mathrm{i} - c_0^2 + 2a_2\mathrm{i}}{16\alpha^2}.
\end{aligned} \tag{62}$$

Substituting these approximations in the dispersion relations we obtain the following expressions for the coefficients c_i

$$c_0 = -\frac{\mathrm{i}S}{2(1+m)}, \tag{63}$$

$$c_1 = \mathrm{i}\frac{\Gamma^e}{2(1+m)} - \frac{3}{16}\frac{\mathrm{i}(1+r)S^2}{(1+m)^3}, \tag{64}$$

$$c_2 = -g\frac{(1-r)\mathrm{i}}{2(1+m)} + \frac{3}{8}\frac{(1+r)\mathrm{i}\Gamma^e S}{(1+m)^3} + \frac{(-5a_1r - 3ma_1r + 3a_2 + 5ma_2)S}{8(1+m)^3}$$
$$-\frac{\mathrm{i}(1+20m+20mr^2+m^2r^2+34mr)S^3}{128m(1+m)^5}, \tag{65}$$

where

$$\Gamma^e = \begin{cases} \varepsilon\tilde{E}_1^2 & \text{for the EH-If case,} \\ \dfrac{\tilde{E}_1^2\varepsilon(1-\varepsilon)^2}{(\varepsilon+1)} & \text{for the EH-Ip case.} \end{cases} \tag{66}$$

The effect of the electric field represented by the quantity Γ^e is to destabilize the interface since this quantity is always nonnegative. If the heavier fluid is on the bottom such that gravity stabilizes the system as the electric field is raised, then there is a critical value when the interface first becomes unstable. For the EH-Ip configuration Γ^e vanishes when $\varepsilon = 1$. This is to be expected, since in this configuration the free charge $Q = \varepsilon\tilde{E}_1 - \tilde{E}_2$ is zero and the polarization charge $Q_p = \varepsilon(\tilde{E}_1 - \tilde{E}_2)$ is also zero when $\varepsilon = 1$. To this order of approximation, the term, containing the streaming coefficients a_1 and a_2, is purely real so that, in the short wavelength limit, the streaming does not have any effect on the stability of the flow. If these coefficients are large, we must obtain higher order terms in order to determine the effects of the streaming for short wavelength instabilities.

Similarly, in the long wavelength limit, asymptotic analyses yield

$$c = \frac{a_2 - ra_1}{\alpha(1+r)} - \frac{(1-r)g}{a_2 - a_1r} - \frac{(a_1+a_2)^2(r-1)^2\sqrt{m}\sqrt{r}(\mathrm{i}-1)}{\sqrt{2}(a_2-a_1r)^{3/2}(1+r)^{3/2}(\sqrt{mr}+1)} + \mathrm{o}(1). \tag{67}$$

Therefore, in the long wavelength limit, the growth rate of the instabilities is determined by inertia terms only. In general, the electric field does not affect the stability behaviour of the flow.

4.1. The principle of exchange of stabilities

For a given set of stability parameters, the temporal evolution of each disturbance mode is governed by the sign of the imaginary part of c, c_i. If $c_i < 0$ for all wavenumbers, the disturbances decay exponentially and the flow is classified as stable. On the other hand, if $c_i > 0$ then separates the stable and the unstable modes of disturbances. In the marginal states, two different behaviours are observed depending on whether c_r is zero or not. If the marginal state is represented by $c_r = 0$, then it is characterized by static instability and we say that the *principal of exchange of stability* is valid. If the marginal state corresponds to $c = c_r \neq 0$, then we will have oscillatory instability. This is called *over stability*.

We now consider the possibility that the marginal states of the above limiting cases of our problem are characterized by static instability. Therefore, as $c \to 0$ in Eq. (58) we obtain the following conditions for the incipience of static instability for the (EH-If) charge configuration

$$\alpha^2 - \frac{\varepsilon \tilde{E}_1^2}{S}\alpha + \alpha^{*2} + \mathrm{i}(a_2 - a_1)\frac{Z_1}{S}\alpha = 0, \tag{68}$$

where

$$Z_1 = \frac{m(2J_1A_2 + 2A_1J_2 + A_1'A_2 - A_2'A_1)}{A_2J_1 + mA_1J_2}, \tag{69}$$

and

$$\alpha^{*2} = \frac{(1-r)g}{S}. \tag{70}$$

Similarly, for the (EH-Ip) configuration we obtain the following condition

$$\alpha^2 - \frac{\tilde{E}_1\tilde{E}_2(1-\varepsilon)^2}{(\varepsilon+1)S}\alpha + \alpha^{*2} + \mathrm{i}(a_2 - a_1)\frac{Z_1}{S}\alpha = 0, \tag{71}$$

where the Airy functions are evaluated with $c = 0$ in their argument.

In the absence of streaming, $a_1 = a_2 = 0$ so the Z_1 term in the above equation does not make any contribution. In this case, the minimum electric fields required for the incipience of instability obtained from Eqs. (68) and (71) are

$$\tilde{E}_1^* = \left(\frac{2\alpha^* S}{\varepsilon}\right)^{1/2} \tag{72}$$

for the (EH-If) case and

$$P^* = \left(\frac{2\alpha^* S(1+\varepsilon)}{\varepsilon}\right)^{1/2} \tag{73}$$

for the (EH-Ip) case. When the electric field is raised to these critical values, the first unstable mode occurs at the critical wavenumber α^*. This is consistent with the result found in [17].

However, in the presence of streaming, the Z_1 term does not necessarily vanish and the principle of exchange of stability is valid only if $\mathrm{Re}(Z_1) = 0$. If the incipience of instability occurs at large values of α, then by utilizing the asymptotic expressions (62) we obtain the following condition for the exchange of stabilities

$$\alpha^3 - \frac{\Gamma^e}{S}\alpha^2 + \alpha^{*2}\alpha - 2a_1^2 V = 0, \tag{74}$$

where

$$V = \frac{(1-m)(r-m^2)}{S(1+m)}. \tag{75}$$

If a_1/α^* is small, then the incipience of static instability occurs at

$$\hat{\alpha} = \alpha^* - 4\,\frac{a_1^2 V}{\alpha^{*2}} + o\left(\left(\frac{a_1}{\alpha^*}\right)^2\right) \tag{76}$$

and the corresponding critical electric fields will be

$$\hat{\tilde{E}}_1 = \tilde{E}_1^* + \frac{\tilde{E}_1^*(\alpha^* - \hat{\alpha})}{8\alpha^*} \tag{77}$$

for the (EH-If) case and

$$\hat{P}^* = P^* + \frac{P^*(\alpha^* - \hat{\alpha})}{8\alpha^*} \tag{78}$$

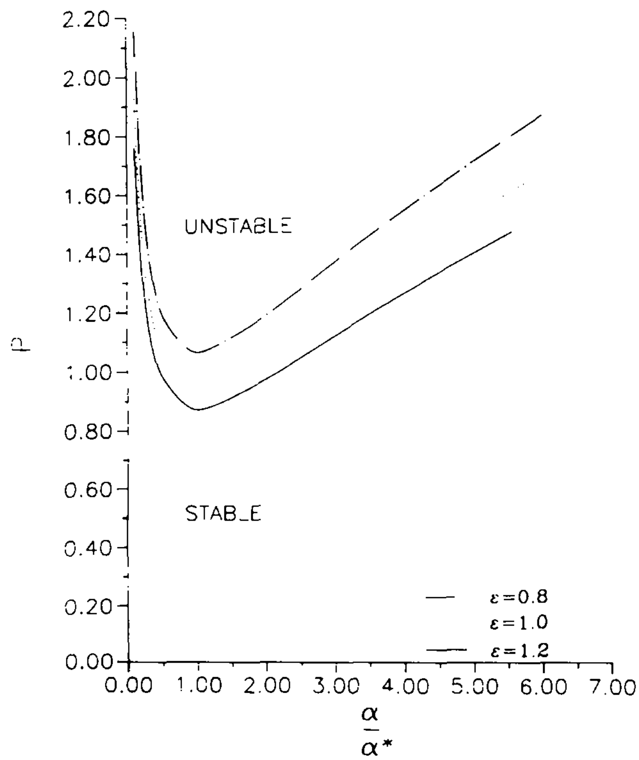

Fig. 2. Marginal stability curves for the EH-If configuration with $\alpha^* = 4.47$, $a_1 - a_2 = 0.5$ and $V = 0.83$.

for the (EH-Ip) case. Therefore, if V is positive, then the critical electric fields required for the incipience of static instability are reduced by the initial streaming and the instability occurs at a lower wavenumber $\hat{\alpha}$.

Figs. 2 and 3 are examples of the marginal stability curves for the (EH-If) and (EH-Ip) configurations, respectively. In both cases, $\alpha^* = \sqrt{20}$, $m = 0.5$, $a_1 - a_2 = 0.5$, and $V = 0.83$. In the absence of the applied electric fields the flow is stable. The curves are computed using the numerical procedures described in [1]. The dispersion relation is directly solved using both Newton's method and an IMSL subroutine based on Muller's method. Results obtained by these two methods differ by less than $O(10^{-7})$. The Airy functions are computed using an algorithm due to Corless et al. [5]. The integration required in the evaluation of J_1 and J_2 was carried out using two different methods. Since the integrands are of the Gauss–Laguerre type, the first method we used was the Gauss–Laguerre quadrature formulae [23]. The second method was an IMSL numerical integration subroutine based on a globally adaptive scheme. It initially transforms the semi-infinite interval into the finite interval [0, 1], and then uses a 21-point Gauss–Kronrod rule to estimate the integral and the associated error. The results found by the two integration methods were in excellent agreement.

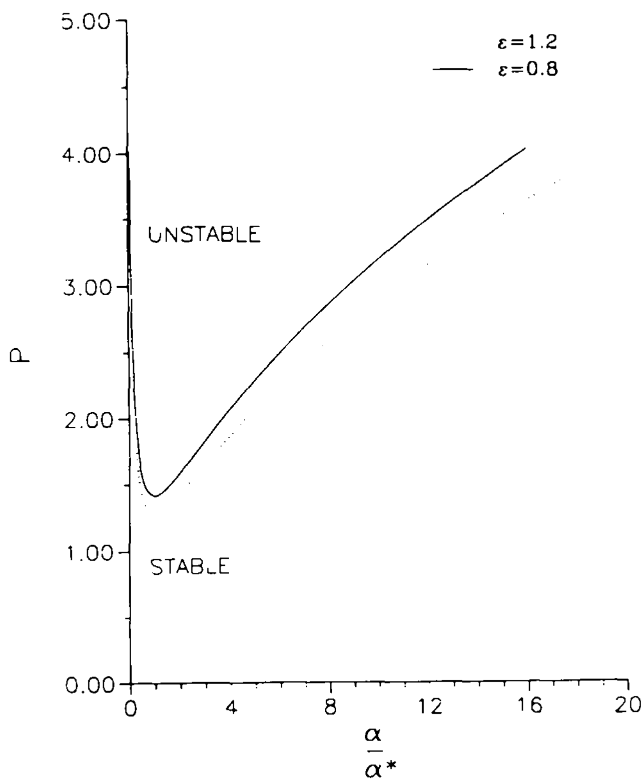

Fig. 3. Marginal stability curves for the EH-Ip configuration with $\alpha^* = 4.47$, $a_1 - a_2 = 0.5$ and $V = 0.83$.

Since α^* is large, the effect of the streaming is small as discussed above. This is demonstrated in Fig. 2 where, as the electric fields increased, Eq. (76) predicts instability to occur at $\alpha/\alpha^* = 0.991$. The critical values of the electric fields corresponding to this wave number are 0.861, 0.944 and 1.056 for $\varepsilon = 1.2$, $\varepsilon = 1.0$ and $\varepsilon = 0.8$, respectively. Even in the presence of such moderate streaming, the agreement between the computed critical values and the predicted values is quite remarkable. Similarly, in Fig. 3, the critical electric fields 1.417 and 1.280 corresponding to $\varepsilon = 0.8$ and $\varepsilon = 1.2$, respectively, predicted by Eq. (78), are in close agreement with the computed curve.

We have also compared the asymptotically computed hyperbolas that are described by Eqs. (68) and (71) with the numerically computed curves. The hyperbolas essentially overlap with the numerically computed curves in both the (EH-If) and the (EH-Ip) cases.

Finally, the destabilizing effects of the streaming for positive values of the group of parameters V, defined by Eq. (75) are demonstrated in Fig. 4. As $a_1 - a_2$ increases, the configurations become more and more unstable until the flow becomes unstable even in the absence of the electric field. Fig. 4 depicts $a_1 - a_2$ and the electric field required for a marginal state for the disturbance of wavenumber α^*. As the streaming increases, the electric field required for destabilizing α^* goes to zero.

Fig. 5 demonstrates that, as $a_1 - a_2$ increases, the equations for the marginal stability curves have non-zero real parts. It is a plot of the real part of c corresponding to the marginal state at α^* as

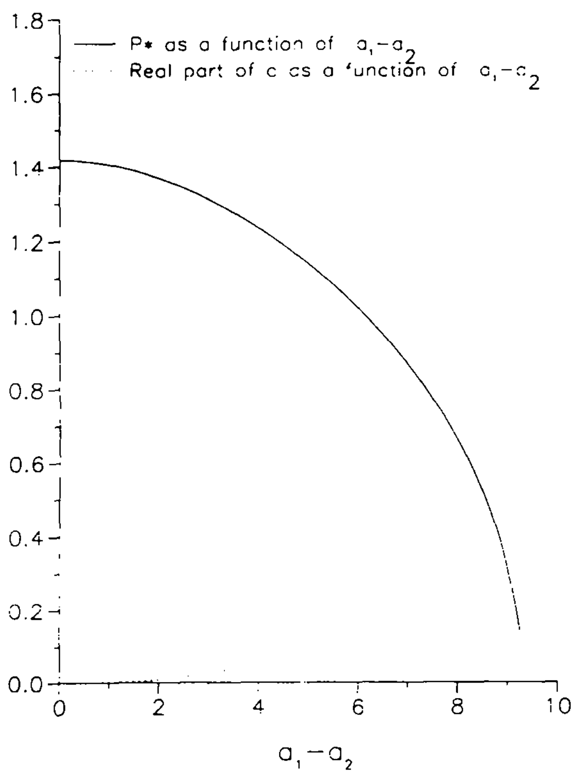

Fig. 4. Critical electric field P^* and the real part of c for marginal stability at $\alpha^* = 4.47$ as a function of $a_1 - a_2$.

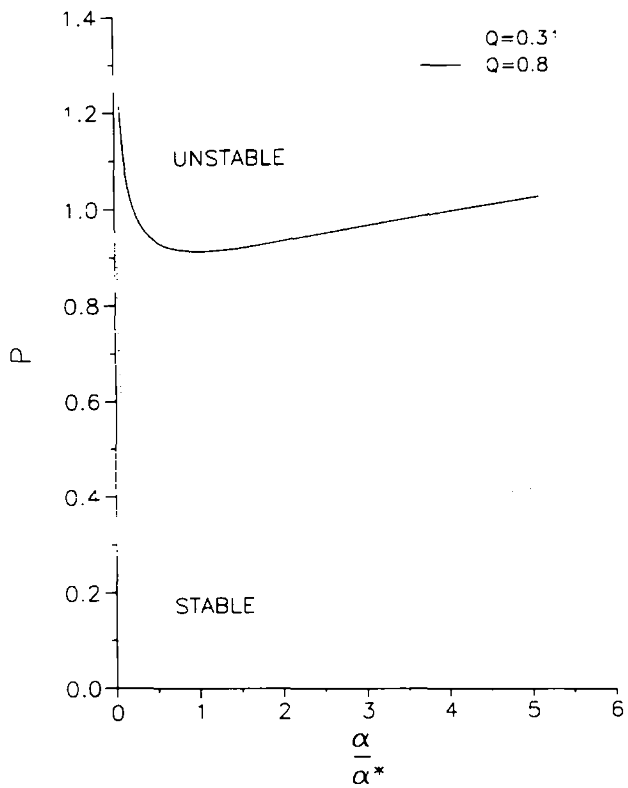

Fig. 5. Marginal stability curves for the case where $1/c = o(\tau)$, $a_1 - a_2 = 0.05$ and two different values of Q.

a function of the parameter $a_1 - a_2$. Consequently, for a large value of this parameter the instability develops in the form of oscillations of increasing amplitude and the stability Eqs. (68) and (71) obtained by setting both the real and imaginary parts of c are not valid.

5. (EH-Ip) configuration with non-zero free charge Q

Here we consider the (EH-Ip) configuration again but the surface now supports free charge Q. As before the time scales of the surface dynamics is relatively shorter compared to the electric charge relaxation times of both fluids. However, unlike the limiting cases discussed in the previous section, here interfacial electrical stresses are present and dominate the surface interactions. A wide range of applications including static and dynamic image reproduction and space propulsion, involve electrical relaxation and electrical shear effects [12]. In this limit, the dispersion relation reduces to

$$F = F_1 + \frac{F_2}{\alpha c} + \frac{P\hat{F}_3 + \mathrm{i}QF_4}{\alpha c(\varepsilon + 1)} + \frac{(\varepsilon \tilde{E}_1^2 - \tilde{E}_2^2)}{\alpha c} F_5 = 0, \tag{79}$$

where $\hat{F}_3$ is the same as F_3 with $\sigma_1 = 0$ and $\sigma_2 = 0$. In the limit where $c \to 0$, the above expression further reduces to

$$\alpha^2 - \frac{\varepsilon\tilde{E}_1^2 + \tilde{E}_2^2}{S}\,\alpha + \alpha^{*2} + \mathrm{i}(a_2 - a_1)\,\frac{Z_2}{S}\,\alpha = 0, \tag{80}$$

where

$$Z_2 = \frac{a_2(\varepsilon\tilde{E}_1 + \tilde{E}_2) - (A_2' - 2J_2)Q}{J_2}. \tag{81}$$

Therefore, in general, an exchange of stabilities is not possible in the presence of streaming.

If there is no streaming of the fluids, then the principle exchange of stability holds and the marginal stability curves are given by

$$\alpha^2 - \frac{\varepsilon\tilde{E}_1^2 + \tilde{E}_2^2}{S}\,\alpha + \alpha^{*2} = 0. \tag{82}$$

Therefore, the minimum electric fields for the incipience of instability must satisfy the relation

$$\varepsilon\tilde{E}_1^2 + \tilde{E}_2^2 = 2\alpha^* S. \tag{83}$$

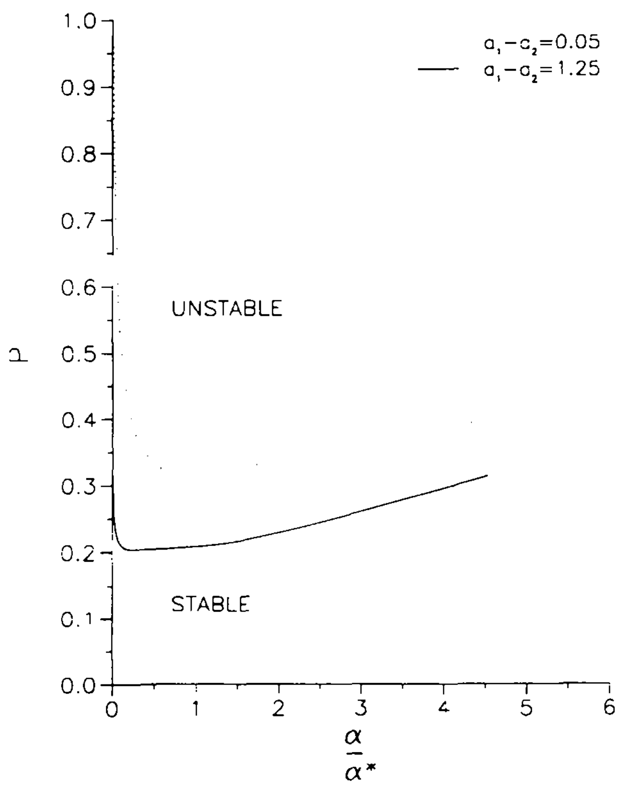

Fig. 6. Marginal stability curves for the case where $1/c = o(\tau)$, $Q = 0.31$ and two different values of $a_1 - a_2$.

In terms of P and Q this condition is given by

$$\varepsilon P^2 + Q^2 - \frac{4\varepsilon PQ}{1+\varepsilon} = \frac{2\alpha^* S(1-\varepsilon)^2}{1+\varepsilon} \tag{84}$$

which is equivalent to the expression (32) of [17]. For any value of ε, Eq. (84) represents a rotated ellipse on the P–Q plane. For values of P and Q inside this ellipse the flow is always stable. However, P and Q values outside the ellipse represent unstable configurations.

For a configuration with $\alpha^* = \sqrt{20}$, $S = 0.1$, $a_1 - a_2 = 0.05$, $\varepsilon = 0.8$ and an equilibrium surface charge of 0.14, Eq. (84) yields the minimum value of P required for the onset of static instability to be about 0.32. Fig. 6 depicts the marginal stability curves for two different values of Q. The curves are computed using the numerical procedure and they are consistent with the predicted analytical values for the critical values of α and P for the incipience of static instability.

As discussed above, however, in the presence of initial motion the induced instability is not necessarily static and the predicted critical values are not valid. Fig. 7 demonstrates this case where we have the same configuration as in Fig. 6 but where $a_1 - a_2 = 1.25$ instead of 0.05. The critical values of α and P are $0.21\alpha^*$ and 0.203, respectively, which are considerably smaller than the predicted values of α^* and 0.32, respectively, in the no streaming limit.

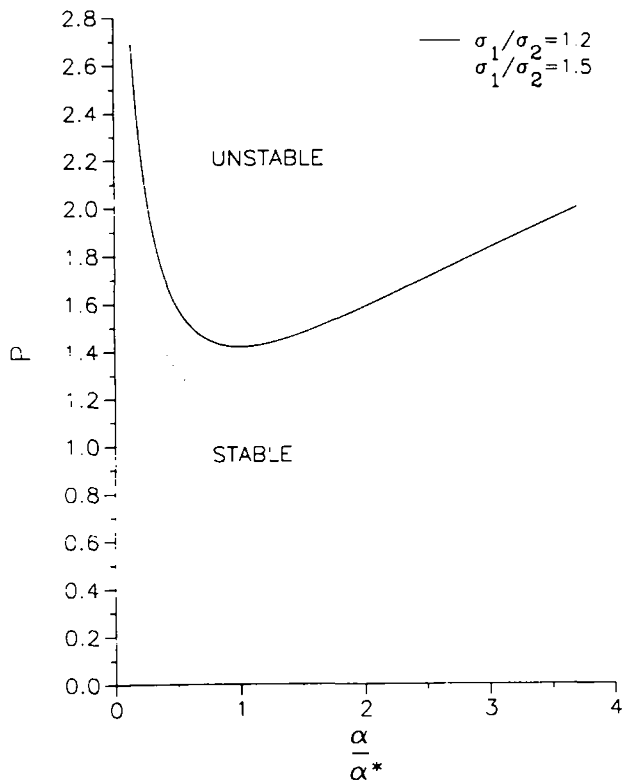

Fig. 7. Marginal stability curves for the case where $c = o(1/\tau)$, $Q = 0.5$ and $a_1 - a_2 = 0.05$.

6. Instantaneous charge relaxation limit

In this section we consider another limiting case involving shear stresses of electrical origin. In contrast to the limiting case discussed in the previous section, here the surface dynamics time scale is relatively long compared to the electric charge relaxation times so that $c = \mathrm{o}(1/\tau)$. Therefore, this configuration represents interfaces between fluids that are highly conducting, for example, an electrolyte and mercury. In this limit, the dispersion relation becomes

$$F = F_1 + \frac{F_2}{\alpha c} - \mathrm{i}\,\frac{P\tilde{F}_3}{\alpha c(\sigma_1 + \sigma_2)} + \frac{QF_4}{\sigma_1 + \sigma_2} + \frac{\varepsilon\tilde{E}_1^2 - \tilde{E}_2^2}{\alpha c} F_5 = 0, \tag{85}$$

where

$$\tilde{F}_3 = (A_2 J_1 + mA_1 J_2)(\tilde{E}_2\sigma_1 - \sigma_2\varepsilon\tilde{E}_1) + (mA_1' J_2 + A_2' J_1 + 2(m-1)J_1 J_2 + \mathrm{i}(\alpha^{-1}(1-r)J_1 J_2)(\varepsilon\tilde{E}_1\sigma_2 + \tilde{E}_2\sigma_1).$$

For short wavelength limits, the asymptotic expressions for the Airy functions and their integrals, given by Eq. (62), can be substituted in the above dispersion relation to give the following approximation for the eigenvalue c:

$$c = b_0 + b_1\alpha^{-1} + \mathrm{o}(\alpha^{-1}), \tag{86}$$

where

$$b_0 = -\frac{\mathrm{i}S}{2(1+m)}, \qquad b_1 = \mathrm{i}\,\frac{(\tilde{E}_1\varepsilon k - \tilde{E}_2)P}{2(1+m)(1+k)} - \frac{3}{16}\,\frac{\mathrm{i}(1+r)S^2}{(1+m)^3}, \tag{87}$$

where

$$k = \sigma_2/\sigma_1. \tag{88}$$

From this expression, we deduce that the growth rate for the short wavelength instability depends on the ratio of the conductivities of the two fluids. If the lower fluid has a much greater conductivity relative to the conductivity of the upper fluid, then the above expression reduces to Eq. (61) with Γ^e corresponding to the (EH-If) limit. On the other hand, if the fluids have nearly equal relaxation times so that there is no equilibrium charge, then the above expression reduces to Eq. (61) with Γ^e corresponding to the (EH-Ip) configuration. Therefore, in the instantaneous relaxation limit the nature of the charge interaction and the stability of the flow are closely related to the conduction and relaxation of the electric charges.

By letting $c \to 0$ in the dispersion relation, we determine the following condition for the onset of static instability:

$$\alpha^2 + \alpha^{*2} - \frac{B\alpha}{S}((\sigma_1 + \sigma_2)(mJ_2 A_1 + J_1 A_2) + Q^2 J_1 J_2) = 0, \tag{89}$$

where

$$B = (P\tilde{F}_3 + Q^2(\varepsilon\tilde{E}_1^2 + \tilde{E}_2^2) + \mathrm{i}(a_2 - a_1)(J_1 A_2(\varepsilon\tilde{E}_1 + \tilde{E}_2)Q + (A_2' J_1 - J_1 J_2)Q^2 + m(2J_1 A_2 + 2A_1 J_2 + A_1' A_2 - A_2' A_1)(\sigma_1 + \sigma_2))).$$

For large values of α^*, Eq. (89) reduces to the following simple condition for the incipience of static instability:

$$\alpha^2 - \frac{(\tilde{E}_1 \varepsilon k - \tilde{E}_2)P}{S(1+k)}\alpha + \alpha^{*2} = 0. \tag{90}$$

Therefore, the critical electric fields for the incipience of instability are given by

$$2\alpha^*(1+k)(1-\varepsilon)S = \varepsilon P^2(k-1) + PQ(1-\varepsilon k). \tag{91}$$

If k is large, then the above expression reduces to Eq. (72) which represents the critical electric field required for the incipience of static instability for the (EH-If) configuration. For a given surface charge Q, the critical P for the incipience of instability is given by

$$P^* = \frac{Q(\varepsilon k - 1)}{2\varepsilon(k-1)} + \frac{\sqrt{Q^2(1-\varepsilon k)^2 + 8\alpha^* S(1+k)(1-\varepsilon)\varepsilon}}{2\varepsilon(k-1)}. \tag{92}$$

If the free charge Q is large enough that the argument of the square root in the above equation is positive, then the conductivity ratio is destabilizing. Otherwise, short wavelength disturbances are stabilized by the conductivity jump.

Fig. 8 depicts the marginal stability curves for a configuration of two fluids with instantaneous relaxation times, an equilibrium surface charge of $Q = 0.5$, $\alpha^* = \sqrt{20}$ and $a_1 - a_2 = 0.05$. The figure shows similarity with the other limiting cases we have seen so far. The incipience of the instability in each case occurs at α^* and the critical values of P for which the first unstable mode occurs at 1.420 and 1.19 for $k = 1.5$ and $k = 1.25$, respectively. These values are very close to the predicted values of 1.418 and 1.212 determined from Eq. (92). This figure demonstrates that, in the instantaneous relaxation limit, the stability of the flow is closely related to the conduction and the relaxation process. The ratio of the conductivities, the equilibrium free charge and the permittivity ratio determine the general stability behaviour of the flow.

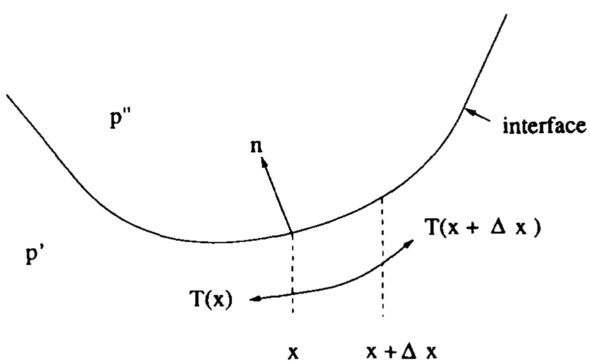

Fig. 8. Definition sketch for developing interfacial boundary conditions.

7. Relaxation time effects

The limiting cases we have considered so far involve configurations whose surface dynamics time scale are either very short or very long compared to the electric charge relaxation times of the fluids. In this section we look at configurations in which the time scale of the surface dynamics is comparable to the electric charge relaxation time ($c = O(\tau)$). These configurations are important since, for most real systems, the limiting cases we have considered are approximations only. Even though these approximations have been quite successful in modeling many real systems [15], relaxation effects are likely to be important in configurations where equilibrium charges are present. Furthermore, it is believed that relaxation time effects have important implications for fluid interactions involving bulk coupling of the fluids [18].

In this configuration the possibility of incipience of static instability (i.e. the principle exchange of stability is valid) is determined by

$$(\alpha^2 + \alpha^{*2})(H_e^2 + 1) - H_e^2(\varepsilon\tilde{E}_1^2 + \tilde{E}_2^2)\alpha - \frac{(\tilde{E}_1\varepsilon k - \tilde{E}_2)P}{S(1+k)}\alpha$$

$$- \frac{P(\tilde{E}_1\varepsilon k + \tilde{E}_2) - (\varepsilon\tilde{E}_1^2 + \tilde{E}_2^2)(1+k)}{1+k} Z_3\alpha + \mathrm{i}(a_2 - a_1)\frac{Z_1(\sigma_1 + \sigma_2) + QJ_1Z_2}{S(mJ_2 + J_1)(\sigma_1 + \sigma_2)}\alpha = 0,$$

where Z_1 and Z_2 are given by (69) and (81), respectively, and

$$Z_3 = \frac{A_2'J_1 + mA_1'J_2 - 2(1-m)J_1J_2}{mJ_2 + J_1}, \tag{93}$$

$$H_e^2 = \frac{Q^2J_1J_2}{(mJ_1 + J_2)(\sigma_1 + \sigma_2)}. \tag{94}$$

The quantity H_e, generalized electric Hartmann number, is a measure of the relative effects of the electric forces and the mechanical forces [18], and it is analogous to the Hartmann number of magneto-hydrodynamics.

Clearly, in the presence of streaming, the imaginary part of the above equation may be nonzero and therefore, the onset of instability may not be static. In the short wavelength limit and where there is relatively small streaming, the condition for the static instability reduces to

$$(\alpha^2 + \alpha^{*2})(H_{el}^2 + 1) - H_{el}^2(\varepsilon\tilde{E}_1^2 + \tilde{E}_2^2)\alpha - \frac{(\tilde{E}_1\varepsilon k - \tilde{E}_2)P}{S(1+k)}\alpha = 0, \tag{95}$$

where the electric Hartmann number H_{el} is given by

$$H_{el}^2 = \frac{Q^2}{2(m+1)(\sigma_1 + \sigma_2)}. \tag{96}$$

Therefore, as the electric field is raised the first unstable mode will occur at α^* again. The critical electric field must be such that

$$2\alpha^*(1 + H_{el}^2) = H_{el}^2(\varepsilon\tilde{E}_1^2 + \tilde{E}_2^2) + \frac{(\tilde{E}_1\varepsilon k - \tilde{E}_2)P}{S(1 + k)}. \tag{97}$$

Therefore, the general stability behaviour of the flow is characterized by the Hartmann number. If the Hartmann number is large compared to unity, then Eq. (95) reduces to Eq. (82) which is the condition for the onset of static instability for the infinite charge relaxation limit. On the other hand, if the Hartmann number is small in comparison to unity, then Eq. (95) reduces to Eq. (90) which is the condition for the onset of static instability for the instantaneous charge relaxation limit.

If the Hartmann number is of order one, then the stability is generally determined by the conductivity ratio, k, the permittivity ratio ε and the Hartmann number. While H_{el} is always destabilizing, the effects of the conductivity ratio are determined by the quantity $(\tilde{E}_1\varepsilon k - \tilde{E}_2)P$. If this quantity is negative, then k is stabilizing. Otherwise it is destabilizing.

8. Concluding remarks

We examined the electrohydrodynamic stability of a shear flow which is subjected to perpendicular electric fields. We investigated various limiting cases and developed conditions for the incipience of static instability (characterized in the marginal state by $c = 0$). As the streaming parameters $a_i \to 0$, our results agreed with previous works [15]. In the long wavelength limit the electric field does not affect the stability behaviour. Short wavelength analysis, however, deduces that the electric field effects are of secondary importance compared to the effects of surface tension. In the presence of initial motion, static stability is not generally possible except in the limit of short wavelengths. Finally, we considered the effect of relaxation times that are comparable to the time scales of surface dynamics. We found that the electric field effects are characterized by the Hartmann number and the ratio of the conductivities. For large Hartmann numbers the threshold for static instability reduces to the threshold found for the infinite charge relaxation limit and for small Hartmann numbers it reduces to the instantaneous charge relaxation limit. While a nonzero Hartmann number causes instability, the effects of the ratio of the conductivities is determined by the specific configurations.

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Appendix A. Derivation of boundary conditions at the interface

A.1. Surface tension force

Let us use rectangular coordinates (x_1, x_2) with velocity (u_1, u_2) and electric field (E_1, E_2). The interface is given by $y = \eta(x, t)$ so the normal and tangential vectors are given by

$$\boldsymbol{n} = (n_1, n_2) = \left(\frac{-\eta_x}{(1+\eta_x^2)^{1/2}}, \frac{1}{(1+\eta_x^2)^{1/2}}\right), \tag{1}$$

$$\boldsymbol{t} = (t_1, t_2) = \left(\frac{1}{(1+\eta_x^2)^{1/2}}, \frac{\eta_x}{(1+\eta_x^2)^{1/2}}\right), \tag{2}$$

see Fig. 8. Then the tension force can be written as

$$\boldsymbol{T}(x) = \gamma \boldsymbol{t}.$$

If we let F to be the normal force per unit length, we have

$$F\,\mathrm{d}s = \boldsymbol{n}\cdot(\boldsymbol{T}(x+\Delta x) - \boldsymbol{T}(x)),$$

where $\mathrm{d}s = (1+\eta_x^2)^{1/2}\Delta x$.

We now substitute for $\boldsymbol{T}(x)$ and let $\Delta x \to 0$. Hence

$$F = \gamma \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}.$$

Since

$$\frac{1}{R} = \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}$$

we get

$$F = 1/R, \tag{3}$$

where R is the radius of curvature.

If there are no viscous or electric stresses at the interface $F = p'' - p'$ then

$$p'' - p' = \gamma/R$$

which is Laplace's formula in two dimensions.

A.2. Stress tensors

The viscous stress tensor for an incompressible fluid can be written in the form

$$T_{ij}^{m} = -p\delta_{ij} + 2\mu e_{ij},$$

where

$$e_{i,j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and μ is the viscosity. See [3, p. 147].

Similarly we can write the electric stress tensor as

$$T^{e}_{ij} = \varepsilon E_i E_j - \tfrac{1}{2}\varepsilon\delta_{ij}E_kE_k + \tfrac{1}{2}\rho\delta_{ij}\left(\frac{\partial\varepsilon}{\partial\rho}\right)_{\theta} E_kE_k,$$

where the subscript θ indicates an isothermal process.

A.3. Normal force

We have already shown that the normal force is equal to γ/R, Eq. (3), so we get

$$[T^{m}_{ij}]n_j + [T^{e}_{ij}]n_j + [\rho g\eta]\delta_{i2} = -\frac{\gamma}{R}n_i, \tag{4}$$

where $[X] = X'' - X'$. Hence the normal component is

$$[T^{m}_{ij}]n_jn_i + [T^{e}_{ij}]n_jn_i + [\rho g\eta]n_2 = -\gamma/R, \tag{5}$$

where n_i is the ith component of the unit normal vector $\boldsymbol{n}$.

Substitute

$$\left[-p + 2\mu e_{ij}n_jn_i + \varepsilon E_iE_jn_jn_i - \tfrac{1}{2}\varepsilon\delta_{ij}E_kE_kn_jn_i - \tfrac{1}{2}\rho\left(\frac{\partial\varepsilon}{\partial\rho}\right)_{\theta}E_kE_k + \rho g\eta n_2\right] = -\frac{\gamma}{R}.$$

Define a reduced pressure by

$$\pi = p - \tfrac{1}{2}\rho\left(\frac{\partial\varepsilon}{\partial\rho}\right)_{\theta}(E_1^2 + E_2^2). \tag{6}$$

Now

$$2\mu e_{ij}n_jn_i = 2\mu\left(\frac{\partial u}{\partial x}n_x^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)n_xn_y + \frac{\partial v}{\partial y}n_x^2\right),$$

where we have used $x_1 = x$, $x_2 = y$, $u_1 = u$, $u_2 = v$, $n_1 = n_x$, $n_2 = n_x$ and $E_1 = E_x$, $E_2 = E_y$.

$$\varepsilon E_iE_jn_jn_i = \varepsilon(E_x^2n_x^2 + 2E_xE_yn_xn_y + E_y^2n_y^2),$$

$$\varepsilon\delta_{ij}E_kE_kn_jn_i = \varepsilon(E_x^2 + E_y^2).$$

Hence the normal boundary condition becomes

$$\left[-\pi + 2\mu\left(\frac{\partial u}{\partial x}n_x^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)n_xn_y + \frac{\partial v}{\partial y}n_x^2\right)\right.$$
$$\left. + \rho g\eta n_y + \varepsilon(E_x^2n_x^2 + 2E_xE_yn_xn_y + E_y^2n_y^2) - \tfrac{1}{2}\varepsilon(E_x^2 + \tilde{E}_x^2)\right] = -\gamma/R.$$

A.4. Tangential force

The tangential component of (4) is

$$[T^{\mathrm{m}}_{ij}]n_j t_i + [T^{\mathrm{e}}_{ij}]n_j t_i + [\rho g \eta]\delta_{i2} t_i = -\frac{\gamma}{R} n_i t_i,$$

where t_i is the ith component of the unit tangential vector $\boldsymbol{t}$.

Now

$$T^{\mathrm{m}}_{ij} n_j t_i = 2\mu e_{ij} n_j t_i = 2\mu\left(\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)n_x n_y + \tfrac{1}{2}(n_y^2 - n_x^2)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)$$

and

$$T^{\mathrm{e}}_{ij} n_j t_i = \varepsilon(E_x^2 - E_y^2)n_x n_y + \varepsilon E_x E_y (n_y^2 - n_x^2),$$

where we have used the fact that $t_x = n_y$ and $t_y = -n_x$.

Hence the tangential boundary conditions become

$$\left[2\mu\left(\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)n_x n_y + (n_y^2 - n_x^2)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \varepsilon(E_x^2 - E_y^2)n_x n_y + \varepsilon E_x E_y (n_y^2 - n_x^2) - \rho g \eta n_x\right] = 0.$$

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